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FIXED POINTS AND CHARACTERIZATIONS OF CERTAIN MAPS

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Let T be a self map on a metric space (X, d) such that

$$d(T(x), T(y)) \leqq (d(x, T(x)) + d(y, T(y))/2 , \qquad x, y \in X .$$

It is proved that: (a) T has a fixed point if T is continuous and X is weakly compact convex subset of a Banach space. (b) All such T which have fixed points can be explicitly determined in terms of d. Related results are obtained.

1. Introduction. In [7], [8], [9], [10], [11], R. Kannan considered the following family of self maps T on a (nonempty) complete metric space (X, d):

$$(1) \qquad d(T(x), T(y)) \leqq \frac{1}{2}(d(x, T(x)) + d(y, T(y))) , \qquad x, y \in X .$$

He obtained a number of results of the following type: T has a (unique) fixed point if X is a weakly compact convex subset of a reflexive Banach space B and for each closed convex subset H of X with $T(H) \subset H$ and $\delta(H) > 0$,

$$(2) \qquad \sup \{d(y, T(y)) : y \in H\} < \delta(H) ,$$

where d is the metric induced by the norm $\| \ \|$ on B and $\delta(H)$ is the diameter of H. Suppose now that X is a weakly compact convex subset of a Banach space B and T is a self map on X which satisfies (1). P. Soardi [17, Theorems I, II] proved that T has a fixed point if either X has normal structure [3] or T has diminishing orbital diameters [2]. In this paper, the following results are obtained: (a) T has a fixed point if it is continuous (with respect to the strong topology). In fact, T has a fixed point if it is continuous along line segments. It may be worthwhile to mention here that it is a well-known open problem that the same conclusion holds for nonexpansive self maps on X [1, p. 217]. (b) T has a fixed point if for any closed convex subset H of X with more than one point and $T(H) \subset H$,

$$\inf \{d(T(y), y) : y \in H\} < \delta(H) .$$

It is obvious that this result generalizes the above results of Soardi and the above result of Kannan. (c) Let T be a self map on X such that there exist a_1, a_2, a_3, a_4, a_5 in $[0, 1]$ for which $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and for all x, y in X,

$$(3)\quad \begin{aligned} d(T(x), T(y)) \leqq a_1 d(x, T(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) \\ + a_4 d(y, T(x)) + a_5 d(x, y) . \end{aligned}$$

Then T has a fixed point u if for some x_0 in X, the sequence $\{x_n\}$, where $x_{n+1} = (x_n + T(x_n))/2$, converges to u. Note that by calculating $(d(T(x), T(y)) + d(T(y), T(x)))/2$ through (3), one may assume that $a_1 = a_2$ and $a_3 = a_4$. Thus when $a_3 = a_4 = a_5 = 0$, T satisfies (1) if it satisfies (3). Also, (3) is satisfied if T is nonexpansive. Serious consideration of those T which satisfy (3) with $a_1 = a_2 = a_3 = a_4 = 0$ was started by M. Edelstein [5]. In [14], [15], and [16], S. Reich considered those T which satisfy (3) with $a_3 = a_4 = 0$ and thus combined in a natural way, those maps investigated by Edelstein and Kannan. Recently, G. Hardy and T. Rogers [6] considered those maps which satisfy (3) with the possibility that none of the coefficients a_i is zero.

For the interest of the reader, we pose the following statement: Every self map T on a weakly compact convex subset X of a Banach space has a fixed point if

$$\| T(x) - T(y) \| \leqq \frac{1}{2}(\| x - T(x) \| + \| y - T(y) \|) , \qquad x, y \in X .$$

From (a), (b), and (c), a counter example, if exists, would be difficult to construct.

Let T be a self map on a metric space (X, d) which satisfies (1). Once one succeeds in proving that T has a fixed point, it is possible to characterize such T by its fixed point. Because of this observation, the family of all self maps T on a metric space (X, d) which have a fixed point and which satisfy (1) can be found explicitly. As an example, the family of all such self maps T on the unit interval is illustrated. There are exactly 2^c such maps, where c is the cardinal of the real line.

2. Fixed point theorems. Let T be a self map on a convex subset X of a normed linear space. T is *continuous along line segments* if for any x, y in X, $\{T(x_t)\}$ converges to $T(x)$ as t tends to 0, where $x_t = (1 - t)x + ty$, $t \in (0, 1)$. Thus T is continuous along line segments if T is continuous.

THEOREM 1. *Let X be a (nonempty) weakly compact convex subset of a normed linear space. Let T be a self map on X which is continuous along line segments and satisfies*

$$\| T(x) - T(y) \| \leqq (\| x - T(x) \| + \| y - T(y) \|)/2 , \qquad x, y \in X .$$

Then T has a unique fixed point in X.

Proof. By Zorn's lemma, there exists a minimal nonempty weakly compact convex subset H of X such that $T(H) \subset H$. Let $x \in H$, $r = \|x - T(x)\|$. It suffices to prove that $r = 0$. Consider

$$W = \{y \in H: \|y - T(y)\| \leqq r\} .$$

Since $x \in W$, $W \neq \emptyset$. Since $\|T^2(z) - T(z)\| \leqq \|T(z) - z\|$ for each z in X, $T(W) \subset W$. Let V be the closure of the convex hull of $T(W)$. We shall prove that $V \subset W$. Let $v \in V$ and $\varepsilon > 0$. Then there exist $y_1, y_2, \cdots, y_n$ in W and $t_1, t_2, \cdots, t_n$ in $[0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$ and

$$\left\|v - \sum_{i=1}^{n} t_i T(y_i)\right\| < \varepsilon .$$

Thus

$$\begin{aligned}
\|v - T(v)\| &\leqq \left\|v - \sum_{i=1}^{n} t_i T(y_i)\right\| + \left\|\sum_{i=1}^{n} t_i T(y_i) - T(v)\right\| \\
&< \varepsilon + \sum_{i=1}^{n} t_i \|T(y_i) - T(v)\| \\
&\leqq \varepsilon + \Big(\sum_{i=1}^{n} t_i(\|y_i - T(y_i)\| + \|v - T(v)\|)\Big)/2 \\
&\leqq \varepsilon + (r + \|v - T(v)\|)/2 .
\end{aligned}$$

So $\|v - T(v)\| \leqq 2\varepsilon + r$. Since ε is arbitrarily chosen, $\|v - T(v)\| \leqq r$, i.e., $v \in W$. Thus $V \subset W$. So

$$T(V) \subset T(W) \subset V .$$

By minimality of H, $V = H$. So $W = H$. Since x is arbitrarily chosen, it follows from $W = H$ that $\|z - T(z)\| = r$ for all z in H. Now let $y = T(x)$. Then

$$\begin{aligned}
r &= \|(1 - t)T(x) + tT(y) - T((1 - t)T(x) + tT(y))\| \\
&\leqq (1 - t)\|T(x) - T((1 - t)T(x) + tT(y))\| \\
&\quad + t\|T(y) - T((1 - t)T(x) + tT(y))\| \\
&\leqq \frac{(1 - t)}{2}(\|x - T(x)\| + \|(1 - t)T(x) + tT(y) \\
&\quad - T((1 - t)T(x) + tT(y))\|) \\
&\quad + \frac{t}{2}(\|y - T(y)\| + \|(1 - t)T(x) + tT(y) \\
&\quad - T((1 - t)T(x) + tT(y))\|) \\
&\leqq \frac{1 - t}{2}(r + r) + \frac{t}{2}(r + r) = r .
\end{aligned}$$

Therefore, all the above inequalities are equalities. So

$$\| T(y) - T((1-t)T(x) + tT(y)) \| = r .$$

Since T is continuous along line segments, we have by letting t tend to 0

$$\| T(y) - T^2(x) \| = r .$$

Since $y = T(x)$, $r = 0$.

THEOREM 2. *Let X be a weakly compact convex subset of a Banach space B. Let T be a self map on X such that*

$$\| T(x) - T(y) \| \leqq (\| x - T(x) \| + \| y - T(y) \|)/2 , \qquad x, y \in X .$$

Suppose that for any closed convex subset H of X with $T(H) \subset H$ and $\delta(H) > 0$,

$$(4) \qquad \inf \{\| y - T(y) \| : y \in H\} < \delta(H) .$$

Then T has a unique fixed point.

Proof. Construct H as in the proof of Theorem 1. Suppose to the contrary that $\delta(H) > 0$. By hypothesis, there exists x in H such that

$$r \equiv \| x - T(x) \| < \delta(H) .$$

Construct V as in the proof of Theorem 1. It was proved that $V = W = H$ and $r = \| y - T(y) \|$ for each y in H. Let $u, v \in H$. Then

$$\begin{aligned} \| T(u) - T(v) \| &\leqq (\| u - T(u) \| + \| v - T(v) \|)/2 \\ &= (r + r)/2 = r . \end{aligned}$$

So $\delta(T(H)) \leqq r$. Thus

$$\delta(H) = \delta(V) = \delta(T(W)) \leqq r < \delta(H) ,$$

a contradiction.

We owe the above argument to Kannan [11] and Soardi [17]. Variants of such argument were used in [4] by R. DeMarr and were later refined by W. A. Kirk [12] for nonexpansive mappings. Then they were used in numerous other articles, e.g., [1], [2], and [13]. We would like to emphasize here that even for reflexive Banach spaces, our results are more general than the corresponding results of Kannan [Theorems 1 and 2, 11].

THEOREM 3. *Let X be a convex subset of a normed linear space B. Let T be a self map on X. Suppose that there exist a_i, $i = 1, 2, 3, 4, 5$, in $[0, 1]$ such that $\sum_{i=1}^{5} a_i = 1$ and for all x, y in X,*

$$\|T(x) - T(y)\| \leqq a_1\|x - T(x)\| + a_2\|y - T(y)\| + a_3\|x - T(y)\| + a_4\|y - T(x)\| + a_5\|x - y\| .$$

Let $x_0 \in X$, $t \in (0, 1)$ and $x_{n+1} = (1-t)x_n + t\,T(x_n)$ for each integer $n \geqq 0$. Suppose that the sequence $\{x_n\}$ converges to a point u in X. Then u is a fixed point of T.

Proof. Let $n \geqq 0$. Then

$$\begin{aligned}\|x_{n+1} - T(u)\| &= \|(1-t)(x_n - T(u) + t(T(x_n) - T(u))\| \\ &\leqq (1-t)\|x_n - T(u)\| + t\|T(x_n) - T(u)\| .\end{aligned} \tag{5}$$

By hypothesis,

$$\begin{aligned}\|T(x_n) - T(u)\| \leqq a_1\|x_n - T(x_n)\| &+ a_2\|u - T(u)\| \\ &+ a_3\|x_n - T(u)\| + a_4\|u - T(x_n)\| \\ &+ a_5\|x_n - u\| .\end{aligned} \tag{6}$$

Since $\{x_m\}$ converges to u, and $x_{n+1} - x_n = t(T(x_n) - x_n)$, $\{T(x_m) - x_m\}$ converges to 0. Thus $\{u - T(x_m)\}$ converges to 0. From (5) and (6), we have by letting n tend to ∞,

$$\begin{aligned}&\|u - T(u)\| \\ &\leqq (1-t)\|u - T(u)\| + t(a_2)\|u = T(u)\| + a_3\|u - T(u)\| \\ &= ((1-t) + t(a_2 + a_3))\|u - T(u)\| .\end{aligned} \tag{7}$$

We may assume that $a_1 = a_2$ and $a_3 = a_4$. Thus $a_2 + a_3 \leqq 1/2$. So from (7),

$$\|u - T(u)\| \leqq \left(1 - \frac{1}{2}t\right)\|u - T(u)\| .$$

Since $1 - (1/2)t \in (0, 1)$, $T(u) = u$.

3. Examples. Let a be a point in a metric space (X, d). X_a will denote the set

$$\{(x, y) \in X \times X : d(y, a) \leqq \frac{1}{2}d(x, y)\} .$$

THEOREM 4. *Let T be a self map on a metric space. Then the following coditions are equivalent:*

(a) *T has a fixed point and*

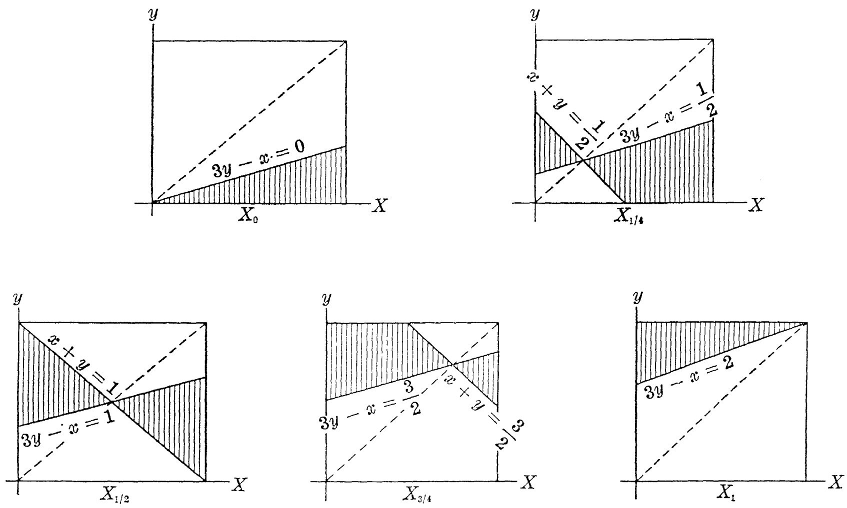
$$d(T(x), T(y)) \leqq (d(x, T(x)) + d(y, T(y)))/2$$

for all x, y in X.

(b) *There exists a in X such that* $T \subset X_a$, i.e.,

$$d(T(x),\, a) \leqq d(x,\, T(x))/2\ , \qquad x \in X\ .$$

To see the use of Theorems 2 and 4, let T be a self map on the unit interval with the usual metric. Then T satisfies (1) if and only if there exists a in X such that $T \subset X_a$. So in order to find all the maps considered by Kannan, it suffices to find all X_a's with a in X. For illustration, we list X_0, $X_{1/4}$, $X_{1/2}$, $X_{3/4}$, X_1 as follows (the shaded parts): Thus X_a is determined by the lines $x + y = 2a$ and



$3y - x = 2a$: If one cuts $X \times X$ along these lines, then X_a is the closure (in $X \times X$) of the union of those parts which do not contain more than one point of the diagonal of $X \times X$.

Once it was difficult to construct enough examples of those maps considered by Kannan. Now it seems too easy to find a lot. It is then only natural to be more demanding. By looking at the above example where $X = [0, 1]$, we find that it is difficult for a self map on X which satisfies (1) to be surjective. In [18], it was proved that if T is a self map on [0, 1] which satisfies (1) and if $0, 1 \in T[0, 1]$, then 1/2 is the fixed point of T. This result is an obvious consequence of Theorems 2 and 4. For the interest of the reader, we list two more consequences of Theorems 2 and 4: (a) Let T be a self map on $X = [0, 1]$ such that $0, 1 \in T([0, 1)]$ and T is not the identity map on [0, 1]. Then the following conditions are equivalent: (i) T satisfies (1). (ii) T satisfies (3). (iii) For any $x, y \in X$,

$$d(T(x),\, T(y))$$
$$\leqq \sup\{d(x, y),\, (d(x, T(x)) + d(y, T(y)))/2,\, (d(x, T(y)) + d(y, T(x)))/2\}\ .$$

(b) Let X be a regular polygon in the two dimensional Euclidean space. Then there exists a self map T of X onto X which satisfies (1) if and only if X has even number of sides.

Added in proof. Let K be a closed convex subset of a Banach space B. K has a *close-to-normal structure* if for any bounded closed convex subset H of K with $\delta(H) > 0$, there exists x in H such that $\|x - y\| < \delta(H)$ for all y in H. The following related results were announced by the author at the International Congress of Mathematicians in 1974. (a) B has a close-to-normal structure if (1) B is separable, (2) B is strictly convex, or (3) B is reflexive and for any sequence $\{x_n\}$ in B, $\{x_n\}$ converges to x in B whenever $\{x_n\}$ converges weakly to x and $\{\|x_n\|\}$ converges to $\|x\|$. (b) A Hilbert space is isomorphic to a Banach space which has no close-to-normal structure if and only if it is separable. (b) together with Theorem 4 in Chi Song Wong, Proc. Amer. Math. Soc., **46** (1974), to appear, solve the problem we posed in the introduction of this paper. Related to (b), we mentioned here that all theorems for normal structure (=completely normal structure) in L. P. Belluce, W. A. Kirk and E. P. Steiner, Normal structure in Banach spaces, Pacific J. Math., **26** (1968) 433-440, can be modified to results for close-to-normal structure.

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