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ABIAN'S ORDER RELATION AND ORTHOGONAL COMPLETIONS FOR REDUCED RINGS

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Chacron has shown that, in a ring R, the relation "$a \leqq b$ iff $ab = a^2$", first studied by Abian, is an order relation iff R is reduced (has no nilpotent elements). Let R be a reduced ring with 1, a set X in R is *orthogonal* if $ab = 0$ for all $a \neq b$ in X and R is *orthogonally complete* if every orthogonal set in R has a supremum with respect to "$\leqq$". A strongly regular ring is shown to be right (and left) self-injective iff it is orthogonally complete. If $R \subset S$ are reduced rings, S is an *orthogonal extension* of R if every element of S is the supremum of an orthogonal set in R; an orthogonal extension which is complete is an *orthogonal completion*. Completions are unique if they exist. An example shows that not all reduced rings have completions but if R is strongly regular, its complete ring of quotients, $Q(R)$, is its completion. Further, if R is reduced, Baer and such that $Q(R)$ is strongly regular then R has a completion which is a partial ring of quotients.

1. Orthogonal completeness and injectivity. The usual order relation in a Boolean ring extends to reduced rings R when expressed as: $a \leqq b$ iff $ab = a^2$ ([1] and [5]). *In what follows all rings referred to will be reduced* (i.e., 0 is the only nilpotent element) *and with* 1. The basic facts about reduced rings required below can be found in [13] and some of these are quoted here for convenience. If $X \subset R$ then the left and right annihilators of X coincide and will be denoted $\mathrm{Ann}_R X$ or $\mathrm{Ann}\, X$. Also the left and right singular ideals are always trivial and, so, the left and right complete rings of quotients, $Q_l(R)$ and $Q_r(R)$, are always regular. Further, $Q_l(R) = Q_r(R)(= Q(R))$ iff $aR \cap bR = 0$ implies $ab = 0$ for all $a, b \in R$. In this case $Q(R)$ is strongly regular (i.e., $Q(R)$ is also reduced). We note also that all idempotents of a ring R are central and that if R is strongly regular it is duo (i.e., all one-sided ideals are two-sided).

The order relation on a ring R makes R into a partially-ordered multiplicative semigroup since $a \leqq b$ and $c \leqq d$ imply $ac \leqq bd$ ([5]). Also, if $a \leqq b$ in R then $ab = ba$ for $a \leqq b$ implies that $(ab - ba)^2 = 0$. Hence all order properties are right-left symmetric.

In the sequel, if X is a subset of a ring R, $\sup_R X$ or $\sup X$ will always refer to the supremum with respect to "$\leqq$". It is shown in [2] that there is an infinite distributive law in reduced rings. That is, if $X \subset R$ and $\sup X = a$ exists then for any $b \in R$, $\sup bX =$

ba and $\sup Xb = ab$. This is a very useful tool.

LEMMA 1. *If R is a ring and X a subset such that for each $x \in X$, $x^n = x$ (some fixed integer n) and if $\sup X = y$ exists then $y^n = y$. In particular the supremum of a set of idempotents, if it exists, is an idempotent.*

Proof. By the infinite distributivity $y^n = y^{n-1}(\sup X) = \sup(y^{n-1}X)$. But since $y = \sup X$ and $x^n = x$ for all $x \in X$, $y^{n-1}x = x^n = x$ and $\sup y^{n-1}X = \sup X = y$.

The following theorem is not only of independent interest, giving the result of Brainerd and Lambek [4] on Boolean rings as a special case, but is also a tool in the remainder of this article. Recall that a subset X of a ring R is *orthogonal* if $ab = 0$ for all $a, b \in X$, $a \neq b$. R is *orthogonally complete* iff every orthogonal set in R has a supremum. The idea of orthogonal completeness is more useful than completeness for rings which are not Boolean since rings which are not Boolean are rarely complete; the only field which is complete is $\boldsymbol{Z}_2$. However, there are interesting orthogonally complete rings such as products of domains. Orthogonal completeness was one of two conditions Chacron, generalizing Abian's theorem, used to characterize direct products of division rings. What follows arose from an attempt to characterize orthogonally complete rings and to generalize the theorem of Brainerd and Lambek ([4]) on the complete ring of quotients of a Boolean ring.

THEOREM 2. *A strongly regular ring R is right self-injective iff it is orthogonally complete.*

Proof. The proof which follows is a direct one but a proof, of about the same length, using the sheaf representation of Pierce ([11]) is also possible.

For one direction we need the following lemmas.

LEMMA 3. *Let R be a Baer ring (every anninilator is generated by an idempotent). If X is a subset of R with an upper bound $a \in R$ then $\sup X = a - ae$ where e is the idempotent so that $\operatorname{Ann} X = eR$.*

Proof. For $x \in X$, $x(a - ae) = x^2$ so $a - ae$ is also an upper bound for X. If b is any upper bound, $x(b - a) = 0$ for all $x \in X$ and so $b - a = er$ for some $r \in R$. Then, $(a - ae)b = (1 - e)ab = (1 - e)(a(b - a) + a^2) = (1 - e)a^2 = (a - ae)^2$ (using throughout that idempotents are central). Hence $a - ae \leqq b$.

LEMMA 4. *Let R be a Baer ring such that $aR \cap bR = 0$ implies $ab = 0$. Then every idempotent of $Q(R)$ is in R and the Boolean algebra of idempotents of R is isomorphic to the complete Boolean algebra of annihilator ideals of R.*

Proof. In [12, Lemma 1.6] it is shown that for a commutative Baer ring R, R contains all the idempotents of $Q(R)$. Since, here, $Q(R)$ is duo, a trivial modification of the proof gives the result in the present situation. For the rest, the arguments of [10, § 2.4], for the commutative case carry over without change.

We can also note that, in general, if $\sup X = a$ exists then $\operatorname{Ann}\{a\} = \operatorname{Ann} X$. Indeed if $ra = 0$ then $\sup rX = ra = 0$ and $rX = \{0\}$ and if $rX = \{0\}$, $\sup rX = ra = 0$. Conversely, if a is an upper bound for X and $\operatorname{Ann}\{a\} = \operatorname{Ann} X$ then a is the supremum. For if b is another upper bound, $X(a - b) = 0$ so $a(a - b) = 0$ and $a \leqq b$.

Returning now to the proof of the theorem, let R be right self-injective. If X is an orthogonal subset of R, $I = \sum_{x \in X} xR$ is a direct sum and $\phi: I \rightarrow R$ defined by $\phi(x) = x^2$ for all $x \in X$, is an R-homomorphism. Hence there is $a \in R$ so that $\phi(x) = ax = x^2$ for all $x \in X$. It follows that a is an upper bound for X and by (3) and (4) X has a supremum.

Conversely, if R is orthogonally complete and $\phi: I \rightarrow R$ an R-homomorphism where I is a large right ideal then we with to lift ϕ to an endomorphism. (It suffices to consider large right ideals by, for example, [10, Exercise 4, p. 93].) Let X be a maximal orthogonal set in I; it is easily seen ([14]) that $\bigoplus_{x \in X} xR$ is also large. Indeed, if $0 \neq r \in R$ there is $s \in R$ with $0 \neq rs \in I$. But, by maximality of X, for some $x \in X$, $rsx \neq 0$ and since a strongly regular ring is duo, $rsx \in xR$.

Now for each $x \in X$ let e_x be its corresponding idempotent, $e_x = xx' = x'x$ where $x^2x' = x$, $x'^2x = x'$. If $\phi(e_x) = a_x$, the set $\{a_x\}_{x \in X}$ is orthogonal since for $x \neq y$, $a_x a_y = \phi(e_x)\phi(e_y) = \phi(e_x)e_x\phi(e_y)e_y = a_x a_y e_x e_y = a_x a_y x'xyy' = 0$. Put $a = \sup\{a_x\}$. For all $x \in X$ it will be shown that $ae_x = a_x e_x$. For $y \neq x$ in X,

$$a_y e_x = \phi(e_y)e_x = \phi(e_y)e_y e_x = 0\ ;$$

hence,

$$ae_x = \sup\{a_y e_x\} = \sup\{0, a_x e_x\} = a_x e_x\ .$$

The result now follows since $\phi(x) = \phi(e_x)x = a_x x = a_x e_x x = ae_x x = ax$ and multiplication by a and the homomorphism ϕ coincide on the large right ideal $\bigoplus_{x \in X} xR$.

COROLLARY 5. *A Boolean ring B is orthogonally complete iff it is self-injective iff it is complete.*

The equivalence "self-injective iff complete" was first proved by Brainerd and Lambek in [4].

Proof. Only the implication "orthogonally complete implies complete" needs to be proved. If X is a subset of B, let Y be a maximal orthogonal subset of XB with supremum a. If for some $x \in X$, $xa \neq x$ then for all $y \in Y$, $y(xa - x) = 0$. This contradicts the maximality of Y so a is an upper bound for X. However, any upper bound of X is easily seen to be an upper bound of Y so a is the supremum of X.

COROLLARY 6. (Renault [13]). *A strongly regular ring is left self-injective iff it is right self-injective.*

Proof. "Orthogonally complete" is right-left symmetric.

In [6] Connell shows that if a commutative ring R has certain roots of unity, the set $R_q = \{r \in R \mid r^q = r\}$, where q is a prime power, forms a ring with the multiplication of R and a suitable addition. If R is orthogonally complete, (1) shows that each R_q will be orthogonally complete. But when R_q is a ring it is regular so, when this occurs, if R is orthogonally complete then R_q is self-injective. In particular, if R is self-injective so is each of the rings R_q.

There is another class of rings which is easily seen to consist of orthogonally complete rings. This extends the fact that a finite Boolean ring is complete.

LEMMA 7. *The supremum of every finite orthogonal set in a ring R exists and is the sum of its elements.*

PROPOSITION 8. *If R is a reduced ring with ascending chain condition on annihilator ideals then R is orthogonally complete.*

Proof. Let $\{a_i\}_{i \in A}$ be orthogonal with A well-ordered. For each $j \in A$ put $I_j = \{r \in R \mid ra_k = 0 \text{ for all } k > j\}$. Since R is reduced this is a properly ascending chain of annihilators forcing A to be finite. Then, (7) gives the result.

From this one can see that there are orthogonally complete rings which are not Baer, any Noetherian ring which is not Baer will do. For example $R = \boldsymbol{Z}[x, y]/(xy)$.

2. Orthogonal extensions and completions. The aim of this section is to investigate when a ring R (always reduced) may be embedded into an orthogonally complete ring S so that each element of S is the supremum of an orthogonal set in R.

DEFINITION 9. If $R \subseteqq S$ are rings then S is called an *orthogonal extension* of R if every element of S is the supremum of an orthogonal set of R. If $R \subset S$ is an orthogonal extension so that S is orthogonally complete then S is called an *orthogonal completion* of R.

LEMMA 10. *If $R \subset S$ is an orthogonal extension then S is isomorphic over R to a subring of $Q_r(R) \cap Q_l(R) = L(R)$ ($L(R)$ is the maximal two-sided ring of quotients* ([9])).

Proof. Since the singular ideals are zero it suffices to show that S is a right and left essential extension of R. But for $0 \neq s \in S$, $a \leqq s$ for some $0 \neq a \in S$, a any nonzero element of an orthogonal set in R of which s is the supremum, and then $0 \neq as = sa = a^2$.

Although $L(R)$ can be seen to be reduced, it is not known to us if it is orthogonally complete so, in what follows, *we will now assume that our rings R are such that $Q_r(R) = Q_l(R) = Q(R)$ is strongly regular*; i.e., $aR \cap bR = 0$ implies $ab = 0$ for all $a, b \in R$ ([13]). Of course any commutative or duo ring has this property.

LEMMA 11. *Let $X \subset R$ be such that $\sup_R X = a$ exists, then* $\sup_{Q(R)} X = a$.

Proof. If $\sup_R X = a$ and $\sup_{Q(R)} X = q$ then $q \leqq a$. Let D be a large right ideal so that $qD \subseteqq R$. For each $d \in D$, $X(a - q)d = 0$ so, since $\mathrm{Ann}_R \{a\} = \mathrm{Ann}_R X$, $a(a - q)d = 0$. Then $a^2 = aq$ and $a \leqq q$.

From this it follows that when dealing with rings between R and $Q(R)$ it is not necessary to consider in which ring a supremum is found. That is, if X is a subset of R with supremum q in $Q(R)$, then for a ring S, $R \subseteqq S \subseteqq Q(R)$, $\sup_S X$ exists if, and only if, $q \in S$.

THEOREM 12. *Let R be reduced and such that $aR \cap bR = 0$ implies $ab = 0$. Then R has maximal orthogonal extensions in $Q(R)$ which have no proper orthogonal extensions, and R has a unique smallest extension C_R in $Q(R)$ which is orthogonally complete.*

Proof. The existence of maximal orthogonal extensions follows by Zorn's lemma and the rest is a consequence of the fact that "orthogonal extension" is transitive. Indeed suppose $R \subseteqq S$ and $S \subseteqq T$ are orthogonal extensions and $t \in T$. Now $t = \sup X$ for some orthogonal Y_X in R. Then $\bigcup_{x \in X} Y_x$ is orthogonal. For, suppose $y \in Y_x$, $z \in Y_{x'}$, $x \neq x'$, then $yxzx' = y^2z^2$. But, using the fact that $Q(R)$ is duo we get $yxax' = 0$. Hence $y^2z^2 = 0$ and using again that $Q(R)$ is duo we get $(yz)^3 = 0$ and $yz = 0$.

Next, $t = \sup \bigcup_x Y_x$. We have $t \geqq x \geqq y$ for each $x \in X$, $y \in Y_x$. Hence t is an upper bound. If t' is another, $t' \geqq y$ for each $y \in Y_x$ (for each $x \in X$) so $t' \geqq x$. Then $t' \geqq t$.

To find a minimal extension which is orthogonally complete, put $C_R = \bigcap S$ for all S, $R \subseteqq S \subseteqq Q(R)$, so that S is orthogonally complete. By (11), C_R is also orthogonally complete.

Clearly any orthogonal extension is in C_R and any orthogonal completion must be C_R. This shows the uniqueness of orthogonal completions if they exist.

EXAMPLE 13. Let R be the ring of all continuous real-valued functions on $[0, 1]$. It will be shown that R does not have an orthogonal completion. By [8, p. 14], $Q(R)$ is the ring of equivalence classes of continuous functions on dense open subsets of $[0, 1]$.

An orthogonal set in R is simply a set of functions whose supports are pairwise disjoint. It is clear that if $q = \sup\{f_\alpha\}$, where $\{f_\alpha\}$ is an orthogonal set in R of more than one nonzero element, then q coincides with f_α on the intersection of the domain of q with the support of f_α and so q must have values arbitrarily close to 0. Hence an element of $Q(R)$ bounded away from 0 cannot be the supremum of a nontrivial orthogonal set in R. Now let $\{f_n\}$ be, for example, the sequence of functions whose graphs are:

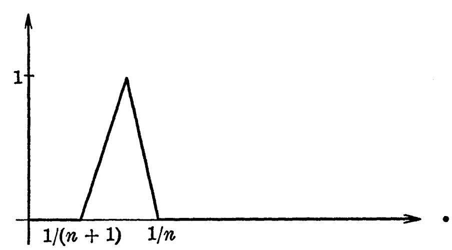

Fig. 1

Let $q = \sup_{Q(R)}\{f_n\}$. Since $q \notin R$ neither is $q + 1$ where 1 is the constant function. But $q + 1$ is bounded away from 0 and must be in any ring between R and $Q(R)$ which contains q.

The next results concern rings having orthogonal completions.

THEOREM 14. *Let R be a strongly regular ring with complete ring of quotients $Q(R)$. Then $Q(R)$ is the orthogonal completion of R.*

Proof. Let D be a large (right) ideal of R and S a maximal orthogonal set of idempotents in D, then $\operatorname{Ann} S = 0$. Indeed if $xS = 0$ then $x = x^2y$ for some y and put $e = e^2 = xy = yx$. Now if $eD \neq 0$, $ed \neq 0$ for some idempotent $d \in D$ and $edS = 0$ would mean that $S \cup \{ed\}$ would contradict the choice of S. Hence $eD = 0$ which implies $e = 0$ and hence $x = 0$.

Now suppose $q \in Q(R)$ and D a large ideal of R so that $qD \subseteq R$. Let S be a maximal orthogonal set of idempotents from D. By the above, SR is large. Now qS is an orthogonal set in R and suppose $\sup_{Q(R)} qS = s$. As shown in (4), $Q(R)$ is Baer and so $\sup_{Q(R)} S$ is an idempotent with trivial annihilator (as in the remark following (4)). Hence $\sup_{Q(R)} S = 1$. Then, $\sup_{Q(R)} qS = q \sup_{Q(R)} S = q$. Hence $Q(R)$ is an orthogonal extension of R which by (2) is orthogonally complete.

LEMMA 15. *Let R be a Baer ring and $B(R)$ its Boolean ring of idempotents. Then if I is a large ideal of $B(R)$, a maximal orthogonal set from I has trivial annihilator in R.*

Proof. Let I be large in $B(R)$ and S a maximal orthogonal set from I. If $rI = 0$ for some $r \in R$, there exists $q \in Q(R)$ with $r^2q = r$. Put $f = f^2 = rq \in B(R)$, since idempotents in $Q(R)$ are in R. Now $fS = 0$ which implies $f = 0$ and $r = 0$.

PROPOSITION 16. *Let R be a reduced Baer ring where $aR \cap bR = 0$ implies $ab = 0$. Then R is orthogonally complete iff for every large ideal I of $B(R)$ and $f \in \operatorname{Hom}_R(IR, R)$ there is $a \in R$ with $f(e) = ea$ for all $e \in I$.*

This proposition says that R is orthogonally complete when certain elements of $Q(R)$ are, in fact, in R. This will be exploited later.

Proof. Suppose R orthogonally complete. Let S be a maximal orthogonal set in I. Since $\operatorname{Ann}_R S = 0$ and SR is an ideal of R, SR is large. Now $f(S)$ is orthogonal so let a be its supremum. Thus $f(e) \leqq a$ for all $e \in S$ and, therefore, $f(e) = f(e)e \leqq ae$. Also, $f(e')e \leqq f(e)$ for all $e, e' \in S$ since if $e = e'$ we have equality and if $e' \neq e$, $f(e')e = 0$. But, $\sup(f(S)e) = (\sup f(S))e = ae$ and so $ae \leqq f(e)$ for all $e \in S$. Combining the inequalities we have $f(e) = ae$ for all $e \in S$. Since $\operatorname{Ann}_R S = 0, f$, as an element of $Q(R)$, equals a and $f(e) = ae$ for all $e \in I$.

Conversely, let S be an orthogonal set in R. Then $SQ(R)$ is an ideal in $Q(R)$ and $SQ(R) \oplus J$ is large for some ideal J of $Q(R)$. Let I be the set of idempotents of $SQ(R) \oplus J$, I is a large ideal of $B(R)$. In fact, $I \subseteq (SQ(R) \cap R) \oplus (J \cap R)$ since an idempotent in I is a sum of an idempotent in $SQ(R)$ with one in J. Hence $IR \subseteqq (SQ(R) \cap R) \oplus (J \cap R)$. Define $\theta: (SQ(R) \cap R) \oplus (J \cap R) \to R$ by $\theta|_{J \cap R} = 0$ and for $s \in S$, $\theta(s) = s^2$. This is well-defined since S is orthogonal. Let $f = \theta|_{IR}$. By hypothesis there exists $a \in R$ so that $f(e) = ea$ for all $e \in I$. For $s \in S$, $s = s^2 q$ for some $q \in Q(R)$ and $sq = e \in I$. Therefore, $s^2 = \theta(s) = \theta(es) = eas = as$. Hence S has an upper bound in R and then, since it is Baer, a supremum.

LEMMA 17. *The set $\mathscr{E}$ of right ideals of R which contain sets of idempotents with trivial annihilator forms a topologizing idempotent filter* ([3]).

Proof. If $D_1, D_2 \in \mathscr{E}$ with $S_i = D_i \cap B(R)$ then $S_1 S_2 \subseteqq D_1 \cap D_2$ since idempotents are central and $\operatorname{Ann} S_1 S_2 = 0$. If $D \in \mathscr{E}$, $D \cap B(R) = S$ then $SR \subseteqq D$ and SR is an ideal in $\mathscr{E}$. Hence for $a \in R$, $a^{-1}D \supseteqq SR \supseteqq S$. Next if $D \in \mathscr{E}$ with $S = D \cap B(R)$ and $J \subseteqq D$ is such that for all $d \in D$, $d^{-1}J \in \mathscr{E}$, then, in particular, for $e \in S$, $e^{-1}J \in \mathscr{E}$. If $S_e = e^{-1}J \cap B(R)$, $T = \bigcup_{e \in S} eS_e \subseteqq J$ has trivial annihilator.

THEOREM 18. *Let R be a reduced Baer ring so that $aR \cap bR = 0$ implies $ab = 0$ and $\mathscr{E}$ the filter of right ideals of R which contain sets of idempotents with trivial annihilator. Let $Q_{\mathscr{E}}$ be the ring of right quotients associated with $\mathscr{E}$, then $Q_{\mathscr{E}}$ is the orthogonal completion of R.*

Proof. We will first use the criterion of (16) to show that $Q_{\mathscr{E}}$ is orthogonally complete. Let I be a large ideal of $B(R) = B(Q(R))$, $f: IQ_{\mathscr{E}} \to Q_{\mathscr{E}}$ and $D = \{r \in IQ \cap R \mid f(r) \in R\}$. D is a right ideal of R and, in fact, $D \in \mathscr{E}$. Indeed, for $e \in I$, $f(e) \in Q_{\mathscr{E}}$ so $f(e)S_e \subseteqq R$ for some set of idempotents with trivial annihilator, S_e (i.e., $f(e)D' \subseteqq R$ for some $D' \in \mathscr{E}$). Then $\bigcup_{e \in I} eS_e \subseteqq D$ so that $D \in \mathscr{E}$. Hence there is a $q \in Q_{\mathscr{E}}$ so that $f|_D = q$. Hence $f(e) = qe$ for all $e \in I$.

Next, every element of $Q_{\mathscr{E}}$ is the supremum of an orthogonal set in R. Let $q \in Q_{\mathscr{E}}$ then $q^{-1}D \subseteqq R$ for some $D \in \mathscr{E}$ with $I = D \cap B(R)$ a large ideal in $B(R)$. By (16), I contains an orthogonal set S with $\operatorname{Ann}_R S = 0$. Then qS is orthogonal in R and has a supremum, say q', in $Q_{\mathscr{E}}$. Then $qe \leqq q'$ for all $e \in S$ and so $qeq' = q^2 e$. Hence, $(qq' - q^2)e = 0$ for all $e \in S$ and, consequently, $qq' = q^2$ giving $q \leqq q'$. Also, $q(qe) = q^2 e = (qe)^2$ so $qe \leqq q$ for all $e \in S$. But since $q' = \sup qS$, $q' \leqq q$. Hence $q = q'$.

Finally a remark about orthogonally complete rings.

THEOREM 19. *Let R be a reduced ring in which $aR \cap bR = 0$ implies $ab = 0$. If R is orthogonally complete then the classical* *ring of right (and of left) fractions $Q_{cl}(R)$ exists and $Q_{cl}(R) = Q(R)$, the complete ring of right quotients.*

Proof. Since $Q(R)$ is, here, a two-sided ring of quotients of R, regular elements of R are invertible in $Q(R)$. If $q \in R$, $qD \subseteqq R$ for some large right ideal D of R. Let S be a maximal orthogonal set from D, which is easily seen to have trivial annihilator in R and in $Q(R)$. Put $\sup S = a \in R$ and a is regular since it has the same annihilator in $Q(R)$ as S (remark after (4)). Hence $\sup qS = qa$ and $qa \in R$ since $qS \subseteqq R$. Putting $qa = b$ we get $q = ba^{-1}$. Similarly for left fractions.

The following shows that the converse of (19) is false, which leaves open the question: Which rings are orthogonally complete?

EXAMPLE 20. Let $R = \prod_I \boldsymbol{Z}$ then $Q(R) = \prod_I \boldsymbol{Q}$. Let $S = \{x \in Q(R) \mid \text{for almost all } i \in I,\ x_i \in \boldsymbol{Z}\}$. Although $Q_{cl}(S) = Q(R) = Q(S)$, it is easily seen that S is not orthogonally complete and, in fact, $Q(S)$ is its orthogonal completion.

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