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ENGEL LIE RINGS WITH CHAIN CONDITIONS

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A result of Max Zorn states that if a Lie ring satisfies the maximal condition for subrings and if each element is a bounded left Engel element then the Lie ring is nilpotent. The purpose of this paper is to extend this result to Lie rings satisfying the general Engel condition and with no infinite strictly ascending chains of abelian subrings. A similar result was obtained by I. N. Stewart for locally nilpotent Lie algebras.

2. Notation and terminology. Let $\mathfrak{r}$ be a noetherian ring (i.e., commutative associative ring with unit and satisfying the ascending chain condition on ideals). Following Barnes [1, 2] we define a *Lie algebra over* $\mathfrak{r}$ to be an $\mathfrak{r}$-module which is a Lie ring and satisfies for x, y in the Lie ring and $r \in \mathfrak{r}$,

$$r[x, y] = [rx, y] = [x, ry] .$$

(Here $[\ ,\ ]$ denotes Lie multiplication.) Let L be a Lie algebra over $\mathfrak{r}$.

If A is a subset of L we write $A \subseteqq L$; if in addition A is an $\mathfrak{r}$-submodule and a Lie subring we write $A \leqq L$ and call A a subalgebra of L. In general $\langle A\rangle$ will denote the subalgebra of L generated by A. If $A = \{a\}$ then

$$\langle a\rangle = \mathfrak{r}a = \{ra \mid r \in \mathfrak{r}\} = \langle\{a\}\rangle ,$$

and we call $\langle a\rangle$ a cyclic $\mathfrak{r}$-module. An $\mathfrak{r}$-module A is said to be finite dimensional over $\mathfrak{r}$ if it is a sum of finitely many cyclic $\mathfrak{r}$-modules. If $A = \{a_1, \cdots, a_n\}$ we define $\langle a_1, \cdots, a_n\rangle = \langle A\rangle$.

Let $A, B \subseteqq L$. We define $[A, B]$ to be the $\mathfrak{r}$-submodule spanned by the products $[a, b]$ for $a \in A$ and $b \in B$. We also define inductively, $[A, {}_0B] = A$ and $[A, {}_{n+1}B] = [[A, {}_nB], B]$. If $x, y \in L$ then $[x, {}_0y] = x$ and $[x, {}_{n+1}y] = [[x, {}_ny], y]$. An $\mathfrak{r}$-submodule H is said to be an ideal if $[H, L] \subseteqq L$; in this case we write $H \lhd L$. If $A \subseteqq L$ then

$$I_L(A) = \{x \in L \mid [A, x] \subseteqq A\}$$

and

$$C_L(A) = \{x \in L \mid [A, x] = 0\} .$$

If A is an $\mathfrak{r}$-submodule then

$$C_L(A) \lhd I_L(A) \leqq L$$

and $I_L(A)/C_L(A)$ is isomorphic to a subalgebra of $\mathrm{End}_{\mathfrak{r}}(A)$; if also A is finite dimensional then (since $\mathfrak{r}$ is noetherian) $\mathrm{End}_{\mathfrak{r}}(A)$ is finite dimensional and so $I_L(A)/C_L(A)$ is finite dimensional over $\mathfrak{r}$.

We will employ the notation of Stewart [3], with the understanding that by Lie algebra we now understand a Lie algebra over $\mathfrak{r}$. So concepts like subideal, ascendant subalgebra, class of Lie algebras, need no further explanation.

We say that L is finitely generated if $L = \langle A \rangle$ for some finite subset A of L; we denote by $\mathfrak{G}$ the class of finitely generated Lie algebras over $\mathfrak{r}$. We define

$$\mathfrak{F},\ \mathfrak{A},\ \mathfrak{N},\ L\mathfrak{N}$$

to be (respectively) the classes of finite dimensional, abelian, nilpotent and locally nilpotent Lie algebras over $\mathfrak{r}$. Then we have as is well known,

$$\mathfrak{G} \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N} = \mathfrak{G} \cap L\mathfrak{N}\,.$$

Let $\mathfrak{X}$ be a class of Lie algebras over $\mathfrak{r}$ and let Δ be any of the relations $\leqq$, $\lhd$, $\lhd^n$, si, asc (see Stewart [3, p. 334–335]). We say that

$$L \in \text{Fin-}\Delta\mathfrak{X}$$

if and only if $H \Delta L$ and $H \in \mathfrak{X}$ implies that $H \in \mathfrak{F}$. (For Fin-$\leqq \mathfrak{X}$ we write Fin-$\mathfrak{X}$.) As is mentioned in [3] the following assertions are equivalent:

(1) Every infinite dimensional $\mathfrak{X}$-algebra L (over $\mathfrak{r}$) has an infinite dimensional abelian subalgebra A, with $A \Delta L$.

(2) $\mathfrak{X} \cap \text{Fin-}\Delta\mathfrak{A} \leqq \mathfrak{F}$.

We say that L satisfies the (general) Engel condition if to each pair x, y of elements of L there corresponds a positive integer $n = n(x, y)$ such that $[x, {}_n y] = 0$.

We denote by $\mathfrak{E}$ the class of Lie algebras over $\mathfrak{r}$ which satisfy the Engel condition. (An element y is said to be a bounded left Engel element if $[L, {}_n y] = 0$ for some $n = n(y)$.)

If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of Lie algebras over $\mathfrak{r}$ then $\mathfrak{X}\mathfrak{Y}$ denotes the class of Lie algebras L (over $\mathfrak{r}$) which have an $\mathfrak{X}$-ideal H such that $L/H \in \mathfrak{Y}$.

3. Preliminary results. The first half of this section is devoted to proving the result:

THEOREM 3.1. *Let $\mathfrak{r}$ be a noetherian ring and let L be a Lie* *algebra over* $\mathfrak{r}$. *Then the following assertions are equivalent*:

(a) *To each finite subset* A *and each* $\mathfrak{G} \cap \mathfrak{N}$-*subalgebra* H *of* L *there corresponds a nonnegative integer* $n = n(A, H)$ *such that*

$$[A, {}_nH] = 0 .$$

(b) $L \in \mathfrak{E}$.

That (a) implies (b) is trivial. To show that (b) implies (a) we make use of some results below most of which appear in some form in Zorn [4].

Let L be a Lie algebra over $\mathfrak{r}$ and let A, B, C be subsets of L.

LEMMA 3.2. *If* $C \subseteqq I_L(B)$ *then* $[A, B, C] \subseteqq [A, B] + [A, C, B]$.

Proof. Immediate from the Jacobi identity and the fact that $[B, C] \subseteqq \mathfrak{r}B = \{\Sigma r_i b_i \mid r_i \in \mathfrak{r} \text{ and } b_i \in B\}$.

COROLLARY 3.3. *If* $C \subseteqq I_L(B)$ *and* $m_1, \cdots, m_k, n_1, \cdots, n_k$ *are nonnegative integers then*

(a) $[A, {}_{m_1}B, {}_{n_1}C] \subseteqq \sum_{i=0}^{n_1} [A, {}_iC, {}_{m_1}B]$ *and*

(b) $[A, {}_{m_1}B, {}_{n_1}C, \cdots, {}_{m_k}B, {}_{n_k}C] \subseteqq \sum_{i=0}^{n_1+\cdots+n_k}[A, {}_iC, {}_{m_1+\cdots+m_k}B]$.

Proof. (a) By induction on $m_1 + n_1$ and by Lemma 3.2, noting that for any i if $A_i = [A, {}_iC]$ then

$$[A_i, {}_{m_1}B, C] \subseteqq [A_i, {}_{m_1}B] + [A_i, {}_{m_1-1}B, C, B] .$$

(b) follows from (a) and induction on k.

Let q be a nonnegative integer and let A, B, C be subsets of L. Then clearly

$$(*) \qquad [A, {}_q B + C] \subseteqq \Sigma[A, {}_{m_1}B, {}_{n_1}C, \cdots, {}_{m_k}B, {}_{n_k}C] ,$$

where the summation is taken over all sets of nonnegative integers $m_1, \cdots, m_k, n_1, \cdots, n_k$ for which $\sum m_i + \sum n_i = q$ (and $k > 0$ and m_1, n_k may be zero but $m_2, \cdots, m_k, n_1, \cdots, n_{k-1}$ (if they exist) are nonzero).

For subsets A and B of L we define A^B to be the smallest $\mathfrak{r}$-submodule containing A and invariant under Lie multiplication by the elements of B. Clearly

$$A^B = \mathfrak{r}A + \sum_{i=1}^{\infty} [A, {}_iB] .$$

Evidently if A and B are contained in some finite dimensional $\mathfrak{r}$-submodule of L (and $\mathfrak{r}$ is noetherian) and for some i $[A, {}_iB] \subseteqq \mathfrak{r}A + \sum_{j=0}^{i-1} [A, {}_jB]$ then A^B is finite dimensional.

We remark that if $C \subseteqq I_L(B)$ then $[A, C, B] \subseteqq [A, B, C] + [A, B]$ and so

$$[A, {}_{n_1}C, {}_{m_1}B] \subseteqq \sum_{i=0}^{n_1} [A, {}_{m_1}B, {}_iC] .$$

(+) Thus if $[A, {}_mB] = 0$ then $[A^C, {}_mB] = 0$ and conversely.

LEMMA 3.4. *Let L be a Lie algebra over a noetherian ring $\mathfrak{r}$ and let A, B, C be subsets of L with $C \subseteqq I_L(B)$. If m, n are non-negative integers such that $[A^B, {}_mB] = 0 = [A^B, {}_nC]$ then*

$$[A^B, {}_{mn}B + C] = 0 .$$

Proof. If $m = 0$ or $n = 0$ then $A^B = 0$ and the result holds trivially. Thus suppose that $m > 0$ and $n > 0$. Put $A_1 = A^B$. From (*) we have

$$[A_1, {}_{mn}B + C] \subseteqq \sum [A_1, {}_{m_1}B, {}_{n_1}C, \cdots, {}_{m_k}B, {}_{n_k}C] ,$$

where $k > 0$, $m_2, \cdots, m_k, n_1, \cdots, n_{k-1} > 0$ and $\Sigma m_i + \Sigma n_i = mn$. Consider a typical term

$$X = [A_1, {}_{m_1}B, {}_{n_1}C, \cdots, {}_{m_k}B, {}_{n_k}C] .$$

By Corollary 3.3(b) we have

$$X \subseteqq \sum_{j=0}^{\Sigma n_1} [A_1, {}_jC, {}_{\Sigma m_i}B] .$$

Hence if $\Sigma m_i \geqq m$, then $X = 0$ (see remark (+) above). Thus assume that $\Sigma m_i < m$; then as $m_2, \cdots, m_k$ are nonzero we must have $k - 1 \leqq \Sigma m_i < m$ and so $k < m + 1$.

Suppose then that each $n_i < n$. Then $\Sigma n_i \leqq k(n - 1) \leqq m(n - 1)$ and so $\Sigma m_i = mn - \Sigma n_i \geqq mn - m(n - 1) = m$, a contradiction. Thus some $n_i \geqq n$. If $i = 1$, then as $[A_1, {}_{m_1}B] \subseteqq A_1$ we have $X = 0$. Suppose $i > 1$; then $[A_1, {}_{m_1}B, {}_{n_1}C, \cdots, {}_{m_i}B] \subseteqq A_1^C$ and $[A_1^C, {}_nC] = 0$ so $X = 0$.

Hence $X = 0$ in all cases and so $[A_1, {}_{mn}B + C] = 0$. This proves the required result.

Evidently the conclusion $[A, {}_{mn}B + C] = 0$ holds for $[A, {}_mB] = [A, {}_nC] = 0$ provided that $B \subseteqq I_L(A)$ and $C \subseteqq I_L(B)$.

Let B, C be subsets of L such that $[B, C] \subseteqq \mathfrak{r}(B \cap C)$. Let $m_1, n_1, \cdots, m_k, n_k$ be nonnegative integers with $m' = \sum_{j=1}^k m_j$ and $n' = \sum_{j=1}^k n_j$. Then it follows from Corollary 3.3 that for any subset A of L,

$$[A, {}_{m_1}B, {}_{n_1}C, \cdots, {}_{m_k}B, {}_{n_k}C] \subseteqq \left(\sum_{i=0}^{n'} [A, {}_iC, {}_{m'}B]\right) \cap \left(\sum_{i=0}^{m'} [A, {}_iB, {}_{n'}C]\right).$$

Thus by (*) and the remarks preceding Lemma 3.4 it follows that if $[A, {}_mB] = [A, {}_nC] = 0$ then $[A, {}_{m+n-1}B + C] = 0$ (where we interpret $[A, {}_{m+n-1}B + C]$ as 0 in case $m = n = 0$). Inducting on k we now have:

If $A, B_1, \cdots, B_k \subseteqq L$ such that $[A, {}_{m_i}B_i] = 0$ for $i = 1, \cdots, k$ and $[B_1 + \cdots + B_i, B_{i+1}] \subseteqq \mathfrak{r}((B_1 + \cdots + B_i) \cap B_{i+1})$ for $i = 1, \cdots, k-1$, then

(**) $$[A, {}_{m_1+\cdots+m_k-k+1}B_1 + \cdots + B_k] = 0 .$$

Proof of Theorem 3.1. We want to prove that (b) implies (a). So let $L \in \mathfrak{E}$ and let A be a finite subset of L. Suppose that $H = \langle x_1, \cdots, x_k \rangle \leqq L$ and H is nilpotent of class c. We induct on c to show that $[A, {}_nH] = 0$ for some $n = n(A, H)$. Suppose that $c = 1$. Then $H = \mathfrak{r}x_1 + \cdots + \mathfrak{r}x_k$. As A is a finite set we can find n_i such that $[A, {}_{n_i}x_i] = 0$, whence $[A, {}_{n_i}\mathfrak{r}x_i] = 0$ for $i = 1, \cdots, k$. Furthermore, $[H, \mathfrak{r}x_i] = 0$ for all i and so by (**), $[A, {}_nH] = 0$, where $n = \Sigma n_i - k + 1$. So the result holds for $c = 1$.

Suppose that $c > 1$ and the result holds for $c - 1$. For each i let $B_i = H^2 + \mathfrak{r}x_i$. Evidently B_i is nilpotent of class not exceeding $c - 1$ and is finitely generated as a subalgebra of L. Hence we can find n_i such that $[A, {}_{n_i}B_i] = 0$, by the inductive hypothesis on $c - 1$. Now $B_i \lhd H$ for each i and so by (**) $[A, {}_nH] = 0$, where $n = \Sigma n_i - k + 1$, since $H = B_1 + \cdots + B_k$. This completes our induction on c and the proof of Theorem 3.1.

REMARK. Evidently Theorem 3.1 holds for Lie algebras defined over an arbitrary commutative ring $\mathfrak{r}$.

Define the classes $\mathfrak{E}^*$ and $\mathfrak{E}_n$ ($n > 0$) of *left Engel algebras* and *n-Engel algebras* respectively by:

$L \in \mathfrak{E}^*$ is and only if to each $x \in L$ there corresponds $m = m(x)$ such that $[y, {}_mx] = 0$ for all $y \in L$ (equivalently, $[L, {}_mx] = 0$);

$L \in \mathfrak{E}_n$ if and only if $[x, {}_ny] = 0$ for all $x, y \in L$. Clearly

$$\bigcup\nolimits_n \mathfrak{E}_n \leqq \mathfrak{E}^* \leqq \mathfrak{E} ,$$

and $\mathfrak{E}^*$ is the class of algebras in which each element is a bounded left Engel element.

Let $\mathfrak{G}_k$ be the class of Lie algebras which can be generated by k elements and $\mathfrak{N}_c$ the class of Lie algebras which are nilpotent of class $\leqq c$.

In much the same way as the proof of Theorem 3.1 we can prove:

THEOREM 3.5. *Let L be a Lie algebra over a commutative ring $\mathfrak{r}$. Then*

(1) *$L \in \mathfrak{E}^*$ if and only if to each $\mathfrak{G} \cap \mathfrak{N}$-subalgebra H of L there corresponds $m = m(H)$ such that $[L, {}_m H] = 0$.*

(2) *$L \in \mathfrak{E}_n$ if and only if there exists $h = h(n, m, c)$ such that for any $\mathfrak{G}_m \cap \mathfrak{N}_c$-subalgebra H of L, $[L, {}_h H] = 0$. ($h(n, m, 1) = m(n-1)+1$ and $h(n, m, c) = m(h(n, m_1, c-1)-1)$ with $m_1 = m(m^c - 1)/(m-1)$.)*

The following result is probably well known:

LEMMA 3.6. *Let $L \in \mathfrak{E}$ and H be a $\mathfrak{G} \cap \mathfrak{N}$ subalgebra of L.*

(a) *If $K \in \mathfrak{G} \cap \mathfrak{N}$ and $K \leqq I_L(H)$ then $H + K \in \mathfrak{G} \cap \mathfrak{N}$.*

(b) *If $H < L$ then there exists $K \in \mathfrak{G} \cap \mathfrak{N}$ with*

$$H < K \leqq I_L(H) .$$

Proof. (a) Evidently $H + K$ is a finite dimensional subalgebra of L. By Theorem 3.1 we can find m and n such that

$$[H + K, {}_m H] = [H + K, {}_n K] = 0 .$$

Thus by Lemma 3.4 we have

$$[H + K, {}_{mn} H + K] = 0$$

and so $H + K \in \mathfrak{G} \cap \mathfrak{N}_{mn+1}$.

(b) If $H < L$ then we can find a finite subset A of L with $A \nsubseteq H$. By Theorem 3.1 we can find m such that $[A, {}_m H] = 0 \subseteq H$. Let k be minimal with respect to $[A, {}_k H] \subseteq H$. If $k = 0$ then $A = [A, {}_0 H] \subseteq H$, a contradiction. So $k > 0$. Now by the definition of k we have $[A, {}_{k-1} H] \nsubseteq H$ and $[A, {}_{k-1} H] \subseteq I_L(H)$. Pick $x \in [A, {}_{k-1} H]$ with $x \notin H$ and let $K = H + \langle x \rangle$. Then $H < K \leqq I_L(H)$ and by part (a) $K \in \mathfrak{G} \cap \mathfrak{N}$ (for $\langle x \rangle$ is abelian and 1-dimensional and contained in $I_L(H)$).

Let A, B be closure operations (see Stewart [3]) and let $\mathfrak{X}$ be a class of Lie algebras (over $\mathfrak{r}$). Define the class $(AB)\mathfrak{X}$ by

$$(AB)\mathfrak{X} = A(B\mathfrak{X}) .$$

If α is an ordinal and $(AB)^\alpha \mathfrak{X}$ has been defined, let

$$(AB)^{\alpha+1}\mathfrak{X} = (AB)((AB)^\alpha \mathfrak{X}) .$$

If λ is a limit ordinal and $(AB)^{\alpha}\mathfrak{X}$ has been defined for all ordinals $\alpha < \lambda$, define

$$(AB)^{\lambda}\mathfrak{X} = \bigcup_{\alpha<\lambda} (AB)^{\alpha}\mathfrak{X} .$$

Let

$$\{A, B\}\mathfrak{X} = \bigcup_{\text{all ordinals } \alpha} (AB)^{\alpha}\mathfrak{X} . \qquad ((AB)^{0}\mathfrak{X} = \mathfrak{X} .)$$

Define the closure operation $\acute{E}$ by $L \in \acute{E}\mathfrak{X}$ if L has an ascending series (from 0 to L) with $\mathfrak{X}$-factors (see Stewart [3]).

THEOREM 3.7. $\mathfrak{E} \cap \{\acute{E}, L\}\mathfrak{A} = L\mathfrak{N}$.

Proof. Use transfinite induction on α to show that

$$(\,'\,) \qquad \mathfrak{E} \cap (\acute{E}L)^{\alpha}\mathfrak{A} \leqq L\mathfrak{N}$$

for all α. Evidently (′) will hold for a limit ordinal λ provided it holds for all ordinals $\alpha < \lambda$. Thus we need only verify the inductive step from α to $\alpha + 1$. Since subalgebras and quotients of Engel algebras are also Engel algebras this boils down to proving that

$$(\,''\,) \qquad \mathfrak{E} \cap \acute{E}L\mathfrak{N} \leqq L\mathfrak{N} .$$

Now the union on an ascending chain of $L\mathfrak{N}$-subalgebras is locally nilpotent and so (″) will follow from showing that whenever $L \in \mathfrak{E}$, $H \lhd L$ and $H, L/H \in L\mathfrak{N}$ then $L \in L\mathfrak{N}$.

This will follow from the following results: Let $L \in \mathfrak{E}$.

(a) If $H, K \in L\mathfrak{N}$ and $K \leqq I_L(H)$ then $H + K \in L\mathfrak{N}$. For every finitely generated subalgebra of $H + K$ is contained in one of the form $C = \langle A, B\rangle$ where A and B are finite subsets of H, K respectively. Now $\langle B\rangle \in \mathfrak{G} \cap \mathfrak{N}$ (for $K \in L\mathfrak{N}$) and so by Theorem 3.1 we can find n such that

$$[A, {}_{n}\langle B\rangle] = 0 .$$

Therefore, $A^{\langle B\rangle} = \mathfrak{r}(\sum_{i=0}^{n-1} [A, {}_{i}\langle B\rangle])$ is a finite dimensional submodule of H (for $K \leqq I_L(H)$) and so $A_0 = \langle A^{\langle B\rangle}\rangle \in \mathfrak{G} \cap \mathfrak{N}$ (since $H \in L\mathfrak{N}$). By Lemm 3.6, $C = A_0 + \langle B\rangle \in \mathfrak{G} \cap \mathfrak{N}$ and (a) is proved.

(b) If $H \in L\mathfrak{N}$, $X \leqq I_L(H)$ and $X^2 \leqq H$ then $X \in L\mathfrak{N}$ and so

$$H + X \in L\mathfrak{N} .$$

For let $x_1, \cdots, x_k \in X$ and define $X_0 = X^2$, $X_{i+1} = X_i + \langle x_{i+1}\rangle$ for $i = 0, 1, \cdots, k-1$. Clearly $X_i \lhd X$ for all X_i. Now $X_0 = X^2 \leqq H$

and so $X_0 \in \mathfrak{N}$. If $X_i \in L\mathfrak{N}$ then by (a) we have $X_{i+1} \in L\mathfrak{N}$. So by induction $X_k \in L\mathfrak{N}$, whence $\langle x_1, \cdots, x_k \rangle \in \mathfrak{N}$ and $X \in L\mathfrak{N}$. By (a) $H + X \in L\mathfrak{N}$.

From (a) and (b) it follows by induction on c that if $H \in L\mathfrak{N}$, $X \leqq I_L(H)$ and $X/X \cap H$ is nilpotent of class c then $X \in L\mathfrak{N}$ and so $H + X \in L\mathfrak{N}$. In particular let $H \lhd L$ with H, $L/H \in L\mathfrak{N}$ and let X be a finitely generated subalgebra of L. Then $X/X \cap H$ is nilpotent so X is locally nilpotent and thus nilpotent. Hence $L \in L\mathfrak{N}$ as required. This completes the proof of the inductive step for ($'$). We note that for $\alpha = 0$ we have $(\acute{E}L)^0\mathfrak{A} = \mathfrak{A} \leqq L\mathfrak{N}$. So ($'$) holds for all α and $\mathfrak{E} \cap \{\acute{E}, L\}\mathfrak{A} \leqq L\mathfrak{N}$. Evidently $L\mathfrak{N} \leqq \mathfrak{E}$ and $L\mathfrak{N} \leqq L\acute{E}\mathfrak{A} \leqq \{\acute{E}, L\}\mathfrak{A}$. This completes the proof of Theorem 3.7.

Clearly $\{\acute{E}, L\}\mathfrak{A}$ contains the class $\acute{E}L\mathfrak{N}$ and the class of Lie algebras with an ascending series whose factors are locally soluble. So Theorem 3.7 includes the well known result of Gruenberg that a locally soluble algebra with Engel condition is locally nilpotent. It also shows that if $L \in \mathfrak{E}$ and P is the sum of all the locally nilpotent ideals of L then P is locally nilpotent, and L/P has no nontrivial locally nilpotent ideals. This latter property is the basis of the solution of the restricted Burnside problem by A. I. Kostrikin.

Finally we remark that all the results in this section hold for Lie algebras defined over an arbitrary commutative ring.

Let Δ be one of the relations $\leqq$, $\lhd^\alpha$ $(\alpha > 0)$, si, asc and let $\mathfrak{X}$ be a class of Lie algebras. We define

$$\acute{E}(\Delta)\mathfrak{X}$$

to be the class of Lie algebras L with an ascending series $\{L_\beta : 0 \leqq \beta \leqq \lambda\}$ with $L_{\beta+1}/L_\beta \in \mathfrak{X}$ for all $\beta < \lambda$ and $L_\beta \Delta L$ for all $\beta \leqq \lambda$. (Note that $L_\beta \lhd L_{\beta+1}$ for all $\beta < \lambda$ and $L_\mu = \bigcup_{\beta<\mu} L_\beta$ for all limit ordinals $\mu \leqq \lambda$.) If Δ is a transitive relation e.g. $\leqq$, si, asc, then $\acute{E}(\Delta)$ is a closure operation. We normally write $\acute{E}\mathfrak{X}$ for $\acute{E}(\leqq)\mathfrak{X}$. We also write $L \in \acute{E}\mathfrak{X}$ when $L \in \acute{E}\mathfrak{X}$ and λ is a finite ordinal. Clearly E is also a closure operation.

4. The main result.

THEOREM 4.1. *Over any noetherian ring* $\mathfrak{r}$,

$$\mathfrak{E} \cap \text{Fin-}\mathfrak{A} = \mathfrak{F} \cap \mathfrak{N} .$$

That $\mathfrak{F} \cap \mathfrak{N} \leqq \mathfrak{E} \cap \text{Fin-}\mathfrak{A}$ (for as $\mathfrak{r}$ is noetherian then every submodule of a finitely generated $\mathfrak{r}$-module is also finitely generated) *is trivial. We will prove the reverse inclusion later on.*

First we need a result of Stewart [3, Corollary 1, p. 337] (there it is stated for Lie algebras defined over a field, but holds as well for Lie algebras over any noetherian ring $\mathfrak{r}$). Define the classes

$$\mathfrak{Q},\ \mathfrak{R}$$

of Lie algebras over $\mathfrak{r}$ by $L \in \mathfrak{Q}$ if $L \in \mathfrak{F}$ or $L \notin$ Fin-$\mathfrak{A}$ (i.e., L has an infinite dimensional abelian subalgebra); $L \in \mathfrak{R}$ if $L \in \mathfrak{F}$ or $C_L(x) \notin \mathfrak{F}$ for some $x \in L$ $(x \neq 0)$. Thus $\mathfrak{Q} \leqq \mathfrak{R}$ and $\mathfrak{Q} \cap$ Fin-$\mathfrak{A} = \mathfrak{F}$.

For a Lie algebra L over $\mathfrak{r}$ we define $Z_0(L) = 0$, $Z_1(L) = C_L(L)$; if α is an ordinal and $Z_\alpha(L)$ has been defined then $Z_{\alpha+1}(L)/Z_\alpha(L) = Z_1(L/Z_\alpha(L))$; for limit ordinals λ, $Z_\lambda(L) = \bigcup_{\alpha<\lambda} Z_\alpha(L)$.

We define the class $\mathfrak{Z}$ by $L \in \mathfrak{Z}$ if $L = Z_\alpha(L)$ for some α.

We denote by

$$\acute{E}(\lhd)\mathfrak{A}$$

the class of Lie algebras which have an ascending series of ideals with abelian factors.

PROPOSITION 4.2. (Stewart [3]). *Over any noetherian ring* $\mathfrak{r}$,

$$L(\mathfrak{N}\mathfrak{A}) \leqq \mathfrak{Q}$$

and in particular $L\mathfrak{N} \cap$ Fin-$\mathfrak{A} = \mathfrak{F} \cap \mathfrak{N}$.

Proof. Sketch.

(i) If $L \in \mathfrak{Z}$ and $0 \neq N \lhd L$ then $N \cap Z_1(L) \neq 0$. (Let α be minimal with respect to $N \cap Z_\alpha(L) \neq 0$; then α is not a limit ordinal, so $\alpha - 1$ exists. Then $[N \cap Z_\alpha(L), L] \subseteqq N \cap Z_{\alpha-1}(L) = 0$ and so $0 \neq N \cap Z_\alpha(L) \leqq Z_1(L)$.)

(ii) If $L \in \mathfrak{Z}$ and A is a maximal abelian ideal of L then $C_L(A) = A$.
($C_L(A)/A \lhd L/A \in \mathfrak{Z}$; thus if $C_L(A)/A \neq 0$ then by (i) we have $x \notin A$ such that $K = \langle x\rangle + A/A \leqq C_L(A)/A \cap Z_1(L/A)$; hence $B = \langle x \rangle + A \lhd L$ and $B^2 = 0$, a contradiction.)

(iii) $\mathfrak{Z} \cap$ Fin-$\lhd\mathfrak{A} = \mathfrak{F} \cap \mathfrak{N}$.
Let $L \in \mathfrak{Z} \cap$ Fin-$\lhd\mathfrak{A}$ and let A be a maximal abelian ideal of L (existence by Zorn's lemma); then $A \in \mathfrak{F}$ and so $L/C_L(A) \in \mathfrak{F}$; but by (ii) $C_L(A) = A$ and so $L \in \mathfrak{F} \cap \mathfrak{Z} = \mathfrak{F} \cap \mathfrak{N}$. The converse is trivial.)

(iv) $\acute{E}(\lhd)\mathfrak{A} \cap$ Fin-$\lhd^2\mathfrak{A} \leqq \mathfrak{F}$.
Let $L \in \acute{E}(\lhd)\mathfrak{A} \cap$ Fin-$\lhd^2\mathfrak{A}$ and let $\{L_\alpha \mid 0 \leqq \alpha \leqq \sigma\}$ be an ascending series of ideals of L with abelian factors in which all the terms are distinct. Consider two cases:

(a) For some finite n, $L_n \notin \mathfrak{F}$; if so let m be minimal with respect to $L_m \notin \mathfrak{F}$. Then $m > 0$ and $L_{m-1} \in \mathfrak{F}$ and so $L_m^2 \in \mathfrak{F}$. Now $C = C_{L_m}(L_m^2) \lhd L$ and $C \in \mathfrak{N}_2$. Thus $C \in \mathfrak{Z} \cap \text{Fin-}\lhd \mathfrak{A} \leqq \mathfrak{F}$. We also have $L_m/C \in \mathfrak{F}$, whence $L_m \in \mathfrak{F}$, a contradiction.

(b) So assume that $L_n \in \mathfrak{F}$ for all $n < \omega$, and $K = L_\omega \notin \mathfrak{F}$. Now $K = \bigcup_{n<\omega} L_n$; suppose that $H_i \lhd L$, $H_i \in \mathfrak{F} \cap \mathfrak{N}$ and $H_i \leqq K$. Then $C_i = C_K(H_i) \notin \mathfrak{F}$ (since $K/C_i \in \mathfrak{F}$ and $K \notin \mathfrak{F}$) and $C_i \lhd L$. We have $C_i = \bigcup L_n \cap C_i$ and so there exists n_i minimal with respect to $C_{n_i} = C_i \cap L_{n_i} \nleqq H_i$, whence $C_{n_i}^2 \leqq C_i \cap L_{n_i-1} \leqq H_i$ and $C_{n_i} \in \mathfrak{N}_2$; furthermore, $C_{n_i} \lhd L$ and $C_{n_i} \in \mathfrak{F}$. Set $H_{i+1} = H_i + C_{n_i}$ and $H_1 = L_1$. Then $H_i < H_{i+1}$. Define $H = \bigcup_{i=1}^{\infty} H_i$. We note that $C_{n_{i+k}}$ centralizes H_{i+1} for all $k \geqq 1$; and $H_{i+1}^2 \leqq H_i$. Finally for any i we have $H = \langle H_{i+1}, C_{n_{i+1}}, C_{n_{i+2}}, \cdots \rangle$ and so $[H_{i+1}, H] \subseteqq H_{i+1}^2 \leqq H_i$. Therefore, $H_i \leqq Z_i(H)$ for all i and $H \in \mathfrak{Z}$. But $H \lhd L$, whence $H \in \text{Fin-}\lhd \mathfrak{A}$ and by (iii) $H \in \mathfrak{F}$, a contradiction since we would have $H = H_i$ for some i and so $C_{n_i} \leqq H_i$.

The rest of the proof follows along the lines of Stewart [3, Lemmas 3.1 and 3.2, p. 336–337].

Proof of Theorem 4.1. Let $L \in \mathfrak{E} \cap \text{Fin-}\mathfrak{A}$. Let H be a maximal locally nilpotent subalgebra of L (existence by Zorn's lemma, since the zero subalgebra and the union of any well ordered chain of locally nilpotent subalgebras are locally nilpotent). Then $H \in \text{Fin-}\mathfrak{a}$ and so $H \in L\mathfrak{N} \cap \text{Fin-}\mathfrak{A}$. Hence $H \in \mathfrak{F} \cap \mathfrak{N}$ by Proposition 4.2. If $H \neq L$ then by Lemma 3.6 there exists $K \in \mathfrak{F} \cap \mathfrak{N}$ $(= \mathfrak{G} \cap \mathfrak{N})$ with $H < K \leqq I_L(H)$. But this contradicts the choice of H as a maximal locally nilpotent subalgebra. Hence $H = L$ and $L \in \mathfrak{F} \cap \mathfrak{N}$. Thus $\mathfrak{E} \cap \text{Fin-}\mathfrak{A} \leqq \mathfrak{F} \cap \mathfrak{N}$ and our proof is complete.

Trivially $\mathfrak{F} \leqq \text{Fin-}\mathfrak{A}$ (strict inclusion since any free $\mathfrak{k}$-Lie algebra with more than one generator is in Fin-$\mathfrak{A}$ but infinite dimensional) and if $H \leqq L \in \text{Fin-}\mathfrak{A}$ then $H \in \text{Fin-}\mathfrak{A}$.

Evidently $E\mathfrak{A}$ is the class of soluble algebras and

$$E\mathfrak{A} \cap \text{Fin-}\mathfrak{A} = \mathfrak{F} \cap E\mathfrak{A} ,$$

by part (iv) of the proof of Proposition 4.2. Thus if $H \in E\mathfrak{A}$ and $H \lhd L$ then $L \in \text{Fin-}\mathfrak{A}$ if and only if $H, L/H \in \text{Fin-}\mathfrak{A}$.

Suppose that $L \in \acute{E}(\lhd)\mathfrak{E} \cap \text{Fin-}\mathfrak{A}$. Then L has an ascending $\mathfrak{E}$-series of ideals,

$$0 = L_0 \lhd L_1 \lhd \cdots L_\lambda = L$$

for some ordinal λ. Suppose if possible that $L \notin \mathfrak{F} \cap E\mathfrak{A}$. Then we can find an ordinal $\alpha \leqq \lambda$, minimal with respect to $L_\alpha \notin \mathfrak{F} \cap E\mathfrak{A}$. Now $L_1 \in \mathfrak{E} \cap \text{Fin-}\mathfrak{A} = \mathfrak{F} \cap \mathfrak{N}$ by Theorem 4.1 so $\alpha > 1$.

If α is not a limit ordinal then $\alpha - 1$ exists and $H = L_{\alpha-1} \in$

$\mathfrak{F} \cap E\mathfrak{A}$. Now $L_\alpha \in \text{Fin-}\mathfrak{A}$ and so by our remark above $L_\alpha/H \in \text{Fin-}\mathfrak{A}$. But $L_\alpha/H \in \mathfrak{E}$ and so by Theorem 4.1 $L_\alpha/H \in \mathfrak{F} \cap \mathfrak{N}$, whence $L_\alpha \in \mathfrak{F} \cap E\mathfrak{A}$, *a contradiction. So α is a limit ordinal. By definition* $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$, and by the definition of α $L_\beta \in \mathfrak{F} \cap E\mathfrak{A}$ for each $\beta < \alpha$. Thus $L_\alpha \in \acute{E}(\lhd)(\mathfrak{F} \cap E\mathfrak{A}) \leqq \acute{E}(\lhd)\mathfrak{A}$. Since also $L_\alpha \in \text{Fin-}\mathfrak{A}$ then by part (iv) of the proof of Proposition 4.2 we have $L_\alpha \in \mathfrak{F} \cap E\mathfrak{A}$, another contradiction. Therefore $L \in \mathfrak{F} \cap E\mathfrak{A}$.

The first part of our proof above shows that $E\mathfrak{E} \cap \text{Fin-}\mathfrak{A} = \mathfrak{F} \cap E\mathfrak{A}$. So we have:

COROLLARY 4.3. *Over any noetherian ring* $\mathfrak{r}$,
(a) $\mathfrak{E} \cap \mathfrak{F} = \mathfrak{F} \cap \mathfrak{N}$.
(b) $\text{Fin-}\mathfrak{E} = \text{Fin-}L\mathfrak{N} = \text{Fin-}\mathfrak{N} = \text{Fin-}\mathfrak{A}$.
(c) $E\mathfrak{E} \cap \text{Fin-}\mathfrak{A} = \acute{E}(\lhd)\mathfrak{E} \cap \text{Fin-}\mathfrak{A} = \mathfrak{F} \cap E\mathfrak{A}$.

REMARK. Let Max and Min denote respectively the classes of Lie algebras over $\mathfrak{r}$ (a commutative ring with unit) satisfying the maximal and minimal condition on subalgebras. Clearly Max $\leqq$ Fin-$\mathfrak{A}$. However, if $\mathfrak{r}$ is not a field we do not necessarily have Min $\leqq$ Fin-$\mathfrak{A}$: e.g., let $L = C_{p^\infty}$, considered as an abelian Lie algebra over the ring of integers. Then $L \in \mathfrak{A} \cap \text{Min}$ but $L \notin \mathfrak{F}$.

Now suppose that $\mathfrak{X}$ is a class of Lie algebras over $\mathfrak{r}$ (a noetherian ring with unit). Then

(§) $\quad \acute{E}\mathfrak{X} \cap \text{Fin-}\mathfrak{A} \leqq \mathfrak{F} \cap E\mathfrak{A}$ if and only if $\mathfrak{X} \cap \text{Fin-}\mathfrak{A} \leqq \mathfrak{F} \cap E\mathfrak{A}$.

The implication in one direction is clear. For the reverse implicacation we evidently need only consider the case $L = \bigcup_{n=0}^{\infty} L_n \in \text{Fin-}\mathfrak{A}$, with $L_n \lhd L_{n+1}$ and $L_n \in \mathfrak{F} \cap E\mathfrak{A}$ for each n. For this we note that L_n asc L for each n and so by Proposition 4.5 (below) we have $H_n = L_n^\omega \lhd L$ and $H_n \in \mathfrak{F} \cap E\mathfrak{A}$ for each n. Thus $H = \bigcup H_n \in \acute{E}(\lhd)(\mathfrak{F} \cap E\mathfrak{A}) \leqq \acute{E}(\lhd)\mathfrak{E}$ and $H \in \text{Fin-}\mathfrak{A}$ and so by Corollary 4.3, $H \in \mathfrak{F} \cap E\mathfrak{A}$. Since also $H \lhd L$ we have $L/H \in \text{Fin-}\mathfrak{A}$ and clearly $L/H \in L\mathfrak{N}$. Thus by Proposition 4.2, $L/H \in \mathfrak{F} \cap \mathfrak{N}$ and so $L \in \mathfrak{F} \cap E\mathfrak{A}$ and (§) is proved.

Next if $\mathfrak{X}$ is a class of Lie algebras over a noetherian ring $\mathfrak{r}$ then

(ß) $\quad \mathfrak{X} \cap \mathfrak{E} \leqq L\mathfrak{N}$ if and only if $\{\acute{E}, L\}\mathfrak{X} \cap \mathfrak{E} \leqq L\mathfrak{N}$.

(ß) follows from the fact that $\mathfrak{E} \cap \acute{E}L\mathfrak{N} \leqq L\mathfrak{N}$ *and for each ordinal* α,

$$\mathfrak{E} \cap \acute{E}L((EL)^\alpha\mathfrak{X}) = \mathfrak{E} \cap \acute{E}L(\mathfrak{E} \cap (\acute{E}L)^\alpha\mathfrak{X}) .$$

From (§), (ß), and Corollary 4.3 we have:

THEOREM 4.4. *Over any noetherian ring* $\mathfrak{k}$,

(1) $\acute{E}\mathfrak{G} \cap \text{Fin-}\mathfrak{A} = \mathfrak{F} \cap E\mathfrak{A}$.

(2) $\mathfrak{G} \cap \{\acute{E}, L\}(\text{Fin-}\mathfrak{A}) = L\mathfrak{N}$.

Evidently $\mathfrak{F} \leqq \text{Fin-}\mathfrak{A}$ and $\mathfrak{A} \leqq \acute{E}\mathfrak{F}$ and so $\{\acute{E}, L\}\mathfrak{A} \leqq \{\acute{E}, L\}\mathfrak{F} \leqq \{\acute{E}, L\}\text{ Fin-}\mathfrak{A}$. Thus Theorem 4.4 is a generalization of Theorem 3.7.

If L is a Lie algebra we define the transfinite lower central series inductively by: $L^1 = L$, $L^{\alpha+1} = [L^\alpha, L]$ and at limit ordinals μ, $L^\mu = \bigcap_{\alpha<\mu} L^\alpha$. The transfinite derived series is defined by: $L^{(0)} = L$, $L^{(\alpha+1)} = [L^{(\alpha)}, L^{(\alpha)}]$, and at limit ordinals μ, $L^{(\mu)} = \bigcap_{\alpha<\mu} L^{(\alpha)}$.

PROPOSITON 4.5. *Let L be a Lie algebra over a commutative ring $\mathfrak{k}$. If H asc L and there exists $H_0 \lhd H$ such that $H/H_0 \in \mathfrak{F}$ and $[L, H_0] \subseteqq H$ then $H^\omega \lhd L$ and $H^{(\omega)} \lhd L$.*

Proof. Let $H = \langle X, H_0\rangle$ where X is a finite dimensional $\mathfrak{k}$-submodule and let $H = K_0 \lhd K_1 \lhd \cdots K_\rho = L$ be an ascending series from H to L. Suppose that A is a finite dimensional $\mathfrak{k}$-submodule of L.

For each nonnegative integer n let α_n be the least ordinal such that $[A, {}_nX] \subseteqq K_{\alpha_n}$. As $[A, {}_nX]$ is finite dimensional for each n it is clear that α_n is not a nonzero limit ordinal. Furthermore, as $H \leqq K_{\alpha_n - 1} \lhd K_{\alpha_n}$, then $\alpha_n > \alpha_n - 1 \geqq \alpha_{n+1}$ (if $\alpha_n \neq 0$). We cannot have an infinite strictly descending chain of ordinals and so $\alpha_n = 0$ for some n, whence $[A, {}_nX] \subseteqq K_0 = H$ and $[A, H^{(n)}] \subseteqq [A, H^{n+1}] \subseteqq [A, {}_{n+1}H] \subseteqq H^2$. Thus for each m, $[A, H^{n+m}] \subseteqq [A, {}_{n+m}H] \subseteqq H^{m+1}$ and so $[A, H^\omega] \subseteqq \bigcap_m H^{m+1} = H^\omega$. We also have $[A, H^{(n+m+1)}] \subseteqq [A, {}_2H^{(n+m)}] \subset [H, H^{(n+m)}] \subseteqq H^{(n+m)}$, whence $[A, H^{(\omega)}] \subseteqq \bigcap_m H^{(n+m)} = H^{(\omega)}$. Since A was arbitrarily chosen we see that $H^\omega \lhd L$ and $H^{(\omega)} \lhd L$. (ω is the first infinite ordinal.) This proves Proposition 4.5.

For the proof of (§) we take $H_0 = 0$ and note that $\mathfrak{F} \leqq \mathfrak{G}$.

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