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APPROXIMATION AND INTERPOLATION FOR SOME SPACES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

A. STRAY

Let U be a bounded open subset of the complex plane C such that U and $C\setminus\bar{U}$ are connected. (If $B \subset C$, $\bar{B}$ denotes its closure in C.) $H^\infty(U)$ is the space of all bounded analytic functions defined on U. Let $S \subset U$ be the zero set of a nonzero function in $H^\infty(U)$.

Necessary and sufficient conditions on S are given for the existence of an open set $0 \supset \bar{U}\setminus(\bar{S}\setminus S)$ such that $H^\infty(0)$ and $H^\infty(U)$ have the same restrictions to S. If U is the unit disc $D = \{z : |z| < 1\}$ and S is as above, the following result holds for all the Hardy spaces $H^p(D)$, $0 < p \le \infty$: Given $g \in H^p(D)$, there is a function f analytic in $C\setminus(\bar{S}\setminus S)$ such that $f|_D \in H^p(D)$ and $f = g$ on S.

If S and U are as above, $H^\infty(U)|_S$ denotes the set of restrictions $f|_S$ of all $f \in H^\infty(U)$. If $S = \{z_n\} \subset D$ satisfies $\Sigma_n(1 - |z_n|) < \infty$, Detraz [3] proved the following result

(*): *There exists an open set* $0 \supset \bar{D}\setminus(\bar{S}\setminus S)$ *such that*
$$H^\infty(0)|_S = H^\infty(D)|_S.$$

In this paper we give two extensions of this result. First we show that (*) holds for domains of a somewhat more general type than the unit disc D. Consider the following statement which is very similar to (*):

(**) There exists an open set V such that $\bar{V}\setminus(\bar{S}\setminus S) \subset D$ and
$$H^\infty(V)|_S = H^\infty(D)|_S.$$

It turns out that conditions (*) and (**) are equivalent, even with D replaced by a somewhat more general set.

We shall make some use of the theory of the classical H^p spaces. We refer to [4] or [9] in this connection. Before stating our first result, we mention some more notation. If f is a complex valued function defined for each $z \in B$ we put $\|f\|_B = \sup\{|f(z)|, z \in B\}$. If $U \subset C$ is open, $H^\infty(U)$ is a Banach algebra with sup norm on U and we denote by M the maximal ideal space of $H^\infty(U)$. The maximal ideals $m \in M$ are identified with the multiplicative functionals on $H^\infty(U)$ they correspond to. If $S \subset U$ is relatively closed and I denotes the set of all $f \in H^\infty(U)$ which are zero on

S, we define $\tilde{S} = \{m \in M: m(f) = 0 \quad f \in I\}$. (Cf. p. 345 in [3]). We have a projection $\Pi: M \to \bar{U}$ given by $m \to m(e)$ where $e \in H^\infty(U)$ is the function $z \to z$. For a detailed study of M we refer to [7] and Ch. 10 in [9]. Other results like (*) can be found in [1], [5], [8], [11] and [14].

With the notation as above we now state:

THEOREM 1. *Let U be the interior of a compact set X and assume both U and $\mathbf{C} \setminus X$ are connected. If $S \subset U$ is the zero set of a nonzero function in $H^\infty(U)$, the following statements are equivalent:*
(i) *There exists an open set $0 \supset \bar{U} \setminus (\bar{S} \setminus S)$ such that $H^\infty(0)|_S = H^\infty(U)|_S$*
(ii) *There exists an open set V such that $S \subset V \subset U$, $\bar{V} \setminus (\bar{S} \setminus S) \subset U$ and $H^\infty(V)|_S = H^\infty(U)|_S$*
(iii) $\Pi(\tilde{S}) \subset \bar{S}$.

REMARK. The author is indebted to the referee for an example where (iii) fails. For details of this example see the final remarks. If the boundary ∂U of U is a Jordan arc, it is easy to verify that (iii) holds, but considerably weaker conditions on ∂U also imply (iii).

Proof: If $\bar{S} \supset \partial U$, the theorem trivially holds with $0 = V = U$. Assume now $(\partial U) \setminus \bar{S} \neq \Phi$. We prove the implications (ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii). We assume first that (ii) is true and consider the restriction map $R: H^\infty(0) \to H^\infty(V)|_S$ where $0 \supset \bar{U} \setminus (\bar{S} \setminus S)$ is some open set and where $H^\infty(V)|_S$ has the quotient norm induced from $H^\infty(V)$. We need to prove that R maps $H^\infty(0)$ onto $H^\infty(V)|_S$. It is sufficient to find constants $L > 0$ and $\epsilon \in (0, 1)$ such that the image by R of the L-ball in $H^\infty(0)$ is ε-dense in the unit ball in $H^\infty(V)|_S$. (See for example Lemma 1.4. in [11].) Choose f in the unit ball of $H^\infty(V)|_S$. By (ii) and the open mapping theorem there is a constant c_1 independent of f, and $f_1 \in H^\infty(U)$ such that $f_1|_S = f$ and $\|f_1\|_U \leq c_1$. By Lemma 3.2 in [11] we can choose 0 such that for each $g \in H^\infty(U)$ there exists $g_1 \in H^\infty(0)$ such that

(1) $\|g_1\|_0 \leq c_2 \|g\|_U$
(2) $\|g - g_1\|_V \leq (2c_1)^{-1} \|g\|_U$

where c_2 is independent of g. (That we actually can apply Lemma 3.2 in [11] in this situation follows from well known estimates of analytic capacity. See for example the proof of Theorem 7.4 on page 213 in [6]). If we replace g by (f_1) in (1) and (2), we see that with $\epsilon = 1/2$ and $L = c_1 c_2$, Lemma 1.4 in [11] can be applied.

To see that (i) $\Rightarrow$ (iii) we first observe that the restriction map R

defined above is not one-to-one. If it was, $||f||_0$ and $||f||_U$ would be equivalent norms on $H^\infty(0)$ by (i) and the open mapping theorem, and that is absurd. Hence there is some function $h \in H^\infty(0)$ which is zero on S but not identically zero in U. Choose $m \in M$ such that $\Pi(m) = z_0 \in \bar{U} \setminus \bar{S}$. Since h is analytic near z_0 we can write $h - h(z_0) = (z - z_0)h_1$ where $h_1 \in H^\infty(0)$. If we apply m on the right side we get zero and therefore $m(h) = h(z_0)$. Since we clearly can assume $h(z_0) \neq 0$ we have proved that $m \notin \tilde{S}$ and (iii) follows.

It remains to prove that (iii) $\Rightarrow$ (ii) and here we apply Carleson's lemma. (See [2] or on page 203 in [4].) Let $\varphi: U \to D$ be a conformal map and put $S_1 = \varphi(s)$. By (iii) S_1 must be countable and we let B denote the Blaschke product corresponding to S_1. For definition and basic properties of Blaschke products we refer to [4] page 20 or [9] page 66. From these properties it is easy to see that $V_1 = \{z : |B(z)| < 2^{-1}\}$ satisfies $\bar{V}_1 \setminus (\bar{S}_1 \setminus S_1) \subset D$ and Carleson's lemma ([4] page 203) combined with a simple normal family argument, gives that $H^\infty(D)|_{S_1} = H^\infty(V_1)|_{S_1}$. If we define $V = \varphi^{-1}(V_1)$, it only remains to prove that $\bar{V} \setminus (\bar{S} \setminus S) \subset U$. Put $g = B \circ \varphi$. Choose an arbitrary point $z_0 \in (\partial U) \setminus \bar{S}$. If we can show that $|g(w_n)| \to 1$ whenever $\{w_n\}_{n=i}^{\infty} \subset U$ converges to z_0, the proof will be complete.

Let $\{z_n\}$ be an arbitrary sequence in U converging to z_0. We denote by J the ideal of all $h \in H^\infty(U)$ satisfying $\lim h(z_n) = 0$. We want to show that $g \notin J$. Let m denote some maximal ideal containing J. Since J contains the translation $z \to z - z_0$ we get that $\Pi(m) = z_0$. If $g \in J$ and $f \in H^\infty(U)$ vanishes on S, we can write $f = gf_1$, with $f_1 \in H^\infty(U)$. (see Thm. 2.8 on page 24 in [4]) and hence we get $m(f) = m(g)m(f_1) = 0$. This implies $m \in \tilde{S}$ which is impossible by (iii) and since $\Pi(m) = z_0$. We can therefore assume that $|g| > t$ on $U_t = \bar{U} \cap \{z : |z - z_0| < t\}$ for some $t > 0$.

The proof is completed using some well known facts about $H^\infty(U)$ which we shall not prove. But references will be given below. We fix a point $w \in U$ and let λ denote the harmonic measure on ∂U which represents w. There is a (unique) function $g^* \in L^\infty(\lambda)$ whose harmonic extension to U equals g. (See for example [15] page 26.) We now claim:

(a) Since $|B| = 1$ a.e. on ∂D with respect to linear measure, $|g^*| = 1$ a.e. with respect to λ.

(b) Define g_1 on ∂U_t by $g_1 = g$ on $(\partial U_t) \cap U$ and $g_1 = g^*$ on $(\partial U_t) \setminus U$. Then the harmonic extension of g_1 to U_t equals the restriction g_2 of g to U_t. We can also assume that $|g_1| = 1$ on $(\partial U_t) \setminus U$.

Since $|g| > t$ on U_t we have from Jensen's inequality ([6] page 33–34) and (*b*) that the harmonic extension of $\log|g_1|$ to U_t equals $\log|g_2| = \log|g|$. But if $\{w_n\} \subset U$ converges to z_0, we get that $\log|g_2(w_n)| \to 0$ since z_0 is regular for the Dirichlet problem for U_t. Since $g_2 = g$ in U_t this completes the proof that (iii) $\Rightarrow$ (ii). The claims (a) and (b) above are easy to justify using well known theory about harmonic measure and algebras of analytic functions. A convenient reference is the introductory part of [7]. (See in particular Lemma 2.2 and Lemma 4.4 in [7].)

We shall now prove that (*) holds for all the Hardy spaces $H^p(D)$, $0 < p \leq \infty$ and with $0 = C\setminus(\bar{S}\setminus S)$. We first prove a general result which may be of independent interest.

THEOREM 2. *Let A be a Banach space of functions on D with norm $N(\cdot)$. Assume A contains the polynomials in z and there exists constants M_n, $n = 1, 2, \ldots$ such that:*

$$(1) \qquad N(p|_D) \leq M_n \sup\{|p(z)| : |z| \leq 1 + n^{-1}\} \textit{ for } n = 1, 2, \ldots$$

if p is a polynomial. For each $z \in D$ assume the map $f \to f(z)$ is continuous on A.

Let $S \subset D$ and assume there exists an open set $0 \supset \bar{D}\setminus(\bar{S}\setminus D)$ such that each $g \in A|_S$ extends to a function f analytic in 0 such that $f|_D \in A$. Then such a function exists which even extends to be analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$.

REMARKS. Note that (1) implies $f|_D \in A$ whenever f is analytic in a neighbourhood of $\bar{D}$ and that we have estimates like (1) also for such functions.

Proof of Theorem 2. Denote by A_1 all analytic functions in 0 whose restriction to D are in A. We topologize A_1 by saying that a sequence $\{f_n\} \in A_1$ converges to $f \in A_1$ if and only if $N((f_n - f)|_D) \to 0$ and $\|f - f_n\|_K \to 0$ if K is a compact subset of 0.

With this topology A_1 is a Frechet space and we can apply the open mapping theorem to the restriction map $A_1 \to A|_S$ where $A|_S$ has the quotient norm induced from A. $A|_S$ is then a Banach space since the set of functions in A vanishing on S must be closed by hypothesis. Choose an open set $0_1 \supset \bar{D}\setminus(\bar{S}\setminus D)$ such that $\bar{0}_1\setminus\bar{D} \subset 0$. By the open mapping theorem there exists a constant M and constants M_K for each compact subset K of $\bar{0}_1\setminus(\bar{S}\setminus D)$ such that each g in the unit ball of $A|_S$ extends to $h \in A_1$ such that

(i) $N(h|_D) \leq M$

(ii) $|h| \leq M_K$ on K if $K \subset \bar{0}\setminus(\bar{S}\setminus D)$ is compact

Now redefine h by setting $h \equiv 0$ in $C\setminus\bar{0}_1$. When we in the rest of the proof of Theorem 2 claim that a property holds independent of h, we shall mean that this property holds for all $h \in A_1$ satisfying (i) and (ii) as above and extended to $\mathbf{C}$ as above.

We can and shall assume 0_1 has the following property:

(2) $C\setminus\bar{0}_1$ is connected and there exists a constant L such that each $z \in C\setminus\bar{0}_1$ can be connected to a point in $\bar{S}$ by an arc $\gamma_z \subset C\setminus\bar{0}_1$ such that (length of γ_z) $\leq L$ dist $(z, \bar{S}\setminus D)$.

With the notation as above the following lemma completes the proof of Theorem 2:

LEMMA 1. *Given $t > 0$ there exists constants C_K for each compact subset K of $C\setminus(\bar{S}\setminus S)$ such that for each function h as above we can find h_1* analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$ *such that* $h_1|_D \in A$ with the following properties:

(a) $N((h - h_1)|_D) \leq t$

(b) $|h_1| \leq \mathbf{C}_K$ *on each compact subset K of* $\mathbf{C}\setminus(\bar{S}\setminus D)$.

Indeed if Lemma 1 is proved, Theorem 2 follows by the same iteration argument as in the proof of Lemma 1.4 in **[11]**.

The first part of the proof of Lemma 1 is very similar to the proof of Lemma 3.2 in **[11]**, but for completeness we give most of the details.

Let $\{K_n\}_{n=1}^{\infty}$ be compact sets, $\{V_n\}_{n=1}^{\infty}$ open sets with the following properties:

(i) $K_n \subset V_n, n = 1, 2, \ldots$

(ii) $\bar{V}_n \cap \bar{D} = \Phi, n = 1, 2, \ldots$

(iii) $\bar{V}_n \cap \bar{V}_m = \Phi$ if $|n - m| > 1$

(iv) $(\partial 0_1)\setminus\bar{D} = \bigcup_n K_n$

(v) For each compact set $F \subset C\setminus(\bar{S}\setminus D)$, $F \cap \bar{V}_n = \Phi$ if n is sufficiently large.

Fix n: Put $K = K_n$, $V = V_n$ and let $\varepsilon = \varepsilon_n$ be a positive number. Let $\delta > 0$ be given. Then cover C by open discs $\Delta_k = \Delta(z_k, \delta)$ (of radius δ and centered at z_k) and choose continuously differentiable functions ϕ_k (supported at Δ_k) as in the scheme for approximation described on page 210 in **[6]**.

Let $T_{\phi k}$ be the integral operator on $L^{\infty}(dxdy)$ defined by

$$T_{\varphi_k}(f)(w) = \frac{1}{\pi}\iint \frac{f(w) - f(z)}{w - z}\frac{\partial\varphi}{\partial z}dxdy$$

$$= f(w)\varphi_k(w) + \frac{1}{\pi}\iint \frac{f(z)}{z - w}\frac{\partial\varphi_k}{\partial z}dxdy$$

We mention that $T_{\phi_k}(f)$ is analytic outside the support of ϕ_k and wherever f is and that $T_{\phi_k}(f)$ is continuous wherever f is. Also $f - T_{\phi_k}(f)$ is analytic in the interior of the set where ϕ_k attains the value 1. (See on page 28–29 in [6] for more details.)

Put $G_k = T'_{\phi_k}(h)$ where h is as above. We are only interested in those k for which $\bar{\Delta}_k \cap K \neq \Phi$. Assume this happens if and only if $1 \leq k \leq N$.

Then $h - \Sigma_1^N G_k$ is analytic near K since $\Sigma_1^N G_k = T_{(\Sigma_1^N \phi_k)}(h)$ and $\Sigma_1^N \varphi_k \equiv 1$ in a neighborhood of K. We can assume $\delta > 0$ is so small that $\{z : |z - z_k| \leq 2\delta\} \subset V$ for $1 \leq k \leq N$.

Now there exist functions H_k, $k = 1, \ldots, N$ analytic outside a compact subset of $D_k = \{w : |w - z_k| < 2\delta\} \setminus \bar{0}_1$ such that $G_k - H_k$ has a triple zero in the Taylor expansion at infinity, and in our situation (since $C \setminus \bar{0}_1$ is connected) we obtain $\|H_k\| < C_1 \|h\|_V$ where C_1 is an absolute constant. (See [6], Theorem 7.4 on page 213 and the proof of it). The important fact is that C_1 is independent of h.

We now list the facts which will be needed to prove Lemma 1.

(a) One can choose δ depending only on ε and dist $(K, \mathbf{C} \setminus V)$ so small that the function $f = \Sigma_1^N (G_k - H_k)$ satisfies

$$\|f\|_{\mathbf{C} \setminus V} < \varepsilon \|h\|_V$$

and we also have $\|f\|_\infty \leq C_2 \|h\|_V$ where C_2 is independent of h. ($\|f\|_\infty$ denotes ess.sup. of $|f|$ with respect to plane measure.)

(b) The functions H_k can be written as

$$H_k = \alpha_k(h) F_{k,1} + \beta_k(h) F_{k,2}$$

where $F_{k,1}$ and $F_{k,2}$ both are analytic outside a compact subset of D_k, they are independent of h and $\|F_{k,1}\|_\infty + \|F_{k,2}\|_\infty \leq 20$.

Here $\alpha_k(h)$ and $\beta_k(h)$ are complex numbers depending linearly on h and we have

$$|\alpha_k(h)| + |\beta_k(h)| \leq C_3 \|h\|_V \tag{3}$$

where C_3 is independent of h. (See the proof of Theorem 3.1 in [11] for more details about this.)

The functions $F_{k,1}$ and $F_{k,2}$ can now be approximated as well as we please in $\mathbf{C} \setminus D_k$ by rational functions $R_{k,1}$ and $R_{k,2}$ with their poles in D_k $k = 1, 2, \ldots, N$ so that if we define

$$f^* = \sum_{k=1}^{N} G_k - \alpha_k(h) R_{k,1} + \beta_k(h) R_{k,2} \tag{4}$$

then we have

$$\|f^*\|_{\mathbf{C}\setminus V} < \varepsilon \|h\|_V < \varepsilon C_V \tag{5}$$

where C_V is a constant depending only on V. The existence of C_V comes from property (ii) of h listed above, and since $\bar{V} \cap \bar{D} = \Phi$.

Note that from the remark following Theorem 2 there exists a constant C'_V also depending only on V such that from (5) we have

$$N(f^*|_D) \leq \varepsilon C'_V \|h\|_V < \varepsilon C_V C'_V. \tag{6}$$

Let now n vary and carry out this construction with $V = V_{2n-1}$ and $\varepsilon = \varepsilon_n$, $n = 1, 2, \dots$. In this way we obtain functions f_n^*, $n = 1, 2, \dots$ with the same properties as f^* has above. We can choose ε_n independent of h such that

$$\|f_n\|_{\mathbf{C}\setminus V_{2n-1}} + N(f^*_n|_D) < t(2^{-2})2^{-n} \tag{6'}$$

where t is the number in Lemma 1.

Now define $h' = h - \Sigma_n f_n^*$. By (6) and property (iii) of $\{V_n\}$, h' has the following property

$$h'|_D \in A \text{ and } N((h' - h)|_D) < t \cdot 2^{-2} \tag{7}$$

We now wish to repeat this construction with h replaced by h' and V_{2n-1} by V_{2n}, $n = 1, 2, \dots$. We have to be a bit careful because h' can be unbounded in V_{2n} for some n. But for $n = 1, 2, \dots$ it is easy to see that we can find open sets $W_n \subset V_{2n}$ such that $K_{2n} \subset W_n$ and such that none of the rational functions $R_{k,1}$ or $R_{k,2}$ used in the definition of f_n^*, $n = 1, 2, \dots$ has poles in W_n. But then it follows that there exists constants E_n, $n = 1, 2, \dots$ independent of h and h' such that

$$\|h'\|_{W_n} \leq E_n \quad \text{for} \quad n = 1, 2, \dots. \tag{8}$$

We can now repeat the above construction with h replaced by h' and V_{2n-1} replaced by W_n for $n = 1, 2, \dots$. We obtain functions g_n^* analytic in $\mathbf{C}\setminus W_n$ in the same way as we obtained f_n^*.

Define $h^* = h - \Sigma_n f_n^* - \Sigma_n g_n^*$. In the same way as we obtained (7) we get

$$h^*|_D \in A \quad \text{and} \quad N((h^* - h)|_D) < t \cdot 2^{-1}. \tag{9}$$

From the properties of the T_φ-operator mentioned above one can also deduce that h^* is analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$ except for the poles of the rational functions $R_{k,1}$ and $R_{k,2}$ corresponding to each f_n^* and each g_n^*.

Let now K be a compact subset of $\mathbf{C}\setminus(\bar{S}\setminus D)$ and let Σ_n' denote summation over those n for which $\bar{W}_n \cap K = \Phi$ and $V_{2n-1} \cap K = \Phi$. It is easy to see that there exists a constant E_K depending on K but not on h such that

$$\|h - \sum_n{}' (f_n^* + g_n^*)\|_K \leq E_K. \tag{10}$$

We conclude that our function h^* satisfies almost Lemma 1. We get rid of the rational functions $R_{k,1}$ and $R_{k,2}$ by the following lemma

LEMMA 2. *Suppose $\eta > 0$ is given. Let p be a rational function with poles only at the points $z_1, \ldots, z_m$ in $\mathbf{C}\setminus\bar{0}_1$. Then there exists a function s analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$ and an open set $W \subset \mathbf{C}\setminus(\bar{S}\setminus D)$ such that*

(i) *$s|_D \in A$ and $\|s - p\|_{\mathbf{C}\setminus W} + N((s - p)|_D) < \eta$*

(ii) $\operatorname{dist}(z, \bar{S}\setminus D) < 2L \max_{1\leq k\leq m} \operatorname{dist}(z_k, \bar{S}\setminus D)$ *for*

each $z \in W$, where L is as in condition (II) mentioned above.

Proof. It is clearly sufficient to prove this lemma when $m = 1$. We choose a polygonal arc $\gamma = \gamma_{z_1}$ as in condition (2).

Divide γ into subarcs γ_k with endpoints z_k and z_{k+1}, $k = 1, 2, \ldots$ such that z_{k+1} is the only common point of γ_k and γ_{k+1} for each k.

Choose connected open sets $U_k \supset \bar{\gamma}_n$ for $k = 1, 2, \ldots$ and rational functions p_k, $k = 1, 2, \ldots$ (with $p = p_1$) with poles only at z_k such that

$$\|p_{k+1} - p_k\|_{\mathbf{C}\setminus U_k} + N((p_{k+1} - p_k)|_D) < \eta\, 2^{-k}$$

$k = 1, 2, \ldots$. Since each U_k is connected and since we can assume $\bar{U}_k \cap \bar{D} = \Phi$ this is easy to obtain. We can also assume $\bar{U}_k \cap K = \Phi$ if k is sufficiently large and K is a given compact subset of $\mathbf{C}\setminus(\bar{S}\setminus D)$. Since the length of γ is less than $L \operatorname{dist}(z_1, \bar{S}\setminus D)$ it is easy to see that we can choose $U_k\, k = 1, 2, \ldots$ such that $W = \cup_k U$ satisfies (ii). But then p_k converges to a function s which satisfies our requirements.

It is now relatively easy to complete the proof of Lemma 1. Each of the functions f_n^* and g_n^* can be written as finite sums of the form (4). (For g_n^* one must replace h by h' in (4).) The rational functions $R_{k,1}$ and $R_{k,2}$ are independent of h and we have also bounds for the constants $\alpha_k(h)$ and

$\beta_k(h)$ which are independent of h. (See [3] and the remark following (5).) If one applies Lemma 2 with care and approximate the functions $R_{k,1}$ and $R_{k,2}$ by functions $S_{k,1}$ and $S_{k,2}$ analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$ using that lemma, we get "new" functions f_n^{**} and g_n^{**} by replacing $R_{k,1}$ and $R_{k,2}$ by $S_{k,1}$ and $S_{k,2}$ in the expressions of the form (4) for f_n^* and g_n^*. Define then

$$:h_1 = h - \sum_n (f_n^{**} + g_n^{**}). \tag{11}$$

Note from property (v) of $\{V_n\}$ that if $U \supset (\bar{S}\setminus D)$ is open then there exists a number N such that the poles of the rational functions $R_{k,1}$ and $R_{k,2}$ corresponding to f_n^* and g_n^*, must be contained in U if $n \geq N$. From this fact and (ii) in Lemma 2 it is easy to see that the series (11) will converge uniformly on compact subsets of $\mathbf{C}\setminus(\bar{S}\setminus D)$. From (9) and (10) it follows that h_1 will satisfy Lemma 1 if Lemma 2 is applied carefully. We don't want to go into further details about this.

Using Theorem 2 we shall now prove:

THEOREM 3. *Assume $S = \{z_n\} \subset D$ satisfies $\Sigma_n(1 - |z_n|) < \infty$. If $0 < p \leq \infty$ and $f \in H^p(D)|_S$, there exists g analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$ such that $g|_D \in H^p(D)$ and $g = f$ on S.*

Proof. Assume first Theorem 3 is proved for $1 < p \leq \infty$. If $g \in H^p(D)$ has no zeros in D, g has a k'th root for some integer k such that $kp > 1$. By assumption we can find f in $H^{kp}(D)$ which interpolates this k'th root on S and extends to be analytic in $\mathbf{C}\setminus(\bar{S}\setminus D)$. But $f^k|_D \in H^p(D)$ and interpolates g on S. Since an arbitrary function in $H^p(D)$ can be written as the sum of two functions in $H^p(D)$ with no zeros in D, ([4], page 79) Theorem 3 will be true for all $p > 0$ if it holds for $1 < p \leq \infty$. By Theorem 2 we need only prove the following for $1 < p \leq \infty$:

(***) *There exists an open set $0 \supset \bar{D}\setminus(\bar{S}\setminus D)$ such that each f in $H^p(D)|_S$ extends to a function h analytic in 0 such that $h|_D \in H^P(D)$.*

If $p = \infty$ this is just the result (*) proved by Detraz [3]. Her methods seem to work also if $1 < p < \infty$, but some additional results from the theory of H^p-spaces are needed. We give here a different proof for $1 < p < \infty$.

We first need an approximation result for $H^p(D)$ similar to Lemma 3.2 in [11]. If $f \in H^p(D)$, $1 < p < \infty$, $\|f\|_p$ denotes its norm in $H^p(D)$.

LEMMA 3. *Assume $1 < p < \infty$. There exists a constant C_p depending only on p such that for each $\varepsilon > 0$ and each relatively closed set $F \subset D$ we can find an open set $0 \supset \bar{D} \setminus (\bar{F} \setminus D)$ with the following properties:*

Given $f \in H^p$ there exists g analytic in 0 such that $g|_D \in H^p$ and

(a) $\sup\{|f(z) - g(z)|, z \in F\} < \varepsilon \|f\|_p$,

(b) $\|g|_D\|_P \leq C_p \|f\|_p$

(c) *for each set $K \subset 0$ with* $\text{dist}(K, \bar{F} \setminus D) > 0$ *we have* $\sup\{|g(z)|, z \in K\} < C_K \|f\|_p$ *where C_K is independent of f.*

To prove Lemma 3 it is convenient first to establish the following:

LEMMA 4. *Assume $1 < p < \infty$ and $f \in H^p(D)$. If φ is a measurable function on the unit circle T we define*

$$S\varphi f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Re} f(e^{i\theta}) \varphi(e^{i\theta}) \, d\theta$$

if z is outside the closed support supp φ *of φ. Assume* $0 \leq \varphi \leq 1$.

If $K \subset \mathbf{C}$ and $\text{dist}(K, \text{supp } \varphi) > 0$ *we have* $\sup\{|S\varphi f(z)|, z \in K\} \leq M_p$ $\text{dist}(K, \text{supp } \varphi)^{-1} \|f\|_P$ *where M_p is a constant depending only on p.*

Proof of Lemma 4. Since we on T have Re $S\varphi f = \varphi$ Ref, Lemma 4 is an immediate consequence of a well known theorem on M. Riesz ([4], Thm. 4.1, page 54) and HΦlder's inequality.

Proof of Lemma 3. We choose open plane sets V_j, $j = 1, 2, \ldots$ satisfying:

(i) $T \setminus \bar{F} \subset \cup_1^\infty V_j$,

(ii) $\bar{V}_j \cap \bar{V}_i = \Phi$ if $|i - j| > 1$,

(iii) $\bar{F} \cap \bar{V}_j = \Phi$ for $j = 1, 2, \ldots$,

and (iv) if $K \subset \mathbf{C} \setminus (\bar{F} \setminus D)$ is compact there are at most finitely many j such that $K \cap \bar{V}_j \neq \Phi$.

We also choose functions $\varphi_j \in C^\infty(T)$ such that $0 \leq \varphi_j \leq 1$, supp $\varphi_j \subset V_j$ and $\Sigma_1^\infty \varphi_j = 1$ on $T \setminus \bar{F}$.

Given $f \in H^p$ we put $f_j = S\varphi_j(f)$, $j = 1, 2, \ldots$ where $S\varphi_j(f)$ is defined as in Lemma 4. From the arguments used to prove Lemma 4 it is easy to see that we can choose numbers $r_j \in (0, 1)$, $j = 1, 2, \ldots$ independent of f such that the functions $h_j : z \to f(r_j z)$ satisfies

(1): $\sup\{|f_j(z) - h_j(z)| : z \in \mathbf{C} \setminus V_j\} < \varepsilon 2^{-j} \|f\|_p$ for $j = 1, 2, \ldots$.

Define $g = f - \Sigma_{j=1}^{\infty} (f_j - h_j)$. By (1), (a) in Lemma 3 is valid. Consider a point $w \in T \setminus \bar{F}$. There exists by (i) and (ii) a number k and a disc $\Delta(w)$ centered at w such that $\overline{\Delta(w)} \cap \bar{V}_j = \Phi$ if $j \notin \{k, k+1\}$.

Write

$$g = (f - f_k - f_{k+1}) + (h_k + h_{k+1}) + \Big(\sum_{j=k,\,k+1} (h_j - f_j)\Big)$$
$$= F_1 + F_2 + F_3$$

say. Here F_1 can be written as $S\varphi f$ where $\varphi = 1 - \varphi_k - \varphi_{k+1}$ must have compact support disjoint from $\overline{\Delta(w)}$. So F_1 is analytic in $\Delta(w)$ and by Lemma 4 $\sup\{|F_1(z)|, z \in \Delta(w)\} \leq C_w \|f\|_p$ where C_w depends only on dist(supp φ, $\Delta(w)$). Clearly also F_3 is analytic in $\Delta(w)$ and by (1) $\sup\{|F_3(z)|, z \in \Delta(w)\} \leq \varepsilon \|f\|_p$. Put $t = \max\{r_k, r_{k+1}\}$. Then F_2 is analytic in $\{z : |z| < t^{-1}\}$.

Define $D(w) = \Delta(w) \cap \{z : |z| < (1 + t^{-1})2^{-1}\}$. Again by Lemma 4 we obtain $\sup\{|F_2(z)|, z \in D(w)\} \leq C_w^1 \|f\|_p$ where C_w^1 depends only on t.

Let $D_j = D(w_j), j = 1, 2, \ldots$ denote a locally finite covering of $T \setminus \bar{F}$ by such sets. We define $0 = D \cup (\cup_j D_j)$.

To verify (c) in Lemma 3 let $K \subset 0$ have positive distance from $\bar{F} \setminus D$. Then we can write $K = K_1 \cup K_2$ where $\bar{K}_1$ is a compact subset of D and $K_2 \subset \cup_1^N D_j$ for some number N. It is easy to verify (c) on K_1 and K_2 separately.

It remains to verify (b). Consider the point $w \in T \setminus \bar{F}$ again. We have $|\mathrm{Re}\, g(w)| \leq \varepsilon \|f\|_p + |h_k(w)| + |h_{k+1}(w)|$

$$\leq \varepsilon \|f\|_p + \sup_{0<r<1} |f_k(rw)| + \sup_{0<r<1} |f_{k+1}(rw)|$$

$$\leq \varepsilon \|f\|_p + 2 \sup_{0<r<1} u(rw) = \varepsilon \|f\|_p + \eta(w)$$

where u is the harmonic extension to D of $|f|$.

Finally let $w \in \bar{F} \setminus D$. We can clearly assume $\bar{V}_j \cap rz = \Phi$ for all j, all $z \in \bar{F} \setminus D$ and all $r \in (0, 1)$. But this implies

$$|\mathrm{Re}\, g(w)| \leq \varepsilon \|f\|_p + |\mathrm{Re} f(w)|.$$

By a theorem of Hardy and Littlewood $\|\eta\|_p \leq A_p \|f\|_p$ where A_p depends only on p. But then $\|\mathrm{Re}\, g\|_p \leq K_p \|f\|_p$ where K_p depends only p and by the theorem of M. Riesz used in the proof of Lemma 4, (b) follows. The Hardy-Littlewood result is in [4, Thm. 1.9, p. 12].

To complete the proof of the above claim about $H^p(D)$ we need a result similar to (**) for $H^p(D)$ when $1 < p < \infty$.

We need some notation. Let Γ be a simple closed rectifiable curve and denote by 0_Γ the bounded component of $C \setminus \Gamma$. Let μ denote the arc length measure associated with Γ. So $\mu(E)$ is the length of $E \cap \Gamma$ for each Borel set E. If $1 < p < \infty$, $H^P(\Gamma)$ denotes the closure in $L^P(\mu)$ of the polynomials in z. The functions in $H^P(\Gamma)$ can be extended to analytic functions in 0_Γ by Cauchy's integral formula and we shall assume them extended in this way.

LEMMA 5. *Let $S = \{z_n\} \subset D$ satisfy $\Sigma_n(1 - |z_n|) < \infty$. Then there exists a contour Γ such that $\bar{0}_\Gamma \setminus (\bar{S} \setminus S) \subset D$, $0_\Gamma \supset S$ and $H^p(\Gamma)|_S = H^p(D)|_S$ for $1 \leq p \leq \infty$.*

Proof. This result is essentially contained in Carleson's lemma ([4], page 203) and the proof we give has all its basic ideas contained in the proof of Carleson's lemma. Let $B(z)$ be the Blaschke product corresponding to S and let B_N consist of the first N factors in the product defining B. Let

$$S_1 = \{z \in D : |B(z)| \leq 2^{-1}\}. \text{Then } \bar{S}_1 \setminus S_1 = \bar{S} \setminus S.$$

Let now $T \setminus \bar{S}$ consist of the disjoint arcs J_n, $n = 1, 2, \ldots$. For each n we choose a simple arc $I_n \subset D \setminus \{0\}$ with endpoints equal to the endpoints of J_n and with the radial projection onto T equal to J_n. We wish to do this in such a way that the arclength measure associated with $\cup_n I_n$ is a Carleson measure. (See [4] page 157 for definition.) We indicate one way of doing this. Assume for simplicity that $J_n = \{e^{i\theta} : -a < \theta < a\}$ for some $a \in (0, \pi)$. Let $\{a_k\} \subset (0, a)$ and $\{r_k\} \subset (1 - a/\pi, 1)$ be monotonic sequences converging to a and 1 respectively. Assume that $R_k = \{re^{i\theta} : |\theta| < a_k, r_k < r < 1\}$ is disjoint from S_1 and $1 - r_k < a - a_k$ for all k. Define $I_n = D \cap \partial (\cup_k R_k)$. It is easy to verify that $\{I_n\}$ has all the required properties.

Define $\Gamma = (\bar{S} \setminus S) \cup (\cup_n I_n)$. Fix an integer N and choose $f \in H^p(\Gamma)$. As in [4] page 204 and 139–140, we get that the function g_N in $H^p(D)$ of minimal norm which interpolates f on $\{z_1, \ldots, z_N\}$ must satisfy

$$\|g_N\|_p \leq |(2\pi i)^{-1} \int_\Gamma h(z) f(z) (B_N(z))^{-1} dz| \tag{11}$$

for some $h \in H^q(D)$ of norm one and where $p^{-1} + q^{-1} = 1$. Since $|B_N| \geq |B| \geq 2^{-1}$ on Γ and the arc length measure associated with $\Gamma \cap D$ is a Carleson measure we get by using Hϕlder's inequality that

(12) $$\|g_N\|_P \leq C_1 \|f\|_{LP(\mu)} \text{ where } C_1$$

depends only on Γ. (See Theorem 9.3 on page 157 in **[4]**.) A subsequence of $\{g_N\}$ converges uniformly on compact subsets of D to a function g which satisfies Lemma 5.

The result (***) for $1 < p < \infty$ is now easy to prove. It follows from Lemma 3 and Lemma 5 in the same way as we proved (ii) $\Rightarrow$ (i) in Theorem 1.

We finally apply Theorem 2 to a result of Vinogradov **[12]**. Again let $S = \{z_n\} \subset D$. We shall need the following condition on S:

(C) $$\inf_k \prod_{\substack{n=1 \\ n \neq k}} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| > 0.$$

This is a condition which is necessary for solving many interpolation problems. See **[2]**, **[13]** and **[14]** for example.

Denote by BV_1 all sequences $\{a_n\}_{n=1}^{\infty}$ such that $\Sigma_1^{\infty} |a_{n+1} - a_n| < \infty$. BV_1 is a Banach space with norm

$$\|\{a_n\}_{n=1}^{\infty}\| = |a_1| + \sum_1^{\infty} |a_{n+1} - a_n|.$$

We also let B_1 denote the Banach algebra of all analytic functions in D whose derivative belongs to $H^1(D)$. The norm on B_1 is given by $N(f) = \|f\|_D + \|f^1\|_1$.

If $S = \{z_n\}_{n=1}^{\infty} \subset D$ satisfies (C) and converges to 1 non-tangentially, (which means that $|1 - z_n| \leq \lambda(1 - |z_n|)$, $n = 1, 2, \ldots$ for some $\lambda > 0$) Vinogradov proved that $B_1|_S = BV_1$.

Our result is:

THEOREM 4. *Assume $S = \{z_n\}$ satisfies* (C) *and converges to* 1 *non-tangentially. For each $\{a_n\} \in BV_1$ there exists f analytic in $\mathbf{C} \setminus \{1\}$ interpolating $\{a_n\}$ at $\{z_n\}$ such that f is bounded in $\{w : |1 + w| \leq 2\}$ and $f'|_D \in H^1$.*

Proof. We first prove that each $g \in B_1|_S$ extends to a bounded analytic function h in $\{w : |1 + w| < 2\}$ with $h'|_D H^1$.

Define $\Phi(z) = (1 + z)/2$, $z \in \mathbf{C}$. By the theorem of Vinogradov it is sufficient to show that $\{\Phi(z_n)\}_{n=1}^{\infty}$ satisfies (C). (Observe that $f \in B_1 \Rightarrow h = f_0 \Phi B_1$). Clearly $w_n = \Phi(z_n) \rightarrow 1$ non-tangentially.

By a recent result of Kam-Fook Tse **[12]**, Theorem 1, page 352, it is sufficient to find $t > 0$ such that

$$\inf_{i,j} \left| \frac{w_i - w_j}{1 - \bar{w}_j w_i} \right| \geq t.$$

Since $\{z_n\}$ satisfies (C) this is easy and we omit it. But then we can deduce Theorem 4 from Theorem 2.

Final remarks. We now give the example showing that (iii) in Theorem 1 may fail. Let $R = \{z = x + iy : 0 < x < 1, -1 < y < 1\}$ and define $R_n = \{z = x + iy : 2^{-3n-2} \leq x \leq 2^{-3n-1}, |y| > \varepsilon_n\}$ for $n = 1, 2, \ldots$ where $\{\varepsilon_n\}$ is a sequence to be specified. Let $I_n = (2^{-3n-4}, 2^{-3n-2})$ and choose a finite set of points $S_n \subset I_n$ with the following property: If f is an analytic function vanishing on S_n and bounded by one on the rectangle $D_n = \{z = x + iy : x \in I_n, |y| < 1\}$ then $|f(2^{-3n-3} + iy)| < n^{-1}$ if $|y| < 1 - n^{-1}$. Let now $U = R \setminus \cup_n R_n$ and $S = \cup_n S_n$. Clearly $\bar{S} \setminus S = \{0\}$ and if $f \in H^\infty(U)$ then $f(2^{-3n} + iy) \to 0$ as $n \to \infty$ if $|y| < 1$. It follows that $\Pi(\bar{S})$ includes the segment $\{x = 0, -1 < y < 1\}$. It only remains to show that $\{\varepsilon_n\}$ can be choosen such that S is the zero set of a nonzero function h in $H^\infty(U)$. Let g_n correspond to S_n and D_n in the same way as g corresponded to S and V in the proof of Theorem 1. Define $g_n \equiv 1$ outside D_n. Using Vitushkin's scheme for approximation ([**6**], page 210) it is easy to find functions h_n such that $h_n g_n$ is analytic near the endpoints of I_n, h_n is analytic where g_n is and $|1 - h_n(z)| < 2^{-n}$ if dist (z, I_n) is less than $n^{-1} 2^{-3n}$. (Approximate $\log(g_n)$ near the endpoints of I_n and take exponentials and call this function h_n.) Moreover $\sup\{|h_n(z)|, z \in C\} \leq A$ where A is an absolute constant. It follows that the infinite product consisting of all the factors $h_n g_n$, $n = 1, 2, \ldots$ is analytic in $\cup_n D_n$ and in a neighbourhood of the closure of I_n for $n = 1, 2, \ldots$. So if the ε_n tend sufficiently rapidly to zero, h will be in $H^\infty(U)$ and S will be zero set of h.

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UNIVERSITY OF EDINBURG