PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 2, 1974

CLOSURE THEOREMS FOR AFFINE TRANSFORMATION GROUPS

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Let $\mathscr{H}$ be a closed subgroup of the group of linear transformations of R_n onto itself. Let hx denote the image of the point x under the transformation h, and let $\mathscr{G}$ be the transpose group of $\mathscr{H}$: i.e. its elements are associated with matrices which are the transposes of those in $\mathscr{H}$. For f in $L^2(R_n)$, let $Cl\{f; \mathscr{H} \times R_n\}$ denote the closure in the L^2 norm of the linear span of functions of the form $f(hx + t)$ where h is in $\mathscr{H}$, and t is in R_n. Since this space is translation-invariant, it is of the form $L^2(\hat{S})$: i.e. the set of L^2 functions $r(x)$ such that the nonzero set of $\hat{r}$, the Fourier transform of r, is, except for a set of measure zero, included in S. In the first theorem a precise description of S is given, and in the second, a function is constructed in a natural way whose translates alone generate the given space.

S is roughly the orbit of $N(f)$, the nonzero set of $\hat{f}$, under the group $\mathscr{G}$:

$$\bigcup_{g \text{ in } \mathscr{G}} g(N(f)) = \mathscr{G}(N(f)) .$$

However a difficulty arises in that $N(f)$ is determined only to within a set of measure zero, and $\mathscr{G}$ may transform sets of measure zero into nonmeasurable sets. For example, when the rotation group of the plane acts on a nonmeasurable linear set (of the x-axis, say), a nonmeasurable planar set results. Hence some care is required in defining S. Let E_f denote the set of points of density (one) of $N(f)$. Since the exceptional set of $N(f)$ has measure zero, every point of E_f has density one with respect to E_f. The orbit of the set E_f under the group $\mathscr{G}$ will be used as S, and as part of our first theorem, it will be shown that S is measurable.

The fact that closed translation-invariant subspaces of L^2 are of the form $L^2(\hat{S})$ is due to L. Schwartz [3]. The characterization of $Cl\{f; \mathscr{H} \times R_n\}$ in Theorem 1 reduces to the familiar Wiener theorem when $\mathscr{H}$ consists only of the identity and has been proved by S. R. Harasymiv for L^p spaces and for general distribution spaces [**1**, **2**] when $\mathscr{H}$ is the diagonal group. The second theorem involves the construction of a function p, arising naturally from f by an integration over $\mathscr{G}$, such that the translates of p generate the same space: i.e. such that $Cl\{p; R_n\} = Cl\{f; \mathscr{H} \times R_n\}$.

2. The closure theorem.

THEOREM 1. *Let $\mathcal{H}$ be a closed group of linear transformations of R_n onto itself with transpose group $\mathcal{G}$. For f in $L^2(R_n)$, let E_f be the set of points of density of $N(f)$. Then*
(i) $S = \mathcal{G}(E_f)$ *is measurable, and*
(ii) $Cl\{f;\ \mathcal{H} \times R_n\} = L^2(\hat{S})$.

For x in $\mathcal{G}(E_f)$, let $g = g(x)$ be any element of $\mathcal{G}$ such that x is in $g(E_f)$. For such a point, there exists $\delta = \delta(g) > 0$ such that, if $B_r(x)$ is the ball of radius r about x as center, then

$$m\{B_r(x) \cap g(E_f)\} \geq 2\delta m\{B_r(x)\}$$

for sufficiently small r. Here m denotes Lebesgue measure. Hence $B_r(x) \cap g(E_f)$ contains a closed set $A_r(x)$ such that

$$m\{A_r(x)\} \geq \delta m\{B_r(x)\}.$$

It may be assumed that x itself belongs to $A_r(x)$. The sets $A_r(x)$, $r \leq r(x)$, x in $\mathcal{G}(E_f)$, constitute a family of closed sets covering $\mathcal{G}(E_f)$ in the sense of Vitali. Thus there exists a countable subfamily covering $\mathcal{G}(E_f)$ except for a set of measure zero. But since each set $A_r(x)$ is contained in $\mathcal{G}(E_f)$, then $\mathcal{G}(E_f)$ is measurable. This completes the proof of part (i).

For reference below, let $A_r^{(n)}(x)$, $n = 1, 2, \ldots$ denote the countable subcover, and let $\{g_n\}$ denote the corresponding sequence in $\mathcal{G}$. Thus the difference set

$$\mathcal{G}(E_f) \sim \bigcup_n g_n(E_f) \tag{1}$$

is of measure zero.

Now the proof of part (ii) of the theorem follows standard lines. In particular, if k is orthogonal to the space $Cl\{f;\ \mathcal{H} \times R_n\}$, then $\hat{k}(x)\hat{f}(g_n x) = 0$ for every g_n of the sequence in (1). Hence $\hat{k}$ must vanish for almost every point of $\mathcal{G}(E_f) = S$.

3. An equivalent space of translates. Since $L^2(\hat{S})$ is translation-invariant, it is generated by the translates of any function p such that the sets $N(p)$ and S differ by a set of measure zero. Such a function arises naturally from an integration with respect to Haar measure on $\mathcal{G}$. Consider $\hat{f}(gx) = \hat{f}(g, x)$ as a function on the product space $\mathcal{G} \times R$ where the measure associated with $\mathcal{G}$ is μ, the right invariant Haar measure, and the measure associated with R_n is m, Lebesgue measure. It is shown below that $\hat{f}(gx)$ is measurable on this product space. Now let β be a strictly positive measurable function of $\mathscr{G}$ such that $|g^{-1}|^{1/2}\beta(g)$ is in $L^1(\mathscr{G})$. Here $|g^{-1}|$ denotes the absolute value of the determinant associated with g^{-1}. Assuming the measurability of the function $\hat{f}(gx)$, let us consider the function

$$\hat{p}(x) = \int_{\mathscr{G}} |\hat{f}(gx)| \beta(g)\, d\mu(g)\,. \tag{2}$$

In view of Minkowski's inequality, $\hat{p}$ is in $L^2(R_n)$, and its inverse Fourier transform, p, is the one we wish to consider in our second theorem.

THEOREM 2. *Let f be in $L^2(R_n)$ and S the orbit under $\mathscr{G}$ of the set of points of density of $N(f)$. Then*
(i) *$\hat{f}(gx)$ is measurable on $\mathscr{G} \times R_n$, and*
(ii) *the function p, with Fourier transform $\hat{p}$ defined by (2), is in $L^2(R_n)$, and $Cl\{p; R_n\} = L^2(\hat{S})$.*

Let $\mathscr{O}$ be an open set in the complex plane. Then $(\hat{f})^{-1}(\mathscr{O})$ is measurable in R_n and can be expressed as $B \cup C$, where B is a Borel set and C has measure zero. Let α be the map from $\mathscr{G} \times R_n$ defined as $\alpha(g, x) = gx$. We have

$$\alpha^{-1}\{(\hat{f})^{-1}(\mathscr{O})\} = \{\alpha^{-1}(B)\} \cup \{\alpha^{-1}(C)\}.$$

Since α is continuous, $\alpha^{-1}(B)$ is a Borel set in $\mathscr{G} \times R_n$. It is thus enough to prove that $\alpha^{-1}(C)$ is of measure zero: i.e. that α^{-1} is absolutely continuous as a set function. Let K be a compact set in $\mathscr{G}$, and let D be an open set containing C such that $m(D) < \epsilon = \epsilon(K)$. A routine calculation shows that

$$(\mu \times m)\{\alpha^{-1}(D) \cap (K \times R_n)\} \le \varepsilon \int_K |g^{-1}|\, d\mu(g)\,.$$

Thus the outer measure of $\alpha^{-1}(C) \cap (K \times R_n)$ is zero. Since K is an arbitrary compact set of $\mathscr{G}$, $\alpha^{-1}(C)$ is measurable and of measure zero. This completes the proof of (i).

It has already been noted that $\hat{p}$, and hence p, are in $L^2(R_n)$. To complete the proof of part (ii), it is enough to show that $N(p)$, the nonzero set of $\hat{p}$, and S differ by a set of measure zero. Let D be the set of points x such that, for any neighbourhood of x, $\mathscr{N}(x)$,

$$\int_{\mathscr{N}(x)} \hat{p}(y)\, dy > 0\,.$$

The sets D and $N(p)$ differ by a set of measure zero so that it is enough to show the same for D and S.

Our first step is to show that D is invariant with respect to $\mathscr{G}$: i.e.

$$gD \subset D \text{ for all } g \text{ of } \mathscr{G}. \tag{3}$$

Since $\mathscr{G}$ is a group, this is equivalent to saying that $gD = D$ for all g of $\mathscr{G}$. It is clear that S is invariant in this same sense. Since the complement of an invariant set is also invariant, and since the intersection of two invariant sets is invariant, it will follow from (3) that both difference sets, $S \sim D$ and $D \sim S$, are invariant. The proof is then completed by integration of $\hat{p}$ over these sets.

Let x be in D, g_0 in $\mathscr{G}$, and $\epsilon > 0$. Since μ is right invariant Haar measure, it follows that

$$\int_{|g_0x-y|<\varepsilon} \hat{p}(y)\,dm(y) = |g_0| \int_{|g_0(x-z)|<\varepsilon} dm(z) \int_{\mathscr{G}} |\hat{f}(gz)|\,\beta(gg_0^{-1})\,d\mu(g)\,.$$

By the continuity of g_0, there is a $\delta > 0$ such that if $|x - z| < \delta$, then $|g_0(x - z)| < \epsilon$ so that the above integral exceeds

$$|g_0| \int_{|x-z|<\delta} dm(z) \int_{\mathscr{G}} |\hat{f}(gz)|\,\beta(gg_0^{-1})\,d\mu(g)\,. \tag{4}$$

Since $\beta(gg_0^{-1}) > 0$ for all g of $\mathscr{G}$, and since

$$\int_{|x-z|<\delta} dm(z) \int_{\mathscr{G}} |\hat{f}(gz)|\,\beta(g)\,d\mu(g) > 0$$

the integral of (4) must be positive. Thus g_0x is also in D, and so D is invariant in the sense of (3).

Let $A = D \sim S$, which, in view of (3), is invariant. Let q be a strictly positive function of $L^2(R_n)$. Then

$$\int_A \hat{p}(x)\,q(x)\,dm(x) = \int_{\mathscr{G}} \beta(g)\,d\mu(g) \int_A |\hat{f}(gx)|\,q(x)\,dm(x)\,.$$

If this integral is positive, then for some g of $\mathscr{G}$, $gA \cap E_f$ has positive measure. Since $gA = A$, and since $E_f \subset S$, then $A \cap S$ has positive measure. Since the latter is impossible, the integral is zero. Since $q(x) > 0$ for all x, and since $p(x) > 0$ for almost every x of D, this implies that $A = D \sim S$ has zero measure.

Let $B = S \sim D$. Then

$$0 = \int_B |\hat{p}(x)|\,dm(x) = \int_{\mathscr{G}} \beta(g)\,|g^{-1}|\,d\mu(g) \int_{gB} |\hat{f}(x)|\,dm(x)\,.$$

Since β is a strictly positive function, then for almost every g of $\mathscr{G}$,

$$\int_{gB} |\hat{f}(x)|\, dm(x) = 0 . \tag{5}$$

Since B is invariant, (5) is true for all g of $\mathscr{G}$, with $gB = B$. If $m(B) > 0$, then for some g_n of the sequence in (1), $m(B \cap g_n E_f) > 0$. By the invariance of B, $m(B \cap E_f) > 0$. But this contadicts (5) with g taken as the identity.

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Received February 21, 1973. The work on this paper was supported by National Science Foundation Grant GP 9053.

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