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STIELTJES DIFFERENTIAL-BOUNDARY OPERATORS, II

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The differential boundary system

$$Ly = (y + H[Cy(0) + Dy(1)] + H_1\Psi)' + Py ,$$

$$Ay(0) + By(1) + \int_0^1 dK(t)y(t) = 0 ,$$

$$\int_0^1 dK_1(t)y(t) = 0 ,$$

and its adjoint system are written as Stieltjes integral equation systems with end point boundary conditions. Fundamental matrices are exhibited and, from these, a spectral analysis and a Green's matrix are produced. These are used to achieve spectral resolutions in both self-adjoint and nonself-adjoint situations.

1. **Introduction.** This article is a continuation of [2] and [6] which showed the density of the domain of L in $\mathscr{L}_n^p[0, 1]$, $1 \leqq p < \infty$, when the boundary functionals satisfied certain conditions, and which derived the dual operator in $\mathscr{L}_n^q[0, 1]$, $1/p + 1/q = 1$, in those circumstances. Rather than repeat those results, we prefer to refer the reader to the articles mentioned. For our purposes here it is sufficient to state that y is an n dimensional vector in $\mathscr{L}_n^p[0, 1]$; A and B are $m \times n$ matrices, $m \leqq 2n$, such that rank $(A\colon B) = m$; C and D are $(2n - m) \times n$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular; K is an $m \times n$ matrix valued function of bounded variation such that the measure it generates satisfies $dK(0) = A$, $dK(1) = B$; K_1 is an $r \times n$ matrix valued function of bounded variation which is not absolutely continuous, satisfying $dK_1(0) = 0$, $dK_1(1) = 0$; H and H_1 are, respectively, $n \times (2n - m)$ and $n \times s$ matrix valued functions of bounded variation, H_1 not absolutely continuous; P is a continuous $n \times n$ matrix; and, finally, Ψ is an s dimensional constant vector.

Because we wish to exhibit the contributions of K, K_1, H, H_1 at 0 and 1 separately, integrals involving their resulting measures will not include contributions at 0 or 1. At all other points, however, we do assume that these functions are regular as defined by Hildebrandt [4]. This results in considerable simplification throughout. Of course, all integrals are Lebesgue or Lebesgue-Stieltjes integrals.

It is convenient to note that the adjoint system has the form

$$L^*z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi)' + P^*z ,$$

$$\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*(t)z(t) = 0 ,$$

$$\int_0^1 dH_1^*(t)z(t) = 0 ,$$

where ϕ is an r dimensional constant vector, and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ satisfy

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix} = \begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_{2n} .$$

2. Integral equation representation. Let us make the following definitions. Let

$$\begin{aligned} \xi_1 &= y , \\ \xi_2 &= Ay(0) + \int_0^t dK(x)y(x) , \\ \xi_3 &= Cy(0) + Dy(1) , \\ \xi_4 &= \int_0^t dK_1(x)y(x) , \\ \xi_5 &= \Psi . \end{aligned}$$

Then the equation $Ly = 0$, together with the boundary conditions is equivalent to the system

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}(t) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}(0) + \int_0^t d\begin{pmatrix} -Q & 0 & -H & 0 & -H_1 \\ K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ K_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}(x)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}(x) ,$$

where $Q(t) = \int_0^t P(x)dx$,

$$\begin{pmatrix} A & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C & 0 & -\frac{1}{2}I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}(0) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ B & I & 0 & 0 & 0 \\ D & 0 & -\frac{1}{2}I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}(1) = 0 .$$

If $M(t)$ represents the Stieltjes measure in the integral equation, then Hildebrandt's $\Delta M^{\pm}(t)$ has zero entries along the diagonal. Hence $I \pm \Delta M^{\pm}$ is always nonsingular.

The adjoint system $L^*z = 0$, together with the boundary conditions is

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{pmatrix}(t) = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{pmatrix}(0) - \int_0^t d\begin{pmatrix}-Q^* & K^* & 0 & K_1^* & 0\\ 0 & 0 & 0 & 0 & 0\\ -H^* & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -H^* & 0 & 0 & 0 & 0\end{pmatrix}(x)\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{pmatrix}(x) ,$$

$$\begin{pmatrix}I & A^* & C^* & 0 & 0\\ 0 & 0 & -D^* & 0 & 0\\ 0 & 0 & I & 0 & 0\\ 0 & 0 & 0 & 0 & I\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{pmatrix}(0) + \begin{pmatrix}0 & 0 & -C^* & 0 & 0\\ I & -B^* & D^* & 0 & 0\\ 0 & 0 & I & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{pmatrix}(1) = 0 .$$

These representations should be compared to those found in [5] which they generalize under certain conditions.

In addition we note that the problem $Ly = \lambda y$ has a similar representation. The only change necessary is to replace $Q(t) = \int_0^t P(x)dx$ by $Q(t) - \lambda t$. The nonhomogeneous problem $Ly = f$ has a representation as a nonhomogeneous integral equation with an additional term

$$F(t) = \int_0^t \begin{pmatrix}f\\ 0\\ 0\\ 0\\ 0\end{pmatrix}(x)dx$$

on the right side.

3. Fundamental matrices. We can express the homogeneous integral problem generated by $(L - \lambda I)y = 0$ together with the boundary conditions in a more compact way by the expressions

$$\xi(t) = \xi(0) + \int_0^t dM_\lambda(x)\xi x ,$$

$$R\xi(0) + S\xi(1) = 0 ;$$

likewise the adjoint system by

$$\eta(t) = \eta(0) - \int_0^t dM_\lambda^*(x)\eta(x) ,$$

$$\tilde{R}\eta(0) + \tilde{S}\eta(1) = 0 .$$

We shall assume in addition that $M_\lambda(t)$ is regular:

$$M_\lambda(t) = 1/2[M_\lambda(t+) + M_\lambda(t-)] ,$$
$$M(0) = M(0+) , \qquad M(1) = M(1-) .$$

Hildebrandt [4] and Vejvoda and Tvrdy [8] have shown that under these conditions the first integral equation has a solution given by $\xi(t) = U_\lambda(0, t)\xi(0)$, where $U_\lambda(s, t)$ is the uniform limit of Picard-like approximations beginning with I (hence U_λ is analytic in λ) satisfying

$$U_\lambda(s, t) = I + \int_s^t dM_\lambda(x) U_\lambda(s, x) .$$

U_λ has the additional properties $U_\lambda(t, t) = I$, and $U_\lambda(r, t) U_\lambda(s, r) = U_\lambda(s, t)$. U_λ is therefore a fundamental matrix when M_λ is absolutely continuous.

Similarly the adjoint equation has a solution given by $\eta(t) = V_{\lambda^*}(0, t)\eta(0)$, where $V_{\lambda^*}(s, t)$ satisfies

$$V_{\lambda^*}(s, t) = I - \int_s^t dM_\lambda^*(x) V_{\lambda^*}(s, x) ,$$

$V_{\lambda^*}(t, t) = I$, $V_{\lambda^*}(r, t) V_{\lambda^*}(s, r) = V_{\lambda^*}(s, t)$.

Since M_λ is regular, it is possible to show that U_λ and V_{λ^*} are related through the formula

$$U_\lambda(s, t) = V_\lambda(t, s) .$$

Hence $U_\lambda(s, t)^{-1} = V_\lambda(s, t)$. Regularity, however, is not inherited from M_λ unless $(\Delta^+ M_\lambda)^2 \equiv 0$. This occurs only when $\Delta^+ K \Delta^+ H \equiv 0$, $\Delta^+ K_1 \Delta^+ H \equiv 0$, $\Delta^+ K \Delta^+ H_1 \equiv 0$, $\Delta^+ K_1 \Delta^+ H_1 \equiv 0$, and will not be necessary.

The fundamental matrices U_λ and V_λ may be easily calculated in the same way as was done in [5]. If $Y(t)$ is a fundamental matrix for $Y' + PY = 0$ satisfying $Y(0) = I$, and

$$\mathscr{H}(t) = \int_0^t e^{-\lambda x} Y(t) Y(x)^{-1} dH(x) ,$$

$$\mathscr{H}_1(t) = \int_0^t e^{-\lambda x} Y(t) Y(t)^{-1} dH_1(x) ,$$

$$\mathscr{K}(t) = \int_0^t dK(x) e^{\lambda x} Y(x) ,$$

$$\mathscr{K}_1(t) = \int_0^t dK_1(x) e^{\lambda x} Y(x) ,$$

$$\mathscr{L}(t) = \int_0^t dK(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH(x) ,$$

$$\mathscr{L}_{01}(t) = \int_0^t dK(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH_1(x) ,$$

$$\mathscr{L}_{10}(t) = \int_0^t dK_1(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH(x) ,$$

$$\mathscr{L}_{11}(t) = \int_0^t dK_1(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH_1(x) ,$$

and $\mathscr{M}(t)$, $\mathscr{M}_{01}(t)$, $\mathscr{M}_{10}(t)$, $\mathscr{M}_{11}(t)$ are defined by the same formulae as $\mathscr{L}(t)$, $\mathscr{L}_{01}(t)$, $\mathscr{L}_{10}(t)$, $\mathscr{L}_{11}(t)$ with only the limits of integration with respect to x changed to from z to t, then

$$U_\lambda(0, t) = \begin{pmatrix} e^{\lambda t} Y(t) & 0 & -e^{\lambda t}\mathscr{H}(t) & 0 & -e^{\lambda t}\mathscr{H}_1(t) \\ \mathscr{K}(t) & I & -\mathscr{L}(t) & 0 & -\mathscr{L}_{01}(t) \\ 0 & 0 & I & 0 & 0 \\ \mathscr{K}_1(t) & 0 & -\mathscr{L}_{10}(t) & I & -\mathscr{L}_{11}(t) \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

and

$$V_\lambda(0, t) = \begin{pmatrix} e^{-\lambda t} Y(t)^{-1} & 0 & Y(t)^{-1}\mathscr{H}(t) & 0 & Y(t)\mathscr{H}_1(t) \\ -\mathscr{K}(t)e^{-\lambda t}Y(t)^{-1} & I & -\mathscr{M}(t) & 0 & -\mathscr{M}_{01}(t) \\ 0 & 0 & I & 0 & 0 \\ -\mathscr{K}_1(t)e^{-\lambda t}Y(t)^{-1} & 0 & -\mathscr{M}_{10}(t) & I & -\mathscr{M}_{11}(t) \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

By applying the boundary condition of U_λ the following theorem immediately follows.

THEOREM 3.1. *If $Y(t)$ is a fundamental matrix for $Y' + PY = 0$ satisfying $Y(0) = I$, then the system*

$$Ly = \lambda y ,$$

$$Ay(0) + By(1) + \int_0^1 dK(t)y(t) = 0 ,$$

$$\int_0^1 dK_1(t)y(t) = 0$$

is compatible if and only if the rank of

$$\begin{pmatrix} A & -I & 0 & 0 & 0 \\ Be^\lambda Y(1) + \mathscr{K}(1) & I & -Be^\lambda\mathscr{H}(1) - \mathscr{L}(1) & 0 & -Be^\lambda\mathscr{H}_1(1) - \mathscr{L}_{01}(1) \\ De^\lambda Y(1) + C & 0 & -De^\lambda\mathscr{H}(1) - I & 0 & -De^\lambda\mathscr{H}_1(1) \\ 0 & 0 & 0 & I & 0 \\ \mathscr{K}_1(1) & 0 & -\mathscr{L}_{10}(1) & I & -\mathscr{L}_{11}(1) \end{pmatrix}$$

is less than $3n + r + s$. If $m = n$, the system is compatible if and only if the determinant of the matrix above is zero.

We shall assume throughout the remainder of this article that $m = n$ in order to derive eigenfunction expansions under various conditions.

4. The Green's matrix. Whenever the homogeneous problem is not comparable, the nonhomogeneous problem possesses a unique solution generated by a Green's matrix, just as is the case for the regular Sturm-Liouville problem. Hildebrandt [4] shows that the solution to

$$\xi(t) = \int_0^t dM_\lambda(s)\xi(s) + \mathscr{F}(t) ,$$

$$\xi(0) = \mathscr{F}(0)$$

is given by

$$\xi(t) = U_\lambda(0, t)\mathscr{F}(0) + \int_0^t U_\lambda(s, t)d\mathscr{F}(s)$$

whenever $\Delta^{\pm}\mathscr{F} \equiv 0$. Since in our situation $\mathscr{F}(t) = F(t) + \xi(0)$, where $F(t)$ is absolutely continuous, $F'(t) = f_0(t) = (f(t), 0 \cdots 0)^T$, we find that the solution can be expressed by

$$\xi(t) = U_\lambda(t, 0)y(0) + \int_0^t U_\lambda(s, t)f_0(s)ds .$$

If $\xi(1)$ is calculated and $R\xi(0) + S\xi(1)$ is set equal to 0, $\xi(0)$ is determined, and the solution takes the form

$$\xi(t) = \int_0^1 \mathscr{G}_\lambda(s, t)f_0(s)ds ,$$

where the Green's function $\mathscr{G}$ is given by

$$\begin{aligned} \mathscr{G}_\lambda(s, t) &= U(0, t)[R + SU_\lambda(0, 1)]^{-1}RU_\lambda(0, s)^{-1},\ s < t , \\ &= -U(0, t)[R + SU_\lambda(0, 1)]^{-1}SU_\lambda(0, 1)U_\lambda(0, s)^{-1},\ s > t . \end{aligned}$$

This is the same formula as that encountered in the regular Sturm-Liouville problem. The Green's function $\mathscr{G}$ possesses the properties, including the adjoint properties, usually attributed to Green's functions.

We note in particular that λ is in the spectrum of the operator L if and only if

$$\det [R + SU_\lambda(0, 1)] = 0 .$$

Since $[R + SU_\lambda(0, 1)]$ is analytic in λ, this implies that either the entire complex plane is in the point spectrum of L, or else the spectrum of L consists only of isolated eigenvalues, accumulating only at ∞.

5. Self-adjoint Stieltjes differential-boundary expansions. It was shown earlier in [6] that the operator $T = iL$ is self-adjoint in

$\mathscr{L}_n^2[0, 1]$ if and only if

1. $P^* = -P$
2. $m = n,\ r = s.$
3. $K = [BD^* - AC^*]H^*$ a.e.
4. $AA^* = BB^*$
5. $H[CC^* - DD^*] = 0$ a.e.
6. $K_1 = MH_1^*$, where M is a nonsingular $r \times r$ matrix.

This being the case, then the spectrum of T is contained in the real axis. Every point with nonzero imaginary part lies in the resolvent. This implies that $\det[R + U_\lambda(0, 1)S] = 0$ only at isolated real points with ∞ their only limit. An application of the spectral resolution theorem for self-adjoint operators on a Hilbert space results in the following.

THEOREM 5.1. *If T is self-adjoint, then*

1. *The spectrum of T consists of a denumerable set of real eigenvalues, accumulating only at ∞.*

2. *Each eigenvalue corresponds to at most n eigenfunctions. Eigenfunctions corresponding to different eigenvalues are orthogonal.*

3. *For each complex number λ, not an eigenvalue, $(T - \lambda I)^{-1}$ exists and can be represented by a unique linear integral operator*

$$(T - \lambda I)^{-1}f(t) = \int_0^1 G_\lambda(s, t)f(s)ds .$$

4. *The Green's function $G_\lambda(s, t)$ satisfies*

a. *As a function of t, $s \neq t$,*

$$(T - \lambda I)G_\lambda(s, t) = 0 .$$

b. $AG_\lambda(s, 0) + BG_\lambda(s, 1) + \int_0^1 dK(t)G_\lambda(s, t) = 0$

a.e. in s.

c. $\int_0^1 dK_1(t)G_\lambda(s, t) = 0$ *a.e. in s.*

d. $G_\lambda(t, s) = G_\lambda^*(s, t)$ *a.e. in s and t.*

e. *The eigenfunctions of T are complete in $\mathscr{L}_n^2[0, 1]$.*

If those corresponding to the same eigenvalue have been made orthonormal (denote them by $\{y_i\}_1^\infty$), then for all f in $\mathscr{L}_n^2[0, 1]$

$$f = \sum_1^\infty (f, y_i)y_i .$$

Operators self-adjoint under a transformation are substantially more complex and will be discussed in a subsequent paper. At this point the existence of such a transformation except in trivial cases is doubtful.

6. Nonself-adjoint Stieltjes differential-boundary expansions. Expansions for nonself-adjoint systems have been derived in certain earlier circumstances. First, for the case where $H = 0$, $H_1 = 0$, $K_1 = 0$ or when $H = 0$, $H_1 = 0$, $K = 0$ (the adjoint of the former), an expansion was derived in [2] using familiar techniques. Second, when $H_1 = 0$, $K_1 = 0$ (so $r = 0$, $s = 0$) and H and K are absolutely continuous, an expansion was derived in [5].

In the present situation troubles arise. The bottom terms in the matrix of Theorem 3.1 do not all asymptotically have nice limits as $\mathrm{Re}(\lambda) \to \infty$, a necessary sort of condition previously. For example, when

$$K_{j/6}(t) = 0,\ 0 \leqq t < \frac{j}{6}\,,$$
$$= 1,\ \frac{j}{6} < t \leqq 1\,,$$

the system

$$Ly = (y + K_{1/6}[y(0) - y(1)] + K_{2/6}\Psi)'$$
$$y(0) + y(1) + \int_0^1 dK_{3/6}y = 0\,,$$
$$\int_0^1 d[K_{4/6} + K_{5/6}]y = 0\,,$$

has eigenvalues which are zeros of the determinant of

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ e^{\lambda} + e^{3\lambda/6} & 1 & -e^{5\lambda/6} - e^{2\lambda/6} & 0 & -e^{4\lambda/6} - e^{\lambda/6} \\ e^{\lambda} + 1 & 0 & -e^{5\lambda/6} - e^{2\lambda/6} & 0 & -e^{4\lambda/6} \\ 0 & 0 & 0 & 1 & 0 \\ e^{4\lambda/6} + e^{5\lambda/6} & 0 & -e^{3\lambda/6} - e^{4\lambda/6} & 1 & -e^{2\lambda/6} - e^{3\lambda/6} \end{bmatrix}.$$

These are $\lambda = (2k + 1)6\pi i$; $k = 0, \pm 1, \cdots$. As $\mathrm{Re}\,\lambda \to -\infty$, however, the matrix has a singular limit.

However, the system

$$Ly = (y + K_{3/6}\Psi)'$$
$$y(0) + y(1) = 0\,,$$
$$\int_0^1 dK_{3/6}y = 0\,,$$

has as its eigenvalue determining matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -e^{\lambda} & 1 & 0 & 0 & -e^{\lambda/2} \\ 1+e^{\lambda} & 0 & -1 & 0 & -e^{\lambda/2} \\ 0 & 0 & 0 & 1 & 0 \\ -2e^{\lambda/2} & 0 & 0 & 1 & -1 \end{bmatrix} .$$

The eigenvalues are easily seen to be $\lambda = 2k\pi i$, $k = 0, \pm 1, \cdots$. The limit of the matrix above as $\operatorname{Re}\lambda \to -\infty$ is nonsingular. Frankly, the author does not entirely understand what is going on.

It is possible to extend the results of [5] under some rather severe restrictions. Let us assume that $H_1 = 0$ and $K_1 = 0$ so that a 3 dimensional vector representation (with $\xi_4 = 0$ and $\xi_5 = 0$) is possible. In addition assume that H is continuous (or by considering the adjoint problem that K is continuous). One system has the form

$$Ly = (y + H[Cy(0) + Dy(1)])' + Py$$

$$Ay(0) + By(1) + \int_0^1 dKy = 0 .$$

If y is replaced by $\tilde{y}$ under the invertable transformation $y = Y\tilde{y}$ $(Y' + PY = 0)$, then we find the equations $Ly = f$, $Ly = \lambda y$ are equivalent to

$$\left(\tilde{y} + \left[Y^{-1}H - \int_0^t Y^{-1}Pdx\right][CY(0)\tilde{y}(0) + DY(1)\tilde{y}(1)]\right)' = Y^{-1}f \text{ or } = \lambda\tilde{y} .$$

The new equations are of the same form as the old, with the same assumptions, with the absence in the second set of the term Py. This results in an equivalent system in which the terms Y and Y^{-1} are missing, a considerable simplification in calculation. We shall henceforth assume that $P = 0$.

The following lemma is the analog of Lemmas 6.4–6.8 of [5].

LEMMA 6.1. (a) $\lim_{\operatorname{Re}(\lambda)\to\infty} \mathscr{H}(t) = 0$ a.e. *In particular* $\lim_{\operatorname{Re}(\lambda)\to\infty} \mathscr{H}(1) = 0$.

(b) $\lim_{\operatorname{Re}(\lambda)\to\infty} e^{\lambda t}[\mathscr{H}(1) - \mathscr{H}(t)] = 0$ a.e.

(c) $\lim_{\operatorname{Re}(\lambda)\to\infty} e^{-\lambda t}\mathscr{K}(t) = 0$ a.e. *In particular* $\lim_{\operatorname{Re}(\lambda)\to\infty} e^{-\lambda}\mathscr{K}(1) = 0$.

(d) $\lim_{\operatorname{Re}(\lambda)\to\infty} [\mathscr{K}(t) \cdot \mathscr{H}(1) - \mathscr{L}(t)] = 0$ a.e.

(e) $\lim_{\operatorname{Re}(\lambda)\to\infty} \mathscr{M}(t) = 0$ a.e. *In particular* $\lim_{\operatorname{Re}(\lambda)\to\infty} \mathscr{M}(1) = 0$.

Proof. Let V_α^β stand for the total variation from α to β.

(a) If $0 < a < t$, then for an appropriate norm

$$\begin{aligned}\|\mathscr{H}(t)\| &= \left\|\int_0^t e^{-\lambda x} d\mathscr{H}(x)\right\| \\ &\leqq \left\|\int_0^a e^{-\lambda x} d\mathscr{H}(x)\right\| + \left\|\int_a^t e^{-\lambda x} d\mathscr{H}(x)\right\| \\ &\leqq V_0^a\|\mathscr{H}\| + e^{-a\lambda} V_a^t\|\mathscr{H}\| .\end{aligned}$$

The first can be made less than half of any preassingned ε if a is sufficiently close to 0. The second is less than $\varepsilon/2$ if $\operatorname{Re}(\lambda)$ is sufficiently large.

(b) $$\begin{aligned}\|e^{\lambda t}[\mathscr{H}(1) - \mathscr{H}(t)]\| &= \left\|e^{\lambda t}\int_t^1 e^{-\lambda x} d\mathscr{H}(x)\right\| \\ &\leqq \left\|e^{\lambda t}\int_{t+\delta}^1 e^{\lambda x} d\mathscr{H}(x)\right\| + \left\|e^{\lambda t}\int_t^{t+\delta} e^{\lambda x} d\mathscr{H}(x)\right\|\end{aligned}$$

when $t \leqq t+\delta \leqq 1$. The second term is less than $V_t^{t+\delta}\|\mathscr{H}\|$. This can be made less than any $\varepsilon/2$ by choosing δ small. The first is bounded by $e^{-\lambda\delta} V_0^1\|\mathscr{H}\|$ which becomes small as $\operatorname{Re}(\lambda) \to \infty$.

(c) This is shown by the same technique as was used in (a).

(d) $$\begin{aligned}\|\mathscr{K}(t)\mathscr{H}(1) - \mathscr{L}(t)\| &= \left\|\int_0^t d\mathscr{K}(z)\int_z^1 e^{\lambda(z-x)} d\mathscr{H}(x)\right\| \\ &\leqq \left\|\int_0^t d\mathscr{K}(z)\int_{z+\delta}^1 e^{\lambda(z-x)} d\mathscr{H}(x)\right\| \\ &\quad + \left\|\int_0^t d\mathscr{K}(z)\int_z^{z+\delta} e^{\lambda(z-x)} d\mathscr{H}(x)\right\| .\end{aligned}$$

The second term is bounded by $V_0^1\|\mathscr{K}\| \cdot \sup_z V_z^{z+\delta}\|\mathscr{H}\|$. Since $\mathscr{H}$ is continuous on [0, 1] this can be made uniformly small if δ is sufficiently close to 0. The first term is then bounded by $e^{-\lambda\delta} V_0^1\|\mathscr{K}\|$ $V_0^1\|\mathscr{H}\|$ which has zero limit as $\operatorname{Re}(\lambda) \to \infty$.

(e) This is shown by the same technique as was used in (d).

It is now possible to determine the location of the eigenvalues of L.

THEOREM 6.2. *The eigenvalues of L are the zeros of the determinant of*

$$\Delta_1 = \begin{pmatrix} A & -I & 0 \\ Be^{\lambda} + \mathscr{K}(1) & I & -Be^{\lambda}\mathscr{H}(1) - \mathscr{L}(1) \\ De^{\lambda} + C & 0 & -De^{\lambda}\mathscr{H}(1) - I \end{pmatrix}.$$

If A is nonsingular, they are bounded on the left in the complex plane. If B is nonsingular, they are bounded on the right in the complex plane. Hence when both A and B are nonsingular, the eigenvalues of L be in a vertical strip.

Since $\det \Delta_1$ *is almost periodic in* $\operatorname{Im}(\lambda)$, *when A and B are nonsingular, the number of zeros lying in a vertical strip* $|\operatorname{Re}(\lambda)| < h$ *also satisfying* $k < \operatorname{Im}(\lambda) < k+1$ *is bounded by some number* *independent of* $\measuredangle$. *For any* $\delta > 0$ *there is a corresponding* $m(\delta) \gg 0$ *such that*

$$|\det \varDelta_1| > m(\delta)$$

for λ *lying in the strip* $|\mathrm{Re}(\lambda)| < h$ *and outside circles of radius* δ *with centers at the zeros of* $\det \varDelta_1$.

Proof. An elementary calculation shows, when A is nonsingular, that as $\mathrm{Re}(\lambda) \to -\infty$, $\det \varDelta_1 = (\det A + o(1))$, which ultimately cannot be zero. Similarly, using Lemma 6.1, when B is nonsingular, as $\mathrm{Re}(\lambda) \to \infty$, $\det \varDelta_1 = -e^{\lambda}(\det B + o(1))$, which is also ultimately nonzero. The final statements follow from [7, pp. 264-269].

We are now in a position to quote directly the results in § 6 of [5]. Please note that the phrases "uniformly in ⋯" appearing there should be replaced by "for all x, ξ in (0, 1)". Actually a.e. will do fine. Such is our present situation. Assuming A and B are nonsingular, we quote:

THEOREM 6.3. *Let* λ_0 *be in the resolvent set for* L. *Let* $\{\lambda_i\}_1^\infty$ *be the eigenvalues of* L (*which for convenience we assume to be simple*). *Let* $\{Y_i\}_1^\infty$ *and* $\{Z_i\}_1^\infty$ *be the associated eigenfunctions and adjoint eigenfunctions, assuming that* $\int_0^1 Z_i^* Y_i dx = 1$. *Then the Green's function for* L, $G_{\lambda_0}(s, t) = \mathscr{G}_{11}(s, t)$ *satisfies*

$$G_{\lambda_0}(s, t) = \sum_{i=1}^{\infty} \frac{Y_i(t) Z_i^*(s)}{\lambda_i - \lambda_0} \quad \text{a.e.}$$

The proof is by contour integration using the asymptotic estimates established in this section as well as that in [5, § 6], suitably avoiding the zeros of $\det \varDelta_1$ as we know is possible.

By integrating $G_{\lambda_0}(s, t) \cdot f(s)$ with respect to s before the contour approaches ∞ and appealing to the Lebesgue dominated convergence theorem, we find:

THEOREM 6.4. *Let* f *in* $\mathscr{L}_n^p[0, 1]$ *be in the domain of* L, *then*

$$f(t) = \sum_{i=1}^{\infty} Y_i(t) \int_0^1 Z_i^*(s) f(s) ds .$$

COROLLARY 6.5. *If* f *in* $\mathscr{L}_n^p[0, 1]$ *is in the domain if* L *and* g *in* $\mathscr{L}_n^q[0, 1]$ *is in the domain of* L^*, *then* (*Parseval's Equality*)

$$\int_0^1 g^*(t) f(t) dt = \sum_{i=1}^{\infty} \int_0^1 g^*(t) Y_i(t) dt \int_0^1 Z_i^*(s) f(s) ds .$$

The problem of expansions in the general case remains open.

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