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CHARACTERS FULLY RAMIFIED OVER A NORMAL SUBGROUP

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Let H be a group and N a normal subgroup. Assume that χ is an irreducible (complex) character of H, and that the restriction of χ to N is a multiple of some irreducible character of N, say θ. Then $\chi_N = e\theta$, and e is called the ramification index. It is easy to see that it always satisfies $e^2 \leqq |H: N|$, and when equality holds, χ is said to be fully ramified over N. It is this "fully ramified case" which will be studied here in some detail. As an application of some of the methods of this paper, we prove the following solvability theorem in the last section. If H has an irreducible character fully ramified over a normal subgroup N and if p^4 is the highest power of p dividing $|H: N|$ for all primes corresponding to nonabelian Sylow p-subgroups of H/N, then H/N is solvable.

1. Fully ramified triples. Groups of type f.r. To simplify notation, say that (H, N, χ) is a fully ramified triple if χ is an irreducible character of H, N is normal in H, and χ is fully ramified over N. It has been conjectured in [13] that H/N is solvable in this case, and some partial results in this direction appear in [12]. We extend this work in Theorem 4.5 below. It is also possible to show that no known simple group can occur as a homomorphic image of H/N, but we will only need to consider a few cases in this paper (see Lemmas 4.1 and 4.3).

Since we are primarily concerned with the factor group H/N, rather than with H itself, the following theorem (due ultimately to I. Schur and A. H. Clifford) is extremely useful.

THEOREM 1.1. *Let H be a group, N a normal subgroup, and χ an irreducible character of H. Let θ be an irreducible constituent of χ_N, and assume $\chi_N = e\theta$* (i.e. *θ is invariant*). *Then, there exists a group H^*, with an irreducible character χ^*, and a normal subgroup N^* having a faithful irreducible character θ^*, such that*

$$\chi^*_{N^*} = e\theta^*$$
$$N^* \textit{ is central in } H^*$$
$$\textit{and } H/N \cong H^*/N^* \,.$$

Moreover, the isomorphism is "natural" in the sense that if K is any normal subgroup of H containing N, and K^*/N^* corresponds *to K/N, then:*

$$\chi_K = m(\psi_1 + \cdots + \psi_t)$$
$$\text{and } \chi^*_{K^*} = m(\psi^*_1 + \cdots + \psi^*_t)\ ,$$

where $\psi_1, \cdots, \psi_t$(resp. $\psi^*_1, \cdots, \psi^*_t$) are the distinct conjugates of some irreducible constituent of χ_K(resp. $\chi^*_{K^*}$). In particular, (H, K, χ) is a fully ramified triple if and only if (H^*, K^*, χ^*) is a fully ramified triple.

Many other properties hold than those listed above, but they will not be needed. A proof may be found in [9].

If (H, N, χ) is a fully ramified triple in which N is a central subgroup, then it is easy to see that N must be the center, since $|H: N| = \chi(1)^2 \leqq |H: Z(H)| \leqq |H: N|$. Groups with this property have been referred to in the literature as groups of central type, and in view of Theorem 1.1, there is no essential difference between fully ramified triples and groups of central type.

Lemma 2.3 (a) gives a way of constructing new fully ramified triples from old ones. Unfortunately, these new triples need not correspond to groups of central type, even when the original triple does. Because of this, we state our results for triples, rather than groups of central type.

Define a group G to be of type f.r. if G is isomorphic to H/N, for some fully ramified triple (H, N, χ). Groups of type f.r. have been characterized in [12] as those groups G having a factor set α, over the multiplicative group of complex numbers, such that the corresponding twisted group algebra $C[G]_\alpha$ is simple (or equivalently, has center $\cong C$). We shall have no occasion to use this characterization here.

The next theorem may be used to construct examples of fully ramified triples. It will not be needed in any of the later sections, but it does restrict the kinds of properties that hold in groups of type f.r. In particular, if $\mathscr{P}$ is any property of groups which is inherited by subgroups, then the existence of any solvable group not satisfying $\mathscr{P}$ implies the existence of a (solvable) group of type f.r. not satisfying $\mathscr{P}$.

In the following, $\pi(K)$ denotes the set of prime divisors of the order of K.

THEOREM 1.2. *Let G be any solvable group. Then there is a fully ramified triple (H, Z, χ) with G isomorphic to a subgroup of H/Z. Furthermore, such a triple may be chosen with $\pi(H) = \pi(Z) = \pi(G)$, χ faithful, $Z = Z(H)$ and $|Z|$ square-free.*

Proof. Choose $M \lhd G$ with $|G: M| = p$, a prime. By induction, there is a fully ramified triple $(K, Z(K), \zeta)$ with ζ faithful, M isomorphic to a subgroup of $K/Z(K)$, $\pi(K) = \pi(Z(K)) = \pi(M)$, and $|Z(K)|$ squarefree. If $p \nmid |K|$, replace $(K, Z(K), \zeta)$ by $(K \times C, Z(K) \times C, \zeta \divideontimes \lambda)$, where C is a cyclic group of order p, and λ is a faithful linear character of C. We may therefore assume $\pi(K) = \pi(Z(K)) = \pi(G)$. Let

$$W = \{(z_1, \cdots, z_p) \in Z(K) \times \cdots \times Z(K) \mid \prod z_i = 1\} .$$

Then $W \subseteqq Z(K \times K \times \cdots \times K)$, and we may form the quotient

$$U = (K \times \cdots \times K)/W .$$

Let $\eta = \zeta \divideontimes \cdots \divideontimes \zeta \in \mathrm{Irr}\,(K \times \cdots \times K)$, and note that $W = \ker \eta$, so we may view $\eta \in \mathrm{Irr}\,(U)$. It is easy to check that $Z(U) = (Z(K) \times \cdots \times Z(K))/W$, and $\eta(1)^2 = |U: Z(U)|$. Also $Z(U) \cong Z(K)$, so its order is square-free and $\pi(U) = \pi(G)$.

Let $\langle b \rangle$ be a cyclic group of order p. Fix an element $z \in Z(K)$ of order p, and construct an automorphism a of $U \times \langle b \rangle$ as follows:

$$((x_1, x_2, \cdots, x_p)W, b^i) \xrightarrow{a} ((z^i x_p, x_1, x_2, \cdots, x_{p-1})W, b^i) ,$$

for $x_1, \cdots, x_p \in K$ and $0 \leqq i < p$. It is easy to check that a is well defined, and is an automorphism of order p. Using this automorphism, construct the usual semi-direct product $H = (U \times \langle b \rangle) \rtimes \langle a \rangle$. Notice $Z(H) = Z(U)$.

Extend $\eta \in \mathrm{Irr}\,(U \times \langle b \rangle)$ so that $\ker \tilde{\eta} = \langle b \rangle$. Now $\langle b \rangle$ is not normalized by a, so $\tilde{\eta}$ is not an invariant character. It follows that $\chi = \tilde{\eta}^H$ is irreducible, and $\ker \chi = \mathrm{Core}_H(\langle b \rangle) = 1$.

Now:

$$\chi(1)^2 = (p\tilde{\eta}(1))^2 = p^2\eta(1)^2 = p^2|U: Z(U)| = |H: Z(H)| ,$$

so $(H, Z(H), \chi)$ is a fully ramified triple. By construction, $\pi(G) = \pi(H) = \pi(Z(H))$, and $|Z(H)|$ is square-free.

It remains only to check that G is isomorphic to a subgroup of $H/Z(H)$. From the construction of H, the group $H/Z(H)$ is isomorphic to the direct product of a cyclic group of order p (generated by the image of b in $H/Z(H)$) with the wreath product $(K/Z(K)) \wr \langle a \rangle$. As M is isomorphic to a subgroup of $K/Z(K)$, it follows $M \wr \langle a \rangle$ is isomorphic to a subgroup of $(K/Z(K)) \wr \langle a \rangle$. Finally, $G \leqq M \wr (G/M) \cong M \wr \langle a \rangle$, and this completes the proof. (Elementary properties of the wreath product which were used may be found in [8]. See especially pp. 98–99).

2. Restriction to normal subgroups. Let (H, Z, χ) be a fully ramified triple, and K a normal subgroup of H containing Z. More can be said about the irreducible constituents of χ_K than is already contained in Clifford's theorem. The explicit statement is Lemma 2.3 below.

We begin first with a lemma describing what happens when K is not assumed to be normal. If α and β are characters of the same group, write $\alpha \leqq \beta$(or $\beta \geqq \alpha$) if $\beta - \alpha$ is zero or a character.

LEMMA 2.1. *Let (H, Z, χ) be a fully ramified triple and let L be a subgroup of H containing Z. Write*

$$\chi_L = \sum_{i=1}^{t} a_i \zeta_i$$

for positive integers $a_1, \cdots, a_t$ and distinct irreducible characters $\zeta_1, \cdots, \zeta_t$ of L. Let θ denote the unique irreducible constituent of χ_Z so that $\chi_Z = e\theta$ and $e^2 = |H: Z|$. Let $b_i = |L: Z|a_i/e$ for $i = 1, \cdots, t$. Then:

(a) $e\theta^L = |L: Z|\chi_L$.

(b) $\theta^L = \sum_{i=1}^t b_i\zeta_i$. *In particular,* $e \mid |L: Z|a_i$ *for* $i = 1, \cdots, t$.

(c) $\zeta_{iZ} = b_i\theta$ *and* $\zeta_i^H = a_i\chi$ *for* $i = 1, \cdots, t$.

(d) $\sum_{i=1}^t a_i^2 = |H: L|$ *and* $\sum_{i=1}^t b_i^2 = |L: Z|$.

(e) *Suppose $t = 1$. Then (L, Z, ζ_1) is a fully ramified triple, and $\chi_L = a_1\zeta_1$, while $\zeta_1^H = a_1\chi$, with $a_1^2 = |H: L|$. Suppose additionally that $L \lhd H$. Then (H, L, χ) is a fully ramified triple.*

Proof. Since $e\chi = \theta^H$, the character χ vanishes off of Z. But $(e\theta^L)_Z = e|L: Z|\theta = |L: Z|\chi_Z$. Thus (a) holds. Conclusion (b) is immediate from (a) and the definition of the coefficients b_i. Now $\zeta_{iZ} \leqq \chi_Z = e\theta$. By Frobenius reciprocity, $\zeta_{iZ} = b_i\theta$. Similarly, $\zeta_i^H \leqq \theta^H = e\chi$, and $\zeta_i^H = a_i\chi$.

Now $(\chi_L)^H = |H: L|\chi$ as both sides vanish on $H - Z$, while on Z they equal $|H: L|e\theta$. Also $(\theta^L)_Z = |L: Z|\theta$ holds, again because θ is invariant. We conclude,

$$|H: L| = ((\chi_L)^H, \chi) = (\chi_L, \chi_L) = \sum_{i=1}^{t} a_i^2$$

and

$$|L: Z| = ((\theta^L)_Z, \theta) = ((\theta^L, \theta^L) = \sum_{i=1}^{t} b_i^2 ,$$

proving (d).

When $t = 1$, then (b), (c) and (d) imply conclusion (e).

A slight variation of the next lemma appears in [12], but is given here for completeness.

LEMMA 2.2. *Let (H, Z, χ) be a fully ramified triple, and let L be a Hall π-subgroup of H for some set π of primes. Thus, LZ/Z is a Hall π-subgroup of H/Z. Write $\chi_Z = e\theta$ where $\theta \in \mathrm{Irr}(Z)$, $e^2 = |H: Z|$. Then χ_{LZ} is a multiple of some unique irreducible character, say ζ, of LZ, and (LZ, Z, ζ) is a fully ramified triple. If $Z = Z(H)$, then $(L, L \cap Z, \zeta_L)$ is also a fully ramified triple.*

Proof. Let $\chi_{LZ} = \sum_{i=1}^t a_i\zeta_i$ as in Lemma 2.1, with LZ in place of L. Write $e = e_\pi e_{\pi'}$ where $e_\pi^2 = |LZ: Z|$ and $e_{\pi'}^2 = |H: LZ|$. By Lemma 2.1 (*b*), $e_\pi e_{\pi'} \,|\, |LZ: Z| a_1$. Therefore $e_{\pi'} | a_1$, and in particular, $e_{\pi'} \leqq a_1$. Now use the first equation from Lemma 2.1 (d):

$$\sum_{i=1}^t a_i^2 = |H: LZ| = e_{\pi'}^2 \leqq a_1^2 .$$

Thus $t = 1$ and the character $\zeta = \zeta_1$ is the only irreducible constituent of χ_{LZ}. Lemma 2.1(e) shows that (LZ, Z, ζ) is a fully ramified triple.

Finally, if Z is central, then $LZ = L \times Z_1$ where Z_1 is an abelian π' group. Thus ζ_L is irreducible and $\zeta(1)^2 = |LZ: Z| = |L: L \cap Z|$. Also $(\zeta_L)_{L\cap Z} = \zeta(1)\theta_{L\cap Z}$ so $(L, L \cap Z, \zeta_L)$ is a fully ramified triple, and the proof is complete.

If p is a prime and G is a group, let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Also, for any integer n, let n_p denote the p-part of n.

LEMMA 2.3. *Let (H, Z, χ) be a fully ramified triple with $Z = Z(H)$ and let K be a normal subgroup of H containing Z. Let R be a subgroup of H containing Z with $R/Z \in \mathrm{Syl}_p(H/Z)$, and let ζ be the unique irreducible constituent of χ_R guaranteed by Lemma 2.2. Write*

$$\chi_K = a(\tau_1 + \cdots + \tau_t)$$
$$\zeta_{R\cap K} = b(\sigma_1 + \cdots + \sigma_s)$$

where the τ_i and σ_j are the distinct conjugates of an irreducible constituent of χ_K and $\zeta_{R\cap K}$ respectively. As in Clifford's theorem, choose the unique $\psi_1 \in \mathrm{Irr}(\mathscr{I}_H(\tau_1))$ and $\tilde{\psi}_1 \in \mathrm{Irr}(\mathscr{I}_R(\sigma_1))$ with

$$\psi_1^H = \chi,\ (\psi_1)_K = a\tau_1$$

and

$$\tilde{\psi}_1^R = \zeta,\ (\tilde{\psi}_1)_{R\cap K} = b\sigma_1 .$$

Then:

(a) $a^2t = |H: K|$ *and* $b^2s = |R: R \cap K|$. *Moreover*, $(\mathscr{I}_H(\tau_1), K, \psi_1)$ *and* $(\mathscr{I}_R(\sigma_1), R \cap K, \psi_1)$ *are both fully ramified triples.*

(b) $b = a_p$, $s = t_p$, $\tau_1(1)_p = \sigma_1(1)$ *and under suitable ordering*, $R \cap \mathscr{I}_H(\tau_1) = \mathscr{I}_R(\sigma_1)$.

(c) *H contains a subgroup T containing K which satisfies* $|H: T| = s$, $T \cap R = \mathscr{I}_R(\sigma_1)$ *and* $TR = H$.

(d) *If* H/K *is a simple group, then* $\mathrm{Core}_R(\mathscr{I}_R(\sigma_1))$ *is either* R *or* $R \cap K$.

Proof. (a) By Lemma 2.1 (d), $|H: K| = a^2t$. Now,

$$a^2t = |H: K| = |H: \mathscr{I}_H(\tau_1)| \cdot |\mathscr{I}_H(\tau_1): K| = t|\mathscr{I}_H(\tau_1): K|$$

so $a^2 = |\mathscr{I}_H(\tau_1): K|$. Also $(\psi_1)_K = a\tau_1$ and this means that $(\mathscr{I}_H(\tau_1), K, \psi_1)$ is a fully ramified triple. The rest of (a) now follows by applying the above to the fully ramified triple (R, Z, ζ).

(b) By Lemma 2.1 (b), there are integers u and v so that $\theta^{R \cap K} = u(\sigma_1 + \cdots + \sigma_s)$, while $\theta^K = v(\tau_1 + \cdots + \tau_t)$. Hence, there are nonnegative integers $a_1, \cdots, a_t$ so that

$$\sigma_1^K = \sum_{i=1}^{t} a_i\tau_i .$$

Now, $(\tau_1)_{R \cap K} \leqq \chi_{R \cap K} = (\chi_R)_{R \cap K} = (|H: R|^{1/2})\zeta_{R \cap K} = |H: R|^{1/2}b(\sigma_1 + \cdots + \sigma_s)$. (The second equality follows from Lemma 2.2 and Lemma 2.1 (e).) Hence, there are nonnegative integers $b_1, \cdots, b_s$ so that

$$(\tau_1)_{R \cap K} = \sum b_j\sigma_j .$$

Comparing degrees: $|K: R \cap K|\sigma_1(1) = (\Sigma a_i)\tau_1(1)$ and

$$\tau_1(1) = (\Sigma b_j)\sigma_1(1) .$$

The second equation implies $\sigma_1(1) | \tau_1(1)$ and thus, $\tau_1(1)/\sigma_1(1)$ divides $|K: R \cap K|$ by the first equation. But $\sigma_1(1)$ is a power of p and $|K: R \cap K|$ is prime to p. It now follows that $\tau_1(1)_p = \sigma_1(1)$.

From (a) we have $a^2t = |H: K|$ and $b^2s = |R: Z|$. As $|R: Z|$ is the order of a Sylow p-subgroup of H/K, we get $a_p^2t_p = b^2s$. We have already derived that $\chi_{R \cap K} = |H: R|^{1/2}b(\sigma_1 + \cdots + \sigma_s)$. Since $\chi_K = a(\tau_1 + \cdots + \tau_t)$, we have by comparing degrees:

$$|H: R|^{1/2}bs\sigma_1(1) = at\tau_1(1) .$$

Equating p parts:

$$bs\sigma_1(1) = a_pt_p\tau_1(1)_p .$$

But $\tau_1(1)_p = \sigma_1(1)$, so $a_pt_p = bs$. We already had $a_p^2t_p = b^2s$, so $a_p = b$ and $t_p = s$.

The group $\mathscr{I}_R(\sigma_1)$ stabilizes σ_1 and acts on the set of irreducible constituents of $\sigma_1^K = \Sigma a_i\tau_i$. As $R \cap K$ is contained in $\mathscr{I}_H(\tau_i)$ for all i, it follows that all orbits of $\mathscr{I}_R(\sigma_1)$ on $\{\tau_1, \cdots, \tau_t\}$ have p-power size. Clearly $a_i = a_j$ if τ_i and τ_j lie in the same orbit, and we may write $\sigma_1^K = \sum_{\mathscr{O}} a_{\mathscr{O}}(\sum_{\tau \in \mathscr{O}}\tau)$, where the outer sum extends over all orbits, and $a_{\mathscr{O}}$ is the common value of a_i for any $\tau_i \in \mathscr{O}$.

Comparing degrees, $|K: K \cap R|\sigma_1(1) = (\Sigma_{\mathscr{O}} a_{\mathscr{O}}|\mathscr{O}|)\tau_1(1)$. Now $\sigma_1(1) = \tau_1(1)_p$ and $|K: K \cap R|$ is prime to p, so $\Sigma a_{\mathscr{O}}|\mathscr{O}| \not\equiv 0 (\mathrm{mod}\, p)$. Thus, there exists an orbit $\mathscr{O}$ with $a_{\mathscr{O}}|\mathscr{O}| \not\equiv 0 (\mathrm{mod}\, p)$. But this means $\mathscr{O} = \{\tau_j\}$ for some j, and $a_j \neq 0$ (so that $\tau_j \leqq \sigma_1^K$). Choose notation so that $\tau_j = \tau_1$. Hence τ_1 is invariant under $\mathscr{I}_R(\sigma_1)$, and thus

$$\mathscr{I}_R(\sigma_1) \subseteqq \mathscr{I}_H(\tau_1) \cap R .$$

From (a), $(\mathscr{I}_H(\tau_1), K, \psi_1)$ is a fully ramified triple, so a_p^2 is the order of a Sylow p-subgroup of $\mathscr{I}_H(\tau_1)/K$. Now $|\mathscr{I}_R(\sigma_1): R \cap K| = b^2 = a_p^2 \geqq |(\mathscr{I}_H(\tau_1) \cap R)K: K| = |\mathscr{I}_H(\tau_1) \cap R: R \cap K|$. But we had

$$\mathscr{I}_R(\sigma_1) \subseteqq \mathscr{I}_H(\tau_1) \cap R ,$$

so equality holds, and this completes the proof of (b). In fact the last argument shows slightly more, namely

$$\mathscr{I}_R(\sigma_1)K/K \in \mathrm{Syl}_p\,(\mathscr{I}_H(\tau_1)/K) .$$

(c) Let $N = N_H(R \cap K)$. As $(R \cap K)/Z \in \mathrm{Syl}_p\,(K/Z)$, the Frattini argument yields $NK = H$. Now N acts on the irreducible constituents of $\theta^{R\cap K}$. Hence N permutes the set $\{\sigma_1, \cdots, \sigma_s\}$. Now $R \subseteqq N$ and R acts transitively on this set, so N acts transitively. Clearly, $R \cap K$ is in the kernel of this action. Moreover, $(N \cap K)/(R \cap K)$ is a normal subgroup of $N/(R \cap K)$ having order prime to p. The set of characters therefore breaks up into k distinct $N \cap K$-orbits, each containing l elements where $l \mid |N \cap K: R \cap K|$ so $(p, l) = 1$. But $s = kl$ is a power of p, so $l = 1$ and $k = s$. This means $N \cap K$ is contained in the kernel of the action. Hence $S = \mathscr{I}_N(\sigma_1)$ contains $N \cap K$ and has index s in N. Thus, $T = SK$ has index s in H, and $TR = H$ is clear. Finally $T \cap R = S \cap R = \mathscr{I}_R(\sigma_1)$.

(d) In the notation of (c), $N/(N \cap K) \cong H/K$ and so $N/(N \cap K)$ is simple. We may assume $s > 1$, in which case N acts transitively on $\{\sigma_1, \cdots \sigma_s\}$ with kernel $N \cap K$. $\mathrm{Core}_R\,(\mathscr{I}_R(\sigma_1))$ is contained in the kernel, so $\mathrm{Core}_R\,(\mathscr{I}_R(\sigma_1)) \subseteqq R \cap (N \cap K) = R \cap K$.

The following consequence of Lemma 2.3 generalizes a theorem appearing in [11].

COROLLARY 2.4. *Let (H, Z, χ) be a fully ramified triple. If $Z \subseteqq K \lhd H$ and χ is induced from a character on K, then H/K is solvable.*

In particular, if K is solvable, then so is H.

Proof. Continuing with the above notation, let $\chi_K = a(\tau_1 + \cdots + \tau_t)$. Then $t = |H: \mathscr{I}_H(\tau_1)| = |H: K| = a^2 t$, so $a = 1$. By the previous lemma, H/K possesses a subgroup of index t_p for every prime divisor of t. The solvability of H/K now follows by Philip Hall's theorem (see p. 662 of [8]).

A special case of the next result appears as Theorem 5 of [12]. It is extremely useful in showing that many simple groups do not occur as homomorphic images of groups of type f.r.

THEOREM 2.5. *Suppose (H, Z, χ) is a fully ramified triple, $Z \subseteq K \lhd H$ and $G = H/K$. Let P be a Sylow p-subgroup of G and assume P is cyclic. Then P has a p-complement M in G. If G is simple, we also have*:

(a) *The prime p is unique*, i.e., *all other Sylow q-subgroups for $q \neq p$ are non-cyclic.*

(b) *P is a self-centralizing T.I. set in G.*

(c) *G acts doubly transitively on the cosets of M.*

Proof. By Theorem 1.1, and the remarks following that theorem, we may assume $Z = Z(G)$, so that Lemma 2.3 becomes applicable. Let $R/Z \in \mathrm{Syl}_p(H/Z)$, and let $\zeta, \sigma_1, \cdots, \sigma_s$, and $\tau_1, \cdots, \tau_t$ be as in Lemma 2.3. Now $\mathscr{I}_R(\sigma_1)$ has a character which is fully ramified over $R \cap K$, and the factor group $\mathscr{I}_R(\sigma_1)/R \cap K$ is cyclic. Thus, all irreducible constituents of $\sigma_1{}^{\mathscr{I}_{R(\sigma_1)}}$, including the fully ramified one, are extensions of σ_1. (See p. 54 of [3].) This can only happen if $\mathscr{I}_R(\sigma_1) = R \cap K$, so $s = |R/(R \cap K)| = |P|$. (This could also be seen by applying Theorem 1.1 to the group $\mathscr{I}_R(\sigma_1)$.) By Lemma 2.3 (c), $H/K = G$ has a subgroup M of index s, and this is clearly a p-complement in G. Suppose now G is simple. If a Sylow q-subgroup, say Q, of G is cyclic for some other prime q, then Q acts faithfully on the $|P|$ cosets of M, so that $|Q| < |P|$. Interchanging the roles of P and Q yields $|Q| > |P|$, and this contradication establishes (a).

To prove that P is a self-centralizing T.I. set in G, it suffices to show $C_G(\Omega_1(P)) = P$. Let $C = C_G(\Omega_1(P))$. As $P \cap M = 1$ and $PM = G$, we have $C = P(C \cap M)$, so that $C \cap M$ is a p-complement in C. Now $N_C(P)$ acts on P and centralizes $\Omega_1(P)$, and hence centralizes P by Fitting's lemma (see p. 178 of [6]). But then by Burnside's transfer theorem (see p. 419 of [8], also p. 252 of [6]), C has a normal p-complement, which must be $C \cap M$. Now $G = CM$ and $C \cap M$ is normal in C, so the normal closure of $C \cap M$ is contained in M. Hence, $C \cap M \subseteq (C \cap M)^G = 1$, as G is simple. Thus $C = P$, proving (b).

If $|P| = p$, then G must be doubly transitive on the cosets of M by a theorem of Burnside's (see p. 609 of [8]). If $|P| > p$, then P is a B-group (p. 65 of [15]) and it suffices to show that G is primitive on the cosets of M. Suppose G is not primitive, so there exists a subgroup L with $M < L < G$. But then $1 < P \cap L$ and $P \cap L$ is normalized by P. Since $G = PL$, it follows that the normal closure of $P \cap L$ is contained in L, contradicting the simplicity of G. (This last assertion can also be proved by considering the Brauer tree of the principal p-block of G. It can be proved that the principal character can be connected only to a nonexceptional character, and the double transitivity follows.)

3. **Special elements.** Let (H, Z, χ) be a fully ramified triple, and let K be a normal subgroup of H containing Z. Also, let ξ denote an irreducible constituent of χ_K. Information about the group H/K was obtained in the previous section by considering the possible indices for the inertia group of ξ. In this section, we obtain information about the group K/Z by considering elements of K at which ξ does not vanish (for all possible ξ). Under the right conditions, K/Z will have a proper normal subgroup. The main application of the methods of this section are contained in Corollary 3.6.

The following concept first appears in [5], and a slight variation of it appears in [9].

DEFINITION. Let $N \lhd G$, and let $\psi \in \mathrm{Irr}(N)$ be invariant under G. For every $x, y \in G$ with $[x, y] = x^{-1}y^{-1}xy \in N$, define the complex number $\langle\langle x, y\rangle\rangle$ as follows. Extend ψ to $\hat{\psi}$ on $\langle N, y\rangle$. Now x normalizes the group $\langle N, y\rangle$, and fixes ψ, so $\hat{\psi}^x$ is another extension of ψ. It follows that $\hat{\psi}^x = \lambda\hat{\psi}$, where λ is a linear character of $\langle N, y\rangle$ with N in its kernel. Moreover, λ is uniquely determined, i.e., is independent of the choice of the extension $\hat{\psi}$. Define $\langle\langle x, y\rangle\rangle$ to be $\lambda(y)$.

The definition above of course depends on ψ. Properties of the map $\langle\langle , \rangle\rangle$ may be found in [9]. In particular, $\langle\langle x, y\rangle\rangle$ is multiplicative in x and y whenever it is defined, and $\langle\langle x, y\rangle\rangle = \langle\langle y, x\rangle\rangle^{-1}$ if $\langle\langle x, y\rangle\rangle$ is defined.

We are now ready to define special elements.

DEFINITION. Let $N \lhd G$ and $\psi \in \mathrm{Irr}(N)$, with ψ invariant in G. Let $\langle\langle , \rangle\rangle$ be defined as above. Say that $g \in G$ is *special* if $\langle\langle x, g\rangle\rangle = 1$, for all x satisfying $xN \in C_{G/N}(gN)$.

If g is special, then so is every conjugate of g, and every element the of the coset gN. We may therefore speak of the special classes of G/N. The following theorem and its proof appear in [5].

THEOREM 3.1. *Let $N \lhd G$ and $\psi \in \mathrm{Irr}(N)$ be G-invariant. Define special conjugacy classes of G/N as indicated above. Then, the number of distinct irreducible constituents of ψ^G is the same as the number of special classes of G/N.*

Because of Theorem 1.1, the case that $N \subseteqq Z(G)$, and ψ is a faithful linear character of N, deserves to be singled out. In this case, the computation of $\langle\!\langle x, y\rangle\!\rangle$, for $x, y \in G$ with $[x, y] \in N$, becomes easier to carry out: Let ψ be an extension of ψ to $\langle N, y\rangle$. This is an abelian group, so $\hat{\psi}$ is linear. Moreover, $\hat{\psi}^x = \lambda\hat{\psi}$ for $\lambda \in \mathrm{Irr}(\langle N, y\rangle)$ with $N \subseteqq \ker\lambda$. All characters appearing in this equation are linear, and so we may solve for λ: $\lambda = \hat{\psi}^{-1}\hat{\psi}^x$.

Evaluating at y yields:

$$\begin{aligned}\langle\!\langle x, y\rangle\!\rangle = \lambda(y) &= \hat{\psi}^{-1}(y)\hat{\psi}^x(y) = \hat{\psi}(y^{-1})\hat{\psi}(xyx^{-1}) = \hat{\psi}(y^{-1}xyx^{-1}) \\ &= \hat{\psi}([y, x^{-1}]) = \hat{\psi}([x, y]) = \psi([x, y]) .\end{aligned}$$

Thus $\langle\!\langle x, y\rangle\!\rangle = \psi([x, y])$, for all $x, y \in G$ with $[x, y] \in N$. As ψ is faithful, we may identify $\langle\!\langle\,,\rangle\!\rangle$ with $[\,,]$, defined for all pairs of elements satisfying $[x, y] \in N$. In particular, it is easy to see that $x \in G$ is special if and only if $C_G(x) = C_G(xN \bmod N)$.

The following easy consequence of Theorem 3.1 will be useful later:

COROLLARY 3.2. *Let (H, Z, χ) be a fully ramified triple. Then H/Z contains no self-centralizing cyclic subgroups, unless $H = Z$.*

Proof. As usual, we may assume $Z = Z(H)$, and then we may identify $\langle\!\langle\,,\rangle\!\rangle$ with $[\,,]$. If θ is the unique constituent of χ_Z, then χ is the unique constituent of θ^H. By Theorem 3.1, there is only one special class of H/Z, and this must be the class of the identity element. Suppose $\langle gZ\rangle$ is a self-centralizing subgroup of H/Z. Then $[x, g] \in Z$ implies $x \in \langle Z, g\rangle$, and since this last group is abelian, $[x, g] = 1$. But then g is special, and since $\bar{1}$ is the only special class of H/Z, it follows that $g \in Z$. Hence,

$$H = C_H(g) \subseteqq \langle g, Z\rangle = Z\,, \quad \text{so } H = Z\,.$$

LEMMA 3.3. *Let $\psi \in \mathrm{Irr}(N)$, where $N \lhd G$ and ψ is faithful. Assume $N \subseteqq Z(G)$, so that ψ is invariant in G, and special elements of G are defined. Let χ be a constituent of ψ^G. If g is not special, then $\chi(g) = 0$.*

Proof. For g not special, there exists $x \in G$ with $[x, g] \in N$ and $[x, g] \neq 1$. Since χ is a class function:

$$\chi(g) = \chi(x^{-1}gx) = \chi(gg^{-1}x^{-1}gx) = \chi(g[g, x]) = \chi(g)\psi([g, x]) ,$$

where the last equality follows from the fact that $[g, x]$ is represented as a scalar matrix, with scalar $\psi([g, x])$, in any representation affording χ. But $\psi([g, x]) \neq 1$, as ψ is faithful, so $\chi(g) = 0$.

When $N \subseteq Z(G)$ and ψ is a faithful character of N, the following gives a stronger relation between constituents of ψ^G and the special classes of G/N than is already implied in Theorem 3.1.

THEOREM 3.4. *Let $N \subseteq Z(G)$ and $\psi \in \mathrm{Irr}(N)$, with ψ faithful, and let $\chi_1, \cdots, \chi_m$ be the distinct irreducible constituents of ψ^G. Let $g_1, \cdots, g_m$ be any m elements of G. Then, the matrix $(\chi_i(g_j))$ is nonsingular if and only if $g_1, \cdots, g_m$ represent the m distinct special classes of G/N.*

Proof. (Only if) If $(\chi_i(g_j))$ is nonsingular, then certainly for every j, the jth column is nonzero. By the previous lemma, this means that g_j is special. Let — denote the natural map from G to G/N. We need to check that $\bar{g}_1, \cdots, \bar{g}_m$ lie in distinct conjugacy classes. Suppose $\bar{g}_i$ is conjugate to $\bar{g}_j$. Then $x^{-1}g_ix = ng_j$, for some $x \in G$ and $n \in N$. Then, for every k:

$$\chi_k(g_i) = \chi_k(x^{-1}g_ix) = \chi_k(ng_j) = \psi(n)\chi_k(g_j) ,$$

so that the i-th and j-th columns of the matrix differ by the scalar multiple $\psi(n)$. This can only happen if $i = j$, and we are done with this half of the theorem.

(If) Suppose $g_1, \cdots, g_m$ represent the m distinct special conjugacy classes of G/N. Again let - denote the map $G \to G/N$. Then,

$$\begin{aligned}\delta_{ij} = (\chi_i, \chi_j) &= (1/|G|) \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) \\ &= (1/|G|) \sum_{g \text{ special}} \chi_i(g)\chi_j(g^{-1}) \\ &= (1/|G|) \sum_{\nu=1}^{m} \sum_{g \sim g_\nu} \chi_i(g)\chi_j(g^{-1}) \\ &= (1/|G|) \sum_{\nu=1}^{m} |\bar{G}: C_{\bar{G}}(\bar{g}_\nu)| \cdot |N| \cdot \chi_i(g_\nu)\chi_j(g_\nu^{-1}) .\end{aligned}$$

The third equality follows from Lemma 3.3, and the last follows from the fact that $\chi_i(g)\chi_j(g^{-1})$ is constant on cosets of N. We therefore have:

$$\delta_{ij} = (|N|/|G|) \sum_{\nu=1}^{m} \chi_i(g_\nu) |\bar{G}: C_{\bar{G}}(\bar{g}_\nu)| \chi_j(g_\nu^{-1}) .$$

Writing this last identity in matrix form:

$$I = (|N|/|G|)(\chi_i(g_j)) \operatorname{diag} (|\bar{G}: C_{\bar{G}}(\bar{g}_i)|)(\chi_j(g_i^{-1})) ,$$

where I is the $m \times m$ identity matrix. Hence, $(\chi_i(g_j))$ is nonsingular, and we are done.

THEOREM 3.5. *Let $(H, Z(H), \chi)$ be a fully ramified triple with χ faithful. Let $Z = Z(H) \subseteqq K \lhd H$ and let R be a subgroup of H containing Z with $R/Z \in \mathrm{Syl}_p(H/Z)$. Finally, let θ be unique constituent of χ_Z, and $g_1, \cdots, g_s$ be representatives of the distinct special classes of $(R \cap K)/Z$, computed with respect to θ. Then:*

(a) *The $s|Z|$ elements, zg_i, for $z \in Z$ and $1 \leqq i \leqq s$, are all special in K, and lie in distinct conjugacy classes of K.*

(b) *If $g \in R \cap K$ is a special element of K, then g is special in $R \cap K$. In particular, g is $R \cap K$-conjugate to a unique element of the form zg_i.*

(c) *If $g, h \in R \cap K$ and g is special in $R \cap K$, then $g \sim_K h$ implies $g \sim_{R \cap K} h$.*

REMARK. The above implies that there is a natural correspondence between conjugacy classes of special elements in $R \cap K$ and conjugacy classes of special elements of K which meet $R \cap K$. The correspondence is given by $\mathscr{L} \mapsto \mathscr{L}^K$, where $\mathscr{L}$ is a conjugacy class of $R \cap K$ consisting of special elements, and $\mathscr{L}^K$ is the unique class of K containing $\mathscr{L}$. The inverse is given by $\mathscr{M} \mapsto \mathscr{M} \cap (R \cap K)$, where $\mathscr{M}$ is a class of special elements of K which meets $R \cap K$.

Proof of Theorem 3.5. Following the notation of Lemma 2.3, let $\sigma_1, \cdots, \sigma_s$ and $\tau_1, \cdots, \tau_t$ be the distinct irreducible constituents of $\chi_{R \cap K}$ and χ_K respectively. We know there are s constituents of $\chi_{R \cap K}$ because there are s special classes in $(R \cap K)/Z$. Let $Z[\tau_1, \cdots, \tau_t]$ denote the additive subgroup of the character ring of K generated by $\tau_1, \cdots, \tau_t$, and similarly define $Z[\sigma_1, \cdots, \sigma_s]$. Let r denote the restriction map from $Z[\tau_1, \cdots, \tau_t]$ to $Z[\sigma_1, \cdots, \sigma_s]$. Since $\chi_{R \cap K} = (\chi_R)_{R \cap K}$, it is clear that r maps $Z[\tau_1, \cdots, \tau_t]$ into $Z[\sigma_1, \cdots, \sigma_s]$. Reducing coefficients mod P, we have the following commutative diagram:

$$\begin{array}{ccc} Z[\tau_1, \cdots, \tau_t] & \xrightarrow{r} & Z[\sigma_1, \cdots, \sigma_s] \\ \downarrow & & \downarrow \\ Z_p[\tau_1, \cdots, \tau_t] & \xrightarrow{\bar{r}} & Z_p[\sigma_1, \cdots, \sigma_s] \end{array}$$

Now R acts on $\{\tau_1, \cdots, \tau_t\}$, and because $R \cap K \lhd R$, R acts on

$\{\sigma_1, \cdots, \sigma_s\}$. The group $R \cap K$ is contained in the kernel of both actions, so the p-group $R/(R \cap K)$ acts on both sets. This action may be extended in the natural way to each of the four additive groups above, so that each such group is an $R/(R \cap K)$-module. The second row of groups may be viewed as $Z_p[R/(R \cap K)]$-modules. All maps in the above diagram are $R/(R \cap K)$-homomorphisms. Since $R/(R \cap K)$ is a p-group acting transitively on $\{\sigma_1, \cdots, \sigma_s\}$, the module $Z_p[\sigma_1, \cdots, \sigma_s]$ contains a unique maximal submodule $M = \{\Sigma l_i\sigma_i \mid l_i \in Z_p \text{ and } \Sigma l_i = 0\}$.

As in the (b) part of Lemma 2.3, write

$$r(\tau_1) = (\tau_1)_{R\cap K} = \sum_{j=1}^{s} b_j\sigma_j \ .$$

As $\tau_1(1)_p = \sigma_1(1)$, it follows that $\Sigma b_j \not\equiv 0 \bmod p$. Hence

$$\bar{r}(\tau_1) = \sum_{j=1}^{s} \bar{b}_j\sigma_j \notin M \ ,$$

where $\bar{b}_j$ denotes the residue class of $b_j \bmod p$. Since $\bar{r}$ is a $Z_p[R/(R \cap K)]$-map, it follows that $\bar{r}$ is surjective.

Now define the $t \times s$ matrix $B = (b_{ij})$ as follows:

$$r(\tau_i) = (\tau_i)_{R\cap K} = \sum_{j=1}^{s} b_{ij}\sigma_j \ .$$

Let $\bar{B} = (\bar{b}_{ij})$ be the matrix B with all entries reduced $\bmod p$. Then $\bar{B}$ is the matrix of $\bar{r}$ using the natural bases. Thus $\bar{B}$, and hence B itself, has rank s. Now

$$(\tau_i(g_j)) = B(\sigma_i(g_j)) \ ,$$

where $(\sigma_i(g_j))$ is nonsingular by Theorem 3.4, and B has rank s. Thus, $(\tau_i(g_j))$ has rank s, so that its columns are linearly independent. This means that $g_1, \cdots, g_s$ represent distinct special classes in K/Z.

We now have to check that there is no K-fusion among the elements zg_i. Suppose $zg_i \sim_K z'g_j$. The above implies that $i = j$. Now choose τ_k so that $\tau_k(g_i) \neq 0$. Then

$$\theta(z)\tau_k(g_i) = \tau_k(zg_i) = \tau_k(z'g_i) = \theta(z')\tau_k(g_i) \ ,$$

and so $\theta(z) = \theta(z')$. But θ is faithful because χ is, and so $z = z'$. This proves (a).

Now suppose $g \in R \cap K$ and g is special in K. We have just shown that $g_1, \cdots, g_s$ represent distinct (special) conjugacy classes in K/Z. We may therefore find $\{x_2, \cdots, x_s\} \subseteqq \{g_1, \cdots g_s\}$ such that $g = x_1, x_2, \cdots, x_s$ represent distinct conjugacy classes in K/Z, so that the $t \times s$ matrix $(\tau_i(x_j))$ has rank s, by Theorem 3.4. Now $(\tau_i(x_j)) = B(\sigma_i(x_j))$, and this equation implies that the $s \times s$ matrix $(\sigma_i(x_j))$ is non-singular. By Theorem 3.4 again, $x_1 = g$ is special in $R \cap K$. Hence g is conjugate in $R \cap K$ to some element of the form zg_i. Uniqueness of this element is clear, as these elements are not fused in K even. This proves (b).

Suppose now $g, h \in R \cap K$, g is special in $R \cap K$ and $g \sim_K h$. By (a) above, g is special in K and hence so is h. However, $h \in R \cap K$, so by (b) above, h is special in $R \cap K$. From (b) again, g and h are conjugate in $R \cap K$ to elements of the form zg_i and $z'g_j$ respectively, for some $z, z' \in Z$ and $1 \leqq i, j \leqq s$. Hence $zg_i \sim_K z'g_j$, and from (a) we get $z = z'$, $i = j$. Thus, g and h are fused in $R \cap K$, completing the proof of (c).

As an application of the above non-fusion theorem, we have:

COROLLARY 3.6. *Let (H, Z, χ) be a fully ramified triple. Let $K = O^p(H)Z$, and assume that a Sylow p-subgroup of K/Z is abelian. Then (H, K, χ) is a fully ramified triple, and for ψ the unique constituent of χ_K, the triple (K, Z, ψ) is fully ramified.*

Proof. By applying Theorem 1.1, we may assume $Z = Z(H)$ and that χ is faithful. Let $R/Z \in \mathrm{Syl}_p(H/Z)$, and let $\tau_1, \cdots, \tau_t$ and $\sigma_1, \cdots, \sigma_s$ be as in Lemma 2.3. Then $t = s$ as $|H: K|$ is a power of p. If $t = 1$, this means that (H, K, χ) is a fully ramified triple, and hence so is (K, Z, τ_1), and we are done.

Suppose then $t = s > 1$. Let $N = N_K(R \cap K)$, and let—denote the natural map $K \rightarrow K/Z$. Thus $\bar{N} = N_{\bar{K}}(\overline{R \cap K})$. Since $s > 1$, there is an element $g \in R \cap K$ which is special in $R \cap K$ and $g \notin Z$. If $x \in N$, then $g^x \in R \cap K$, and clearly $g \sim_K g^x$. By Theorem 3.5 (c), g^x is conjugate in $R \cap K$ to g. But $\overline{R \cap K}$ is abelian, so $\bar{g}^{\bar{x}} = \bar{g}$, and this shows:

$$\bar{1} \neq \bar{g} \in \overline{R \cap K} \cap Z(N_{\bar{K}}(\overline{R \cap K})) .$$

However, this implies $O^p(H)Z = O^p(K)Z < K$, (see p. 253 of [6]). Thus, the case $s > 1$ leads to a contradiction, and the corollary is proved.

4. **A solvability theorem.** The final theorem of this section is a solvability theorem for certain groups of type f.r. In order to prove that theorem, it is first necessary to show that certain groups do not occur as homomorphic images of groups of type f.r.

LEMMA 4.1. *Let G be a simple subgroup of A_9* (the alternating group on 9 letters). *Then G is not a homomorphic image of a group of type f.r.*

Proof. Suppose G is such a homomorphic image. Now $|G|$ divides $9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3 = 2^6\cdot 3^4\cdot 5\cdot 7$.

Suppose $5 \mid |G|$. By Theorem 2.5 with $p = 5$, we get $G \leqq A_5$, and so $G \cong A_5$. This contradicts Theorem 2.5 (a), and so $5 \nmid |G|$. By Burnside's $p^a q^b$ theorem (see p. 131 of [6]), $\pi(G) = \{2, 3, 7\}$. Hence, $7 \mid |G|$, and using Theorem 2.5 with $p = 7$, we get $G \leqq A_7$. Thus $|G|$ divides $7\cdot 6\cdot 4\cdot 3 = 2^3\cdot 3^2\cdot 7$, as $5 \nmid |G|$.

Using Theorem 2.5 (a) again, a Sylow 3-subgroup of G cannot be cyclic. We therefore have $|G| = 2^j\cdot 3^2\cdot 7$, for some j. By Burnside's transfer theorem, $P < N_G(P) \leqq N_{A_7}(P)$, where P is a Sylow 7-subgroup of G. This last group has order 21, so $|N_G(P)| = 21$. By Sylow's theorem, $2^j\cdot 3 \equiv 1 \bmod 7$, and this is the final contradiction.

The next fact which is needed is a purely number theoretic statement, due to G. D. Birkhoff and H. S. Vandiver, which first appeared about the turn of the century.

LEMMA 4.2. *Let a and n be integers both greater than one. Then, except for the following two cases, there exists a prime divisor p of $(a^n - 1)$, satisfying $p \nmid (a^m - 1)$ for all m with $1 \leqq m < n$:*

(I) *$n = 2$ and a is a Mersenne number,* i.e. *$a + 1$ is a power of 2.*

(II) *$n = 6$ and $a = 2$.*

A proof of the above lemma for $n \geqq 3$ may be found in [1], where, in fact, a more general version is given. Of course, the case $n = 2$ is a triviality.

The above lemma is extremely useful, when used in conjunction with Theorem 2.5, in eliminating known simple groups from occuring as factor groups of groups of type f.r. However, in this section, we shall only need the following:

LEMMA 4.3. *Let* $\mathrm{PSL}(2, p^n) \leqq X \leqq \mathrm{P\Gamma L}(2, p^n)$, *where p is a prime, and $p^n \geqq 4$. Then X is not the homomorphic image of any group of type f.r.*

Proof. We first note that $\mathrm{PSL}(2, p^n)$ can have no subgroup of index $q^a \neq 1$, where q^a is a prime power less than p^n. This is true because $\mathrm{PSL}(2, p^n)$ contains a proper subgroup of index $m < p^n$, only in the case $p^n = 9$ and $m = 6$ (see p. 214 of [8]). This proves the statement, as 6 is not a prime power.

Suppose X is a homomorphic image of H/Z, where (H, Z, χ) is a fully ramified triple. Let K be the kernel of this homomorphism, and S the inverse image of $\mathrm{PSL}(2, p^n)$. Then, $Z \subseteqq K \subseteqq S \subseteqq H$, where K and S are normal in H, $H/K \cong X$ and $S/K \cong \mathrm{PSL}(2, p^n)$. Clearly, $|H: S|$ divides $2n$, as $|\mathrm{P\Gamma L}(2, p^n): \mathrm{PSL}(2, p^n)| = n(2, p - 1)$.

Suppose there exists a prime q satisfying the following conditions:
(i) $q \mid |\mathrm{PSL}(2, p^n)|$
(ii) $q \neq p$
(iii) q is odd
(iv) $1 + p^n$ is not a power of q
(v) $q \nmid n$.
Then q divides exactly one of $(p^n + 1)$ or $(p^n - 1)$, as q is odd, and a Sylow q-subgroup of S/K is cyclic of order $< p^n$ by (iv). By (v), a Sylow q-subgroup of S/K is also one for H/K, implying that H/K has a q-complement, by Theorem 2.5. But then S/K also has a q-complement, contradicting the first paragraph.

We now prove, under the hypothesis $p^n \geqq 4$, a prime q can always be chosen satisfying (i)–(v) above.

Suppose q is an odd prime dividing $p^n - 1$, but not dividing $p^m - 1$ for any $m < n$ (if $n = 1$, this last condition is vacuously true). Clearly, q satisfies (i)–(iv) above. Now q divides $p^{q-1} - 1$, forcing $n \leqq q - 1$, so that q also satisfies (v). In particular, we are done if $n = 1$, unless $p - 1$ is a power of 2. If $n > 1$, then Lemma 4.2 is applicable (with p in place of a), and any prime satisfying the conclusion of that lemma also satisfies (i)–(v) above. This brings us to one of the following cases:
(a) $n = 1$ and $p - 1$ is a power of 2
(b) $n = 2$ and $p + 1$ is a power of 2
(c) $n = 6$ and $p = 2$.
We consider these cases in turn.

Case (a). Since $p^n \geqq 4$, it follows that $p + 1$ is even, and is not a power of 2. Any odd prime divisor of $p + 1$ satisfies (i)–(v) above, and we are done in this case.

Case (b). Since $p^2 + 1$ is twice an odd number, in this case, let q be an odd prime divisor of $p^2 + 1$. Again, it is readily checked that q satisfies (i)–(v) above.

Case (c). Here $|\mathrm{PSL}(2, p^n)| = 65 \cdot 64 \cdot 63$, and the prime $q = 5$ satisfies the five conditions above.

Let (H, Z, χ) be a fully ramified triple, and assume that H/Z has an abelian Sylow p-subgroup for some prime p. We saw in the previous section (Corollary 3.6) that $(H, O^p(H)Z, \chi)$ is also a fully ramified triple. This suggests the following definition:

DEFINITION. Let Q be a p-group of type f.r. Say that Q is *reductive* if, for every fully famified triple (H, Z, χ) with Q isomorphic to a Sylow p-subgroup of H/Z, the triple $(H, O^p(H)Z, \chi)$ is fully ramified.

By the remarks preceeding the definition, an abelian p-group of type f.r. is reductive. In the following lemma, we extend slightly the class of reductive p-groups of type f.r. The author is unaware of an example of p-group of type f.r. which fails to have this property. We use the classification of groups with dihedral Sylow 2-subgroups in the case $p = 2$ of the following.

LEMMA 4.4. *Let Q be a p-group of order p^4 and of type f.r. Then Q is reductive.*

Proof. Suppose Q is a p-group of order p^4 which is of type f.r., but which is not reductive. Then, there exists a fully ramified triple (H, Z, χ) with a Sylow p-subgroup of H/Z isomorphic to Q, such that the triple $(H, O^p(H)Z, \chi)$ is not fully ramified. By Theorem 1.1, we may assume $Z = Z(H)$. Let $K = O^p(H)Z$ and let $R/Z \in \mathrm{Syl}_p(H/Z)$. Now $K < H$ as (H, K, χ) is not a fully ramified triple. By Corollary 3.6, $(R \cap K)/Z$ is a non-abelian p-group, and so has order $\geqq p^3$. This forces $|(R \cap K)/Z| = p^3$ and $|H\colon K| = p$. Let $C/Z = ((R \cap K)/Z)' = Z((R \cap K)/Z)$. Using Lemma 2.3 and Theorem 3.1, there are p special classes of $(R \cap K)/Z$. Suppose that some element, say g, of $C - Z$ is special. As gZ is central in $(R \cap K)/Z$, we get $[g, R \cap K] \subseteqq Z$. But g is special, and this means $[g, R \cap K] = 1$, so $g \in Z(R \cap K)$. It is clear that $1, g, g^2, \cdots, g^{p-1}$ represent the p distinct special classes in $(R \cap K)/Z$. Let $x \in R \cap K - C$. Then x is not special. However, $C_{(R\cap K)/Z}(xZ) = \langle xZ, C/Z \rangle \subseteqq C_{R\cap K}(x)/Z \subseteqq C_{(R\cap K)/Z}(xZ)$. This contradicts the fact that x is not special, and proves that the only special element of $(R \cap K)/Z$ which lies in C/Z is the identity.

Consider now the case that p is odd. Let $N = N_K(R \cap K)$ so that $\bar{N} = N/Z = N_{\bar{K}}(\overline{R \cap K})$. As p is odd, $(R \cap K)/Z$ is a regular p-group, being of class 2. It follows from the Hall-Wielandt theorem (see p. 447 of [8]), that $\bar{N}$ controls p-transfer, i.e., $O^p(\bar{K}) \cap \bar{N} = O^p(\bar{N})$. As $O^p(K/Z) = K/Z$, we will obtain a contradiction by proving that $O^p(\bar{N}) < \bar{N}$.

Let V denote the transfer homomorphism from N/Z into $(R \cap K)/C$. The map V is computed by

$$V(gZ) = \prod_{t \in T} (tgt^{-1})C\,, \qquad \text{for } g \in R \cap K\,,$$

where T is a right transversal for $R \cap K$ in N. (We used the fact that $R \cap K \lhd N$.) Now let g be any special element of $R \cap K$ with $g \notin Z$. Thus $g \notin C$, from above. For any $t \in N$, tgt^{-1} is $R \cap K$-conjugate to g, by the last part of Theorem 3.5. Thus, $tgt^{-1}C = gC$ for all $t \in$

T, and $V(gZ) = (gC)^{|T|} = g^{|N:R\cap K|}C$. As $|N: R \cap K|$ is prime to p, we have $\bar{g} \in \bar{N}$-ker V. This yields the contradiction $O^p(\bar{N}) < \bar{N}$, and we are done if p is odd.

Suppose now $p = 2$. Then $(R \cap K)/Z$ is non-abelian of order 8, and a nonidentity special elemet, say gZ, of $(R \cap K)/Z$ does not lie in C/Z.

Consider first the case that $(R \cap K)/Z$ is the quaternion group. Again, if $N = N_K(R \cap K)$, the element gZ can only be conjugate to $g^{-1}Z$ in N/Z. This implies $\bar{N}/C_{\bar{K}}(\overline{R \cap K})$ is a 2-group. Clearly, $N_{\bar{K}}(\bar{S})/C_{\bar{K}}(\bar{S})$ is a 2-group for all $\bar{S} < \overline{R \cap K}$, as $\bar{S}$ is cyclic. Thus, $\bar{K}$ has a normal 2-complement by Frobenius' theorem (see p. 253 of [6]). This contradicts $O^2(K)Z = K$, forcing $(R \cap K)/Z$ to be the dihedral group of order 8.

From the classification of groups with dihedral Sylow 2-subgroups, and the fact that $O^2(K/Z) = K/Z$, it follows that K/Z has a factor group isomorphic to Y, where $\mathrm{PSL}(2, p^n) \leqq Y \leqq \mathrm{P\Gamma L}(2, p^n)$ for some odd prime power $p^n \neq 3$, or $Y = A_7$. From this, it follows that K/Z has exactly one chief factor isomorphic to the simple group S, where $S = \mathrm{PSL}(2, p^n)$, or $S = A_7$. Therefore, H/Z has a chief factor isomorphic to S. Let $Z \subseteqq V \subsetneqq U \subseteqq H$, with V and U normal in H, and $U/V \cong S$. Define C by the equation: $C/V = C_{H/V}(U/V)$. Then $C \lhd H$, and $C \cap U = V$. Replacing U and V by UC and C respectively, and continuing this process, if necessary, we may assume $C = V$. The factor group H/V is isomorphic to a group X, which satisfies: $S \leqq X \leqq \mathrm{Aut}(S)$. Since $\mathrm{Aut}(\mathrm{PSL}(2, p^n)) \cong \mathrm{P\Gamma L}(2, p^n)$, Lemma 4.3 forces $S = A_7$. Now, $\mathrm{Aut}(A_7) \cong S_7$, the symmetric group on 7 letters, so that H/Z has a factor group which is either A_7 or S_7. Both of these groups contain a cyclic Sylow 5-subgroup of order 5, but neither group contains a subgroup of index 5. This contradiction to Theorem 2.5 completes the proof of the lemma.

We are now ready to give an application of the above. In [12], G is shown to be solvable if G is of type f.r., and $p^3 \nmid |G|$ for any prime p dividing $|G|$.

THEOREM 4.5. *Let G be a group of type f.r. Assume that G has an abelian Sylow p-subgroup for every prime p satisfying $p^6 \,||\, |G|$. Then G is solvable.*

Proof. Let (H, Z, χ) be a fully ramified triple with $G \cong H/Z$. We proceed by induction on $|H/Z|$, the assertion being trivial if $H = Z$. Suppose that H/Z is not perfect. Then $O^p(H)Z < H$ for some prime p. The hypothesis of the theorem, together with the previous lemma, imply that a Sylow p-subgroup of H/Z is reductive of type f.r.

Therefore, $H/O^p(H)Z$ and $O^p(H)Z/Z$ are of type f.r., and we are done by induction.

Suppose then H/Z is perfect, and let K/Z be a maximal normal subgroup. Then H/K is a non-abelian simple group, and hence has even order by [4]. Also, a Sylow 2-subgroup S of H/K has order $\geqq 4$, as otherwise H/K would have a normal 2-complement.

Suppose S has order 4. Then $H/K \cong \mathrm{PSL}(2, q)$, where q is an odd prime power. But these simple groups are eliminated as possible homomorphic images of H/Z by Lemma 4.3. If $|S| = 8$, then apply Lemma 2.3 for the prime 2. Here $d^2s = 8$, and so $s = 2$ or 8. By the (c) part of that lemma, H/K has a subgroup of index s, which implies $s = 8$. However, this possibility is ruled out by Lemma 4.1. Thus, $|S| \geqq 16$.

If S is non-abelian, then the hypotheses of the theorem imply that $|S| = 16$, and S is isomorphic to a Sylow 2-subgroup of G. By Lemma 2.2, S is of type f.r. However, the only non-abelian groups of order 16 that occur as Sylow 2-subgroups of simple groups are dihedral and semi-dihedral. These types have cyclic self-centralizing subgroups, and by Corollary 3.2, S can have no such subgroup. Therefore, S must be abelian and $|S| \geqq 16$. By Walter's Theorem [14], $H/K \cong \mathrm{PSL}(2, |S|)$. This contradicts Lemma 4.3, and establishes the theorem.

It is possible to show that no known simple group can be a factor group of a group of type f.r. This strongly suggests that a group of type f.r. cannot be perfect. It would be desirable to have a proof of this fact, since it would represent a major step in proving that groups of type f.r. are solvable.

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