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# COMPACT HANKEL OPERATORS AND THE F. AND M. RIESZ THEOREM

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**The F. and M. Riesz theorem asserts that every complex Borel measure on the unit circle whose Fourier coefficients with negative index vanish is necessarily absolutely continuous with respect to Lebesgue measure.**

**The purpose of this note is to give a new proof of Hartman's theorem on compact Hankel operators which clarifies the general context of the theorem. The proof depends only on a few simple operator-theoretic results, Nehari's characterization of bounded Hankel operators, and the aforementioned theorem of F. and M. Riesz.**

A Hankel operator on the Hardy space $\mathscr{H}^2$ of functions on the unit circle is an operator of the form

$$H_\phi : f(e^{it}) \to P_+\phi(e^{-it})f(e^{-it})$$

where $\phi$ is a fixed function in $\mathscr{L}^\infty$ and $P_+$ is the orthogonal projection of $\mathscr{L}^2$ onto $\mathscr{H}^2$. P. Hartman proved in 1958 that if $H_\phi$ above is compact, then $\phi$ can be chosen to be continuous [2].

Let $S$ be the usual unilateral shift on $\mathscr{H}^2$, $S : f(e^{it}) \to e^{it}f(e^{it})$. Nehari had proved earlier that every bounded solution to the operator equation $S^*H = HS$ is of the form $H = H_\phi$, where $\phi \in \mathscr{L}^\infty$, and had further shown that $\phi$ could be chosen so that $\|H\| = \|\phi\|_\infty$ [3].

R. N. Hevener [3] and D. Sarason [5] have previously used the Riesz theorem in obtaining Hartman's characterization of compact Hankel operators. Their procedure involves the factorization of analytic functions. T. L. Kriete motivated this paper by suggesting that Lemma 1 below might be used to make more precise the relationship between compact Hankel operators and the F. and M. Riesz theorem.

Let $\mathscr{C}$ denote the space of complex-valued continuous functions on the unit circle, and $\mathscr{A}_0$ the subspace of $\mathscr{C}$ consisting of those functions which extend continuously to an analytic function on the unit disk vanishing at the origin. With the F. and M. Riesz theorem and the Riesz representation theorem for bounded linear functionals on $\mathscr{C}$, one easily shows that the Hardy space $\mathscr{H}^1$ is the dual space of $\mathscr{C}/\mathscr{A}_0$. Let $\mathscr{H}_0^\infty$ denote the class of $\mathscr{H}^\infty$ functions which vanish at zero. Since $\mathscr{L}^\infty$ is the dual of $\mathscr{L}^1$, $\mathscr{L}^\infty/\mathscr{H}_0^\infty$ is the dual of $\mathscr{H}^1$. (This is the case because $\mathscr{H}_0^\infty$ is the annihilator of $\mathscr{H}^1$.)

Now let $\Psi$ be the mapping from $\mathscr{L}^\infty$ to the space $\mathscr{H}_{\mathscr{B}}$ of bounded Hankel operators on $\mathscr{H}^2$ given by $\Psi : \phi \to H_\phi$. It is elementary to see that $\ker \Psi = \mathscr{H}_0^\infty$. Thus the Nehari theorem states that the mapping $\Psi^\sim : \mathscr{L}^\infty/\mathscr{H}_0^\infty \to \mathscr{H}_{\mathscr{B}}$ given by $\phi + \mathscr{H}_0^\infty \to H_\phi$ is an isomorphism of Banach spaces.

The converse of Hartman's theorem is easy, that is $H_\phi$ is compact if $\phi$ is continuous. The reason is that if $\phi$ is continuous, then $\phi$ is a uniform limit of trigonometric polynomials $\{\phi_n\}$, in which case $H_{\phi_n} \to H_\phi$ in operator norm. Compactness of $H_\phi$ now follows from the observation that each $H_{\phi_n}$ is of finite rank.

We now let $\theta$ denote the restriction of $\Psi$ to $\mathscr{C}$. Since $H_\phi$ is compact when $\phi$ is continuous, $\theta$ maps $\mathscr{C}$ into $\mathscr{H}_{\mathscr{C}}$, the space of compact Hankel operators. Also $\ker \theta = \mathscr{A}_0$, so $\theta^\sim : \mathscr{L}/\mathscr{A}_0 \to \mathscr{H}_{\mathscr{S}}$ defined by $\theta^\sim : \phi + \mathscr{A}_0 \to H_\phi$ is one-to-one. Furthermore, if $\phi \in \mathscr{C}$, the norm of the coset $\phi + \mathscr{H}_0^\infty$ in $\mathscr{L}^\infty/\mathscr{H}_0^\infty$ is the same as the norm of the operator $H_\phi$. (This is the Nehari theorem.) But since the natural mapping of $\mathscr{C}/\mathscr{A}_0$ into $\mathscr{L}^\infty/\mathscr{H}_0^\infty$ is an isometry, $\|H_\phi\|$ is equal to the norm of $\phi + \mathscr{A}_0$ in $\mathscr{C}/\mathscr{A}_0$. Thus the mapping $\theta^\sim$ is an isometry mapping $\mathscr{C}/\mathscr{A}_0$ into $\mathscr{H}_{\mathscr{C}}$. We now have the following commutative diagram:

canonical embedding inclusion

$$\begin{array}{ccc} \mathscr{C}/\mathscr{A}_0 & \longrightarrow & \mathscr{L}^\infty/\mathscr{H}_0^\infty \\ \theta^\sim \downarrow & & \downarrow \Psi^\sim \\ \mathscr{H}_{\mathscr{C}} & \longrightarrow & \mathscr{H}_{\mathscr{B}} \end{array}$$

The mapping $\Psi^\sim$ is an isomorphism. From the two lemmas below it now follows that the range of $\theta^\sim$ is $\mathscr{H}_{\mathscr{C}}$, and this is precisely Hartman's result.

LEMMA 1. *The Banach space $\mathscr{H}_{\mathscr{B}}$ is isomorphic to the second dual space of $\mathscr{H}_{\mathscr{C}}$ in such a way that the inclusion mapping $\mathscr{H}_{\mathscr{C}} \to \mathscr{H}_{\mathscr{B}}$ corresponds to the canonical embedding of a normed space in its second dual.*

*Proof.* This follows from an abstract result due to Gellar and Page **[1]**. Primarily, it is just a matter of checking that every Hankel operator is the limit in the weak operator topology of a sequence of compact Hankel operators. The important point is that this can be checked *without* prior knowledge of which Hankel operators are compact. See Theorems 1 and 2 of **[1]**.

LEMMA 2. *Let $\mathscr{X}$ and $\mathscr{Y}$ be normed linear spaces, and let $A:\mathscr{X}\to\mathscr{Y}$ and $B:\mathscr{X}^{**}\to\mathscr{Y}^{**}$ be linear isometries such that the diagram:*

canonical embedding    canonical embedding

$$\begin{array}{ccc} \mathscr{X} & \longrightarrow & \mathscr{X}^{**} \\ A\Big\downarrow & & \Big\downarrow \\ \mathscr{Y} & \longrightarrow & \mathscr{Y}^{**} \end{array}$$

*is commutative.*

*If the range of $B$ is $\mathscr{Y}^{**}$, then the range of $A$ is $\mathscr{Y}$.*

*Proof.* This is a simple consequence of the Hahn-Banach Theorem and the fact that a normed linear space is weak-star dense in its second dual.

## REFERENCES

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