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ABELIAN GROUPS, A, SUCH THAT HOM $(A,\text{-})$ PRESERVES DIRECT SUMS OF COPIES OF A

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An R-module, A, is *self-small* if Hom$(A,\text{-})$ preserves direct sums of copies of A. Various conditions on the endomorphism ring of a module which guarantee that it is self-small are studied. Various results are proved about subgroups of direct sums or direct products of copies of a self-small abelian group A, which generalize results previously known when A is torsion free of rank one.

0. Introduction. An R-module, A, is *self-small* if $\mathrm{Hom}_R(A,\text{-})$ preserves direct sums of copies of A. Homological arguments show that if A is a self-small R-module with R a commutative ring with 1, then the category of direct summands of direct sums of copies of A is equivalent to the category of projective right $\mathrm{End}_R(A)$-modules ($\mathrm{End}_R(A)$ is the R-endomorphism ring of A). Consequently, direct sum decompositions of direct sums of copies of A may be interpreted in terms of direct sum decompositions of free $\mathrm{End}_R(A)$-modules.

An R-module, A, is self-small in the following cases: (a) A is *small* (i.e., $\mathrm{Hom}_R(A,\text{-})$ preserves arbitrary direct sums of R-modules); (b) $A = \Pi_{i \in I} A_i$, where each A_i is a self-small R-module and $\mathrm{Hom}_R(A_i, A_j) = 0$ if $i \neq j$; (c) $\mathrm{End}_R(A)$ is countable.

If the finite topology on $\mathrm{End}_R(A)$ is discrete, then A is self-small. In certain cases, the converse is true.

COROLLARY I. *Suppose that A is a countably generated R-module. Then A is self-small iff the finite topology on $\mathrm{End}_R(A)$ is discrete. If R is countable, then A is self-small iff* $\mathrm{End}_R(A)$ *is countable.*

A left ideal, I, of $\mathrm{End}_R(A)$ is an *annihilator ideal* if $I = \{f \in \mathrm{End}_R(A) : f(x) = 0 \text{ for all } x \in A \text{ with } Ix = 0\}$.

PROPOSITION II. *Suppose that A is an R-module and that* $\mathrm{End}_R(A)$ *has the minimum condition on left annihilator ideals. Then the finite topology on* $\mathrm{End}_R(A)$ *is discrete and A is the finite direct sum of indecomposable R-modules.*

The remainder of the paper is devoted to self-small abelian groups (although many of the arguments are valid in a more general setting). Self-small torsion abelian groups are finite. Section 3 and examples in §5 demonstrate that self-small torsion free abelian groups are both profuse and diverse. Self-small mixed abelian groups with finite torsion free rank are characterized by Proposition 3.6.

Generalizations of homogeneous separable and completely decomposable torsion free abelian groups are considered in §4. We demonstrate that many of the classical properties of these groups may be viewed as consequences of the fact that endomorphism rings of rank 1 torsion free abelian groups are principal ideal domains.

Let G and A be abelian groups. Define G to be *A-free* if G is isomorphic to a direct sum of copies of A; *A-projective* if G is a summand of an A-free group; and *locally A-projective* [*locally A-free*] if every finite subset of G is contained in an A-projective [A-free] summand of G. An R-module, M, is *locally projective* [*locally free*] if every finite subset of M is contained in an R-projective [R-free] summand of M. Note that if A is torsion free of rank 1, then the class of locally A-projective [A-projective] groups coincides with the class of homogeneous separable [completely decomposable] groups (with type = type of A). Define $S_A(G)$ to be the subgroup of G generated by $\{f(A) | f \in \mathrm{Hom}(A, G)\}$.

THEOREM III. *Suppose that A is an abelian group and that* $\mathrm{End}(A)$ *is discrete in the finite topology. Then the category of locally A-projective abelian groups is equivalent to the category of locally projective right* $\mathrm{End}(A)$*-modules.*

Results of Chase **[3]** and Theorem III suffice to prove:

COROLLARY IV. *Let A be a torsion free abelian group such that* $\mathrm{End}(A)$ *is a principal ideal domain and $A/\mathrm{im}\ f$ is torsion for all* $0 \neq f \in \mathrm{End}(A)$.

(a) *If B is a pure subgroup of an A-free group, G, such that $S_A(B) = B$ and if* $\mathrm{End}(B)$ *is discrete in the finite topology, then B is an A-free summand of G.*

(b) *A group, G, is locally A-free iff $S_A(G) = G$ and G is isomorphic to a pure subgroup of $S_A(\Pi_{i\in I} A_i)$, where $A_i \cong A$ for all $i \in I$.*

(c) *If B is a pure subgroup of a locally A-free group, G, and if $S_A(B) = B$, then B is locally A-free. Moreover, if* $\mathrm{End}(B)$ *is discrete in the finite topology, then B is an A-free summand of G.*

(d) *Countable locally A-free groups are A-free.*

Note that if A is torsion free of rank 1 and type τ, then $\mathrm{End}(A)$ is a principal ideal domain and $S_A(G) = G(\tau)$. Thus Corollary IV includes the classical properties of homogeneous separable groups as a special case (see Fuchs **[7]**).

Fundamental references, for this paper, are Fuchs **[6]** and **[7]**.

1. Self-small modules. Let $\Sigma_{i\in I} \oplus A_i$ be a direct sum of copies of an R-module A. There is a natural monomorphism $e_I: \Sigma_{i\in I} \oplus \mathrm{Hom}_R(A, A_i) \rightarrow \mathrm{Hom}_R(A, \Sigma_{i\in I} \oplus A_i)$ induced by projection maps. Consequently, A is self-small iff for every countable index set I

and every R-homomorphism $\phi: A \to \Sigma_{i\in I} \oplus A_i$, there is a finite subset, I', of I with $\phi(A) \subseteq \Sigma_{i\in I'} \oplus A_i$. Every finitely generated R-module is self-small. On the other hand, if A is an infinite direct sum of R-modules, then A is not self-small.

Observe that endomorphic images (in particular, direct summands) of self-small R-modules are self-small.

Let X and Y be nonempty subsets of A and $\mathrm{End}_R(A)$, respectively. Define $X^* = \{f \in \mathrm{End}_R(A) | f(X) = 0\}$ and $Y^* = \{x \in A | f(x) = 0 \text{ for all } f \in Y\}$. A left ideal, I, of $\mathrm{End}_R(A)$ is an *annihilator ideal* if $I = I^{**}$. An R-submodule, B, of A is a *kernel submodule* if $B = B^{**}$. One can easily verify that X^* is a left ideal of $\mathrm{End}_R(A)$; $Y^* = \cap_{f\in Y} \ker f$ is a submodule of A; $X \subseteq X^{**}$; $Y \subseteq Y^{**}$; a left ideal, I, of $\mathrm{End}_R(A)$ is an annihilator ideal iff $I = X^*$ for some $X \subseteq A$; and a submodule, B, of A is a kernel submodule iff $B = Y^*$ for some $Y \subseteq \mathrm{End}_R(A)$.

LEMMA 1.1. *There is a* 1-1 *order inverting correspondence between the kernel submodules of an R-module, A, and the left annihilator ideals of* $\mathrm{End}_R(A)$.

Proof. The correspondence is given by $B \to B^*$ and $I \to I^*$.

PROPOSITION 1.1. *The following statements are equivalent for an R-module, A*:

(a) *A is not self-small*;

(b) *there is a chain* $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ *of* (proper) *submodules of A such that* $A = \cup_{n=1}^{\infty} A_n$ *and* $A_n^* \neq 0$ *for all* n;

(c) *there is a chain* $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ *of proper kernel submodules of A such that* $A = \cup_{n=1}^{\infty} A_n$;

(d) *there is a chain* $I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ *of nonzero left annihilator ideals of* $\mathrm{End}_R(A)$ *such that* $A = \cup_{n=1}^{\infty} I_n^*$. *In this case*, $\cap_{n=1}^{\infty} I_n = 0$;

(e) *there is an infinite subset, S, of* $\mathrm{End}_R(A)$ *such that* $S \setminus (X^* \cap S)$ *is finite for all finite subsets, X, of A.*

Proof. (a) $\Rightarrow$ (b) Since A is not self-small, there is an R-homomorphism $\phi: A \to \Sigma_{i=1}^{\infty} \oplus B_i$ such that $\Pi_i\phi \neq 0$ for all i, where $\Pi_j: \Sigma \oplus B_i \to B_j = A$ is the projection map for all j. For $n \geqq 1$ let $A_n = \{x \in A | \Pi_i\phi(x) = 0 \text{ for } i > n\}$ so that $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ is a chain of proper submodules of A with $A = \cup_{n=1}^{\infty} A_n$. Clearly, $A_n^* \neq 0$ for all n.

(b) $\Rightarrow$ (c) Replace each A_n by A_n^{**}, thereby obtaining an ascending chain of kernel submodules with $A = \cup_{n=1}^{\infty}(A_n^{**})$ (since $A_n \subseteq A_n^{**}$). Each A_n^{**} is proper, for if $A_n^{**} = A$, then $A_n^* = A^* = 0$, an impossibility by (b).

(c) $\Rightarrow$ (d) A consequence of Lemma 1.1.

(d) $\Rightarrow$ (e) For each n, choose $f_n \in I_n \setminus I_{n+1}$ if $I_n \neq I_{n+1}$ and let $f_n = 0$ if $I_n = I_{n+1}$. Then $S = \{f_n | n = 1, 2, \cdots\}$ is an infinite subset of $\mathrm{End}_R(A)$

(since $\cap_{n=1}^{\infty} I_n = 0$ and each I_n is nonzero). If X is a finite subset of A, then $X \subseteq I_n^*$ for some n (since $A = \cup_{n=1}^{\infty} I_n^*$). Consequently, $I_m = I_m^{**} \subseteq X^*$ for all $m \geqq n$ so that $S \backslash (X^* \cap S) \subseteq \{f_1, \cdots, f_{n-1}\}$, a finite set.

(e) $\Rightarrow$ (a) Define an R-homomorphism $\phi: A \to \Sigma_{f \in S} \oplus A_f$ by $\phi(a) = \Sigma_{f \in S} f(a)$ where $f(a) \in A_f = A$. The hypotheses guarantee that ϕ is well defined. Since S is infinite, $\phi(A) \not\subseteq \Sigma_{f \in S'} \oplus A_f$ for all finite subsets, S', of S. Thus A is not self-small.

REMARK. In the language of Szele **[13]**, (e) may be restated as: $\mathrm{End}_R(A)$ has an infinite 0-system (equivalently, $\mathrm{End}_R(A)$ has an infinite summable system.

COROLLARY 1.3. *Let $\{B_i\}_{i \in I}$ be a family of self-small R-modules with* $\mathrm{Hom}_R(B_i, B_j) = 0$ *for $i \neq j$. Then $A = \Pi_{i \in I} B_i$ is self-small.*

Proof. Assume that A is not self-small and choose a chain $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of proper kernel submodules of A with $A = \cup_{n=1}^{\infty} A_n$. For each $i \in I$, there is a least integer $n(i)$ with $B_i \subseteq A_{n(i)}$ (otherwise, B_i is not self-small by Proposition 1.2.b). If $\{n(i) | i \in I\}$ is bounded, say by m, then for any $0 \neq f \in A_m^*$ ($A_m^* \neq 0$ since $A_m = A_m^{**} \neq A$) it follows that $f(A) = 0$, a contradiction. If $\{n(i) | i \in I\}$ is not bounded, then there is some $a = (b_i) \in A$ with $a \not\in A_m$ for all m, a contradiction (since $A = \cup_{n=1}^{\infty} A_n$).

COROLLARY 1.4. *If A is an R-module such that* $\mathrm{End}_R(A)$ *is countable, then A is self-small.*

Proof. Assume that A is not self-small. By Proposition 1.2.d there is a chain $I_1 \supset \cdots \supset I_n \supset \cdots$ of nonzero left annihilator ideals of $\mathrm{End}_R(A)$ such that $A = \cup_{n=1}^{\infty} I_n^*$ (choose an appropriate subchain to guarantee that $I_n \neq I_{n+1}$ for all n). For each n, choose $g_n \in I_n \setminus I_{n+1}$; let $S = \{g_n\}$; and for each subset, L, of Z^+ (the positive integers) define $g_n^L = g_n$ if $n \in L$ and $g_n^L = 0$ otherwise. Define $g^L \in \mathrm{End}_R(A)$ by $g^L(a) = \Sigma g_i^L(a)$, a well defined homomorphism (since for each $a \in A$, $g_i^L(a) = 0$ for almost all i).

We prove that $g^L = g^K$ iff $K = L$; in which case, $\mathrm{End}_R(A)$ is uncountable, a contradiction. Assume that $K \neq L$ and choose a least integer n with $g_n^L \neq g_n^K$, say $g_n^L = g_n$ and $g_n^K = 0$. Since $g_n \not\in I_{n+1}$, there is some $a \in I_{n+1}^*$ such that $a \not\in \ker g_n$. Thus $g^L(a) - g^K(a) = g_n(a) \neq 0$ and $g^K \neq g^L$.

2. The Finite Topology on $\mathrm{End}_R(A)$.

The *finite topology* on $\mathrm{End}_R(A)$ is defined by letting $\{X^* | X \text{ is a finite subset of } A\}$ be a basis of open neighborhoods of 0. It is known that $\mathrm{End}_R(A)$ is a complete Hausdorff topological ring in the finite topology (see Fuchs **[7]**, p. 221, for the case that $R = Z$; the general argument is similar). The finite topology on $\mathrm{End}_R(A)$ is discrete (i.e., $X^* = 0$ for some finite subset of A) iff $B^* = 0$ for some finitely generated submodule, B, of A.

COROLLARY 2.1. *If the finite topology on* $\mathrm{End}_R(A)$ *is discrete, then* A *is self-small.*

Proof. If A is not self-small, then Proposition 1.2.b leads to a contradiction.

PROPOSITION 2.2. *Suppose that A is an R-module such that the finite topology on* $\mathrm{End}_R(A)$ *is either first countable or locally compact. Then A is self-small iff the finite topology on* $\mathrm{End}_R(A)$ *is discrete.*

Proof. ($\Leftarrow$) Corollary 2.1.

($\Rightarrow$) Assume that $\mathrm{End}_R(A)$ is first countable. Then there is a set $\{X_n \mid n = 1, 2, \cdots\}$ of finite subsets of A such that $X_1^* \supseteq X_2^* \supseteq \cdots$; $\bigcap_{n=1}^{\infty} X_n^* = 0$; and if V is a neighborhood of 0, then $X_n^* \subseteq V$ for some n. If $x \in A$, then $X_n^* \subseteq \{x\}^*$ for some n, so that $x \in X_n^{**}$. Consequently, $A = \bigcup_{n=1}^{\infty} (X_n^*)^*$. By Proposition 1.2.d. and the preceding remarks, $X_n^* = 0$ for some n, i.e., $\mathrm{End}_R(A)$ is discrete.

Assume that $\mathrm{End}_R(A)$ is locally compact. Then there is a compact ideal neighborhood, I, of 0. Since $\mathrm{End}_R(A)$ is Hausdorff, I is both closed and complete. But $I \setminus (X^* \cap I)$ is discrete (since X^* is open) and compact, thus finite, for all finite subsets, X, of A. Since A is self-small, I must be finite (Proposition 1.2.e.). But I is a neighborhood of 0 and $\mathrm{End}_R(A)$ is Hausdorff, so $X^* = 0$ for some finite subset, X, of A.

COROLLARY 2.3. *Suppose that A is a countably generated R-module.*

(a) *A is self-small iff the finite topology on* $\mathrm{End}_R(A)$ *is discrete.*

(b) *If R is countable, then A is self-small iff* $\mathrm{End}_R(A)$ *is countable.*

Proof. (a) Let S be a countable set of generators for A. Then $\{X^* \mid X \text{ is a finite subset of } S\}$ is a countable neighborhood basis of 0; i.e., $\mathrm{End}_R(A)$ is first countable. Now apply Proposition 2.2.

(b) In view of Corollary 1.4, it suffices to prove that if R is countable and A is self-small, then $\mathrm{End}_R(A)$ is countable. By (a), $\mathrm{End}_R(A)$ is discrete, i.e., $X^* = 0$ for some finite subset $X = \{x_1, \cdots, x_n\}$ of A. Define $\phi\colon \mathrm{End}_R(A) \to \Sigma_{i=1}^{n} \oplus A_i$ by $\phi(f) = f(x_1) + \cdots + f(x_n)$, where $f(x_i) \in A_i = A$. Then ϕ is a monomorphism and $\mathrm{End}_R(A)$ is countable (since A is countable).

PROPOSITION 2.4. *Assume that* $\mathrm{End}_R(A)$ *has the minimum condition on left annihilator ideals.*

(a) *The finite topology on* $\mathrm{End}_R(A)$ *is discrete.*

(b) *If* B *is a summand of* A, *then* $\mathrm{End}_R(B)$ *has the minimum condition on left annihilator ideals.*

(c) A *is a finite direct sum of indecomposable* R-*modules.*

Proof. (a) Let $S = \{X^* \mid X$ is a finite nonzero subset of $A\}$ be a collection of left annihilator ideals of $\mathrm{End}_R(A)$. If $X^* = 0$ for some X, then $\mathrm{End}_R(A)$ is discrete. Otherwise, S has a minimal nonzero element, say Y^*. If X is a finite subset of A, then $(Y \cup X)^* \subseteq Y^*$, so by the minimality of Y^*, $Y^* \cap X^* = (Y \cup X)^* = Y^*$. Thus $Y^* \subseteq \cap \{X^* \mid X^* \in S\} = 0$, a contradiction.

(b) Write $A = B \oplus C$ and identify $\mathrm{End}_R(B)$, in the usual way, with a subring of $\mathrm{End}_R(A)$ so that $\mathrm{End}_R(B) \subseteq C^* \subseteq \mathrm{End}_R(A)$. For I, a left annihilator ideal of $\mathrm{End}_R(B)$, let $e(I) = (I^* \oplus C)^* \subseteq \mathrm{End}_R(A)$. Note that $e(\mathrm{End}_R(B)) = C^*$ and that e is an order preserving map from the annihilator ideals of $\mathrm{End}_R(B)$ to the annihilator ideals of $\mathrm{End}_R(A)$. Therefore, it is enough to show that e is monic. Let J be an annihilator ideal of $\mathrm{End}_R(B)$ and let $\phi \in I \setminus J$. Then $\phi \in e(I)$, but $\phi(J^* \oplus C) = \phi(J^*) \neq 0$, i.e., $\phi \notin e(J)$. Consequently, $e(I) = e(J)$ iff $I = J$.

(c) First of all, A must have a nonzero indecomposable summand. Otherwise, there is a chain $A_1 \subset \cdots \subset A_n \subset \cdots$ of submodules of A such that each A_n is a proper summand of A and A_{n+1}. Thus each A_n^* is a nonzero annihilator ideal and $\{A_n^* \mid n = 1, 2, \cdots\}$ has no minimal element, a contradiction.

Write $A = A_1 \oplus B$, where A_1 is a nonzero indecomposable summand of A. Using (b) and the fact that A is self-small, one sees that (c) is true.

3. Self-small Abelian groups.

PROPOSITION 3.1. *Every self-small torsion group is finite.*

Proof. Let A be a self-small torsion group and for each integer n regard $n!$ as an element of $\mathrm{End}(A)$ (i.e., multiplication by n factorial). Define $A_n = \{n!\}^* \subseteq A$ so that $0 = A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ is chain of subgroups of A. Each A_n is bounded, hence a direct sum of cyclic groups. Consequently, $A_n = A$ for some n and A is finite. Otherwise, each A_n is proper, $A_n^* \neq 0$, and A is not self-small (Proposition 1.2.b), a contradiction.

If A is a torsion free abelian group such that $\mathrm{End}(A)$ is countable, then A is self-small (Corollary 1.4). Examples of such groups include all torsion free abelian groups of finite rank, all of the groups constructed by Corner **[4]**, and all rigid groups (see Fuchs **[7]**, p. 124). Other examples, not requiring the countability of $\mathrm{End}(A)$, are given by:

PROPOSITION 3.2. *Let A be a reduced torsion free abelian group.*

(a) *If* $\mathrm{End}(A)$ *is a Dedekind domain, then every nonzero endomorphism of A is monic.*

(b) *If every nonzero endomorphism of A is monic, then* $\mathrm{End}(A)$ *has the minimum condition on annihilator ideals.*

(c) *If B is a torsion free abelian group and if $B \otimes_Z A$ is self-small, then A is self-small.*

Proof. (a) Let $R = \mathrm{End}(A)$ and regard A as a left R-module. Since R has no nontrivial idempotents, A is indecomposable as an abelian group and consequently as an R-module. Suppose that A is not a torsion free R-module. Since R is Dedekind and A is reduced, then A is R-isomorphic to R/I, where I is a nonzero ideal of R (see Kaplansky **[8]**). There is some $0 \neq r \in I$ so that $Rr \subseteq I$ and $r(A) = 0$, a contradiction. Thus A is a torsion free R-module and (a) is proved.

(c) Assume that A is not self-small. By Proposition 1.2.c. there is a chain $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of proper kernel subgroups of A there is a chain $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of proper kernel subgroups of A such that $A = \cup_{n=1}^{\infty} A_n$. Now kernel subgroups are pure ($A_n = A_n^{**} = \cap_{f \in A_n^*}(\ker f)$) and B is torsion free so each $B \otimes_Z A_n$ is proper in $B \otimes_Z A$; $B \otimes_Z A_1 \subseteq \cdots \subseteq B \otimes_Z A_n \subseteq \cdots$; $\cup_{n=1}^{\infty}(B \otimes_Z A_n) = B \otimes_Z A$ and $(B \otimes_Z A_n)^* \neq 0$ for all n (choose $f_n \in \mathrm{End}(A)$ with $f_n(A_n) = 0$ and note that $(1 \otimes f_n)(B \otimes A_n) = 0$). By Proposition 1.2.b., $B \otimes_Z A$ is not self-small, a contradiction.

For a prime, p, let Z_p be the localization of Z at a prime p (i.e., the subring of Q consisting of elements with denominator prime to p).

COROLLARY 3.3. *Let A be a reduced torsion free abelian group.*

(a) *If $Z_p \otimes_Z A$ is a self-small Z_p-module for some prime, p, then A is self-small.*

(b) *If there is a prime p such that the cardinality of A/pA is finite and if $\cap_{n=1}^{\infty} p^n A = 0$, then A is self-small.*

Proof. (a) is a consequence of Proposition 3.2.

(b) Choose $X = \{x_1, \cdots, x_n\}$, where $\{x_1 + pA, \cdots, x_n + pA\}$ is a basis for A/pA (as a Z/pZ *vector space*). *Since* $\cap_{n=1}^{\infty} p^n A = 0$, one can easily see that $X^* = 0$, i.e., $\mathrm{End}(A)$ is discrete.

REMARK. Examples of Corollary 3.3.b include p-pure subgroups of finite direct sums of copies of the p-adic integers.

Conditions guaranteeing that a torsion free abelian group A is self-small may be recast in terms of the *quasi-endomorphism ring of* A, denoted by $\mathscr{E}(A)$. Walker **[14]** observed that $\mathscr{E}(A)$ may be regarded as $Q \otimes_Z \mathrm{End}(A)$ (also see Reid **[11]**). If $f \in \mathscr{E}(A)$, then $f = 1/n \otimes g$ for some $g \in \mathrm{End}(A)$ and $n \in Z$. Define *ker* $f = \{x \in A \mid g(x) = 0\}$; if $X \subseteq A$, let $X_* = \{f \in \mathscr{E}(A) \mid X \subseteq \ker f\}$, a left ideal in $\mathscr{E}(A)$; and if J is a left ideal of $\mathscr{E}(A)$, let $J_* = \cap_{f\in J} \ker f$. Call J an *annihilator ideal of* $\mathscr{E}(A)$ if $J = J_{**}$.

PROPOSITION 3.4. *If A is a torsion free abelian group, then there is a* 1-1 *order preserving correspondence between left annihilator ideals in* End(A) *and left annihilator ideals in* $\mathscr{E}(A)$.

Proof. The correspondence is given by $I \to QI$ and $J \to J \cap \mathrm{End}(A)$, where $\mathrm{End}(A)$ is canonically embedded in $\mathscr{E}(A)$ (via $f \to 1 \otimes f$). Note that left annihilator ideals of $\mathrm{End}(A)$ are pure subgroups of $\mathrm{Hom}(A, A)$. The remainder of the proof is straightforward and is left to the reader.

COROLLARY 3.5. *Let A be a torsion free abelian group.*

(a) End(A) *has minimum condition on left annihilator ideals iff* $\mathscr{E}(A)$ *has minimum condition on left annihilator ideals.*

(b) End(A) *is discrete iff* $\mathscr{E}(A)$ *is discrete.*

(c) *A is not self-small iff there is a chain $J_1 \supset \cdots \supset J_n \supset \cdots$ of nonzero annihilator ideals of $\mathscr{E}(A)$ with $A = \cup_{n=1}^{\infty}(J_n)_*$.*

REMARK. If $\mathscr{E}(A)$ has the minimum condition on left ideals, then A is self-small. This class of groups has been considered by Reid **[11]** and others.

The situation is even more complicated for mixed abelian groups, A. Let tA be the torsion subgroup of A and, for p a prime, let $(tA)_p$ be the p-component of tA. If A is a torsion free abelian group of finite rank, then the *R-type of A* is the quasi-isomorphism class of A/F, where F is a free subgroup of A with A/F torsion (e.g., see Richman **[12]**).

PROPOSITION 3.6. *Suppose that A is a mixed abelian group and that A/tA has finite rank. Then A is self-small iff* (a) *for all primes, p, $(tA)_p$ is finite or zero*; *and* (b) *the R-type of A/tA is p-divisible for all primes p with $(tA)_p \neq 0$.*

Proof. ($\Rightarrow$) Let p be a prime with $(tA)_p \neq 0$. Since A is self-small and A/pA is a direct sum of cyclic groups of order p, it follows that A/pA is finite (otherwise map each summand of A/pA into $(tA)_p \subset A$). But $(tA)_p/p(tA)_p$ is a summand of A/pA, so $(tA)_p$ is bounded. Thus $(tA)_p$ is a bounded pure subgroup of A hence a direct summand of A. Since summands of self-small groups are self-small and since self-small torsion groups are finite, $(tA)_p$ is finite.

Since A is self-small, every endomorphic image of A is self-small. Let $B = A/tA$ and let F be a free subgroup of B with B/F torsion. Now B/F is a homomorphic image of A, so $(B/F)_p$ must be divisible for all but a finite number of primes with $(tA)p \neq 0$ (or else A has an infinite torsion endomorphic image, a con-tradiction). Consequently, $B/F = G \oplus T$ where T is finite and G is p-divisible for all primes p with $(tA)_p \neq 0$. This proves (b).

($\Leftarrow$) We prove that $\mathrm{End}(A)$ is countable and apply Corollary 1.4. Let F be a free subgroup of A such that A/F is torsion. By (b), $A/F+tA \simeq (A/tA)/((F+tA)/tA) = G \oplus T$, where T is finite and G is p-divisible for almost all primes p with $(tA)_p \neq 0$. Moreover, $\mathrm{Hom}(A/F, tA)$ is countable. In view of the exact sequence $0 \to \mathrm{Hom}(A/F, tA) \to \mathrm{Hom}(A, tA) \to \mathrm{Hom}(F, tA)$ and the fact that F is finitely generated and tA is countable, one sees that $\mathrm{Hom}(A, tA)$ is countable. Furthermore, there are exact sequences $0 \to \mathrm{Hom}(A, tA) \to \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A/tA)$ and $0 \to \mathrm{Hom}(A/tA, A/tA) \to \mathrm{Hom}(A, A/tA) \to \mathrm{Hom}(tA, A/tA) = 0$ and $\mathrm{Hom}(A/tA, A/tA)$ is countable. Consequently, $\mathrm{Hom}(A, A)$ is countable, as desired.

COROLLARY 3.7. *Suppose that A is a self-small mixed abelian group. Then for all primes, p, $(tA)_p$ is finite or zero; the cardinality of A/pA is finite for all primes p with $(tA)_p \neq 0$; and if B is a torsion free subgroup of A such that A/B is torsion, then A/B is p-divisible for all but a finite number of primes with $(tA)_p \neq 0$.*

4. Locally A-projective groups.

We first remind the reader of the homological setting described in **[1]**. Let $\mathcal{G}$ be the category of abelian groups, $A \in \mathcal{G}$, $R = \mathrm{End}(A)$ and $\mathcal{M}_R$ the category of right R-modules. There is a left exact functor $H: \mathcal{G} \to \mathcal{M}_R$ defined by $H(G) = \mathrm{Hom}_Z(A, G)$ and a right exact functor $T: \mathcal{M}_R \to \mathcal{G}$ defined by $T(M) = M \otimes_R A$, where A is regarded as a left R-module in the obvious fashion.

There are natural transformations $\theta: TH \to 1_{\mathcal{G}}$ and $\phi: 1_{\mathcal{M}_R} \to HT$, where $\theta_G: \mathrm{Hom}_Z(A, G) \otimes_R A \to G$ is defined by $\theta_G(f \otimes a) = f(a)$ and $\phi_M: M \to \mathrm{Hom}_Z(A, M \otimes_R A)$ is defined by $\phi_M(x)(a) = x \otimes a$.

A group, G, is *finitely A-projective* if G is isomorphic to a direct summand of the direct sum of a finite number of copies of A.

THEOREM 4.1. *Let A be an abelian group.*

(a) *the category of finitely A-projective groups is equivalent to the category of finitely generated projective right* $\mathrm{End}(A)$*-modules.*

(b) *If A is self-small, then the category of A-projective groups is equivalent to the category of projective right* $\mathrm{End}(A)$*-modules.*

Proof. The following observations suffice to prove the theorem (all of which are easy to verify): (i) if G is *finitely A-free* (i.e., a finite direct sum of copies of A) or if G is A-free and A is self-small, then $H(G)$ is a free right R-module; (ii) If M is a free right R-module, then $T(M)$ is A-free; (iii) $\theta_A: TH(A) \to A$ and $\phi_R: R \to HT(R)$ are isomorphisms and (iv) θ and ϕ are natural transformations (details are given in **[1]**; see also Warfield **[15]** for the case that A is torsion free of rank 1).

COROLLARY 4.2. *Let A be an abelian group.*

(a) *If A is self-small, then every A-projective group is A-free iff every projective right* $\mathrm{End}(A)$*-module is free.*

(b) *Every locally A-projective group is locally A-free iff every finite generated projective right* $\mathrm{End}(A)$*-module is free.*

COROLLARY 4.3. *Let $\mathcal{F} = \{A_i\}_{i\in I}$ be a class of countable self-small groups together with a partial ordering on the index set I such that $i \leqq j$ iff* $\mathrm{Hom}(A_i, A_j) \neq 0$. *Define $\mathcal{F}_\epsilon$ to be the class of groups isomorphic to direct sums of groups in $\mathcal{F}$. If every projective right* $\mathrm{End}(A_i)$*-module is free for all $A_i \in \mathcal{F}$, then $\mathcal{F}_\epsilon$ is closed under direct summands.*

Proof. Theorems due to Kulikov-Fuchs (see Charles **[2]**) and Kaplansky (see Fuchs **[6]**, p. 49) reduce the argument to the A_i-projective case. Now apply Corollary 4.2.a.

COROLLARY 4.4. *Suppose that A is torsion free and that* $\mathrm{End}(A)$ *is a principal ideal domain. If B is a subgroup of an A-free group G and $S_A(B) = B$, then B is A-free.*

Proof. A is self-small by (3.2.a) and so $H(B)$ is an $\mathrm{End}(A)$-submodule of the free $\mathrm{End}(A)$-module $H(G)$. Now use the naturality of θ to show $\theta_G: T\cdot H(G) \to G$ is an isomorphism (details are given in **[1]**).

We observe that Corollary 4.3 includes the Baer-Kulikov-Kaplansky theorem (i.e., direct summands of completely decomposable groups are completely decomposable) results of Murley **[10]** and Arnold-Lady **[1]** as special cases. Corollary 4.4 is a generalization of theorems by Baer-Kolettis for the case that A has rank 1 (see Fuchs **[6]**, p. 114].

LEMMA 4.5. *Suppose that A is an abelian group such that* $\mathrm{Hom}_Z(A, G)$ *is a locally projective right* $\mathrm{End}(A)$*-module for all locally A-projective groups G. Then the category of locally A-projective abelian groups is equivalent to the category of locally projective right* $\mathrm{End}(A)$*-modules.*

Proof. Let $\mathcal{L}_A$ be the category of locally A-projective abelian groups and $\mathcal{L}_R$ the category of locally projective right R-modules, where $R = \mathrm{End}(A)$. The hypotheses guarantee that $H: \mathcal{L}_A \to \mathcal{L}_R$ is well defined. To show that $T: \mathcal{L}_R \to \mathcal{L}_A$ is well defined let $M \in \mathcal{L}_R$ and $y_1, \cdots, y_m \in T(M) = M \otimes_R (A)$, where $y_i = \Sigma m_{ij} \otimes a_{ij}$ with $m_{ij} \in M$, $a_{ij} \in A$. Now $\{m_{ij}\}$ is contained in a finitely generated projective summand, P, of M, say $M = P \oplus L$. But $T(M) = T(P) \oplus T(L)$ and $\{y_1, \cdots, y_m\} \subset T(P)$ a finitely A-projective summand of $T(M)$ (Theorem 4.1.a).

Next we prove that HT is naturally equivalent to the identity functor on $\mathscr{L}_R$, i.e., if $M \in \mathscr{L}_R$, then $\phi_M: M \to HT(M)$ is an isomorphism. Assume that $x \in \ker \phi_M$ and embed x in a finitely generated projective summand P of M. Naturality of ϕ gives a commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "\phi_P"] & M \arrow[d, "\phi_M"] \\ 0 \arrow[r] & HT(P) \arrow[r] & HT(M) \end{tikzcd}$$

The bottom row is exact since P is a summand of M. Since P is finitely generated and projective, ϕ_P is an isomorphism. But $x \in P$ and $\phi_M(x) = 0$, so $x = 0$. Consequently, ϕ_M is monic.

To prove that ϕ_M is epic, let $f \in HT(M)$. Since $HT(M) \in \mathscr{L}_R$, f is an element of some finitely generated projective summand, P_1, of $HT(M)$. Let $P = \{x \in M \mid \phi_M(x) \in P_1\}$. Since ϕ is natural, P_1 is the image of $HT(P)$ in $HT(M)$. Moreover, $\phi_p: P \to HT(P)$ is epic and $f \in P_1$, so $f \in \text{image } \phi_M$ as desired.

Finally, we prove that $\theta_G: TH(G) \to G$ is an isomorphism for all $G \in \mathscr{L}_A$. Let $g \in G$ and embed g in a finitely A-projective summand B of G. Now $\theta_B: TH(B) \to B$ is an isomorphism, $g \in \operatorname{im} \theta_B$, and θ is natural, so $g \in \text{image } \theta_G$, i.e., θ_G is epic. To show that θ_G is monic let $y = \Sigma_{i=1}^n f_i \otimes a_i \in \ker \theta_G$, where $f_i \in H(G)$ and $a_i \in A$. Since $H(G) \in \mathscr{L}_R$, $\{f_1, \cdots, f_n\}$ is contained in a finitely generated projective R-summand, P, of $H(G)$. Let $B = \{\theta_G(x) \mid x \in P \otimes_R A \subset TH(G)\}$. It now follows that $P \otimes_R A = \operatorname{Hom}(A, B) \otimes_R A$ and B is finitely A-projective so that $\theta_B: TH(B) \to B$ is an isomorphism and $y \in \ker \theta_B$, i.e., $y = 0$. The proof is now complete.

Proof of Theorem III. In view of Lemma 4.5, it suffices to prove that if G is an locally A-projective group and if $f_1, \cdots, f_n \in \operatorname{Hom}(A, G)$, then $\{f_1, \cdots, f_n\}$ is contained in a finitely generated $\operatorname{End}(A)$-projective summand of $\operatorname{Hom}(A, G)$.

Let B be a finitely generated subgroup of A with $B^* = 0$. By the hypotheses, $f_1(B) + \cdots + f_n(B)$ is contained in a finitely A-projective summand, G_1, of G, say $G = G_1 \oplus G_1'$. Since $\operatorname{Hom}(A, G_1)$ is a finitely generated projective $\operatorname{End}(A)$-summand of $\operatorname{Hom}(A, G)$, it is enough to prove that $f_1, \cdots, f_n \in \operatorname{Hom}(A, G_1)$. Suppose not; choose $a \in A \setminus B$ and $i \leqq n$ such that $\Pi' f_i(a) \neq 0$, where $\Pi': G \to G_1'$ is the projection map. Now $\Pi' f_i(a)$ is contained in a finitely A-projective summand G_2 of G, say $G = G_2 \oplus G_2'$. Furthermore, $\sigma \Pi' f_n(a) \neq 0$ where $\sigma: G \to G_2$ is the projection map. Since G_2 *is* A-projective, there is some $\delta: G_2 \to A$ with $g(a) \neq 0$ where $g = \delta\sigma\Pi' f_n \in \operatorname{End}(A)$. Thus $g \neq 0$ and $g(B) = 0$ (since $f_i(B) \subset G_1$), a contradiction.

We apply the preceding results in the case that $\operatorname{End}(A)$ is a principal ideal domain (hence discrete by Proposition 3.2.a).

Proof of Corollary IV. (a) Since $\operatorname{End}(B)$ is discrete, there is a finite subset X of B with $X^* = 0 \in \operatorname{End}(B)$. Embed X in a finitely A-free summand, C of G say $G = C \oplus D$. We prove that $B \subseteq C$. Suppose not and let $\delta: G \to D$ be the projection map. There is some $\psi: D \to A$ with $\psi\delta(B) \neq 0$ (D is A-free since G is A-free and $R = \operatorname{End}(A)$ is a principal ideal domain). But $S_A(B) = B$ so there is $\phi: A \to B$ with $0 \neq \phi\psi\delta \in \operatorname{End}(B)$. Consequently, $0 \neq \phi\psi\delta \in X^*$ (since $X \subseteq C$) a contradiction.

One can easily show that $\operatorname{Hom}(A, B)$ is a pure R-submodule of $\operatorname{Hom}(A, C)$ (since B is pure in C and $A/im\ f$ is torsion for all $0 \neq f \in R$). But $\operatorname{Hom}(A, C)$ is a finitely generated free R-module and thus $\operatorname{Hom}(A, B)$ is a summand of $\operatorname{Hom}(A, C)$ (recall that R is a principal ideal domain). The naturality of ϕ and Theorem 4.1.a guarantees that $\phi_B: \operatorname{Hom}(A, B) \otimes_R A \to B$ is an isomorphism and that B is an A-free summand of C, hence of G.

(b) ($\Rightarrow$) Let $\mathcal{F}$ be the set of projections of the locally A-free group, G, onto A. For each $\phi \in \mathcal{F}$ define $A_\phi = \phi(A)$ and let $e: G \to \Pi_{\phi \in \mathcal{F}} A_\phi$ be given by $e(x) = (\phi(x))_{\phi \in \mathcal{F}}$. Clearly, e is monic and $G = S_A(G) \subseteq S_A(\Pi_{\phi \in \mathcal{F}} A_\phi)$ (observe that G is locally A-free and that S_A commutes with direct sums). The purity of G is routine (observe that the characteristic of x in G is the minimum of the characteristics of $\phi(x)$ in A_ϕ as ϕ varies over $\mathcal{F}$).

($\Leftarrow$) We first prove that $P = S_A(\Pi_{i \in I} A_i)$ is locally A-free. Note that $\operatorname{Hom}(A, P)$ is R-isomorphic to $\Pi \operatorname{Hom}(A, A_i)$, a locally free R-module (Chase **[3]**). By Theorem III, it suffices to prove that $\theta_p: \operatorname{Hom}(A, P) \otimes_R A \to P$ is an isomorphism. Clearly, θ_p is epic since $S_A(P) = P$. Suppose that $\ker \theta_p \neq 0$; let $0 \neq x \in f_1 \otimes a_1 + \cdots + f_n \otimes a_n \in \ker \theta_p$ with n minimal and $f_i: A \to P$, $a_i \in A$. Since n is minimal, R is a principal ideal domain, and A is a torsion free R-module (Proposition 3.2.a) it follows that $\{a_1, \cdots, a_n\}$ is an R-independent subset of A. Now $\theta_p(x) = f_1(a_1) + \cdots + f_n(a_n) = 0$. For each $i \in I$, let $\Pi_i: P \to A_i = A$ be the projection map so that $\Pi_i f_j \in R$ for all $i \leqq j \leqq n$. But $0 = \Pi_i f_1(a_1) + \cdots + \Pi_i f_n(a_n)$ so that $\Pi_i f_j = 0$ for all $i \in I$, $i \leqq j \leqq n$. Consequently, $f_1 = f_2 = \cdots = f_n = 0$ and $x = 0$, a contradiction. This proves that θ_p is monic.

Now $\operatorname{Hom}(A, G)$ is a pure R-submodule of $\operatorname{Hom}(A, P)$ (since G is pure in P and $A/im\ f$ is torsion for all $0 \neq f \in R$), so $\operatorname{Hom}(A, G)$ is locally R-free (Chase **[3]**). Naturality of θ guarantees that $\theta_G: \operatorname{Hom}(A, G) \otimes_R A \to G$ is an isomorphism so that G is locally A-free (by Theorem III).

(c) A consequence of (a) and (b).

(d) Let G be a countable locally A-free group and write $G = \{x_i \mid i = 1, 2, 3, \cdots\}$. There is an A-free summand G_1 of G, say $G = G_1 \oplus H_1$, with $x_1 \in G_1$. By (c), H_1 is locally A-free. Let y_2 be the projection of x_2 in H_1. If $y_2 = 0$ define $G_2 = G_1$. Otherwise, let G_2 be an A-free summand of H_1 containing y_2. Proceed inductively to construct G_n for each $n > 0$ so that $G = \Sigma_{n=1}^{\infty} \oplus G_n$ and G is A-free.

REMARK. P is locally A-free if $\operatorname{End}(A)$ is a principal ideal domain, but is not A-free if I is infinite and A is reduced. Corollary IV.d. includes Theorem 1 of Murley **[10]**.

The discussion in this section has been restricted to Abelian groups in order to obtain sharpness in examples and applications. However, much remains true for R-modules. Let R be a commutative ring with 1 and A be an R-module. Define A-projective [A-free] in the obvious way and note that the homological setting remains intact. It follows that Theorem 4.1 and Corollary 4.2 hold for an R-module A. Suppose R and $\operatorname{End}_R(A)$ are Dedekind domains. Then it is easily seen that any $\operatorname{End}_R(A)$-injective module is an R-injective ($= R$-divisible) module. This observation together with the arguments involved show that Proposition 3.2.a and Corollary 4.4 hold for an R-module A with R a Dedekind domain.

5. Examples.

EXAMPLE 5.1. There is a self-small torsion free abelian group, A, such that $\operatorname{End}(A)$ is not discrete in the finite topology.

Proof. Let p be a prime, Z_p the localization of Z at p, and let A_p be a free Z_p-module of rank p. Then $A = \Pi_p A_p$ is self-small (Corollary 1.3). Assume that the finite topology on $\operatorname{End}(A)$ is discrete, i.e., $X^* = 0$ for some finite subset, X, of A. Suppose that n is the cardinality of X. For each prime p, let $\pi_p \colon A \to A_p$ be the projection map; the cardinality of $\pi_p(X) \leqq n$. Now $0 = (\pi_p(X))^* \subseteq \operatorname{End}(A_p)$. On the other hand, for each $p > n$, $\pi_p(X)$ is contained in a Z_p-free summand of A_p of rank at most n, so $0 \neq (\pi_p(X))^* \subseteq \operatorname{End}(A_p)$, a contradiction. Thus $\operatorname{End}(A)$ is not discrete.

EXAMPLE 5.2. There is a countable torsion free abelian group, A, such that $\operatorname{End}(A)$ is discrete and $\operatorname{End}(A)$ fails to have the minimum condition on left annihilator ideals. (compare Proposition 2.4.a)

Proof. Corner **[4]**, constructs a countable torsion free abelian group, A, such that $\operatorname{End}(A)$ is countable and A has no nonzero indecomposable summands. Now apply Corollary 2.3 and Proposition 2.4.c.

EXAMPLE 5.3. There is a countable self-small torsion free abelian group, A, such that the cardinality of A/pA is infinite for all primes p (compare Corollary 3.3.b).

Proof. Let $R = Z[X]$, the polynomial ring in an indeterminate X. By Corner **[4]**, there is a countable torsion free abelian group A with $\mathrm{End}(A) = R$. Clearly, A/pA is infinite for all primes, p.

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