THE FIXED POINT PROPERTY FOR TREE-LIKE CONTINUA WITH FINITELY MANY ARC COMPONENTS

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It is shown that if M is a tree-like continuum with a finite number of arc components, then every continuous mapping of M into itself has a fixed point.

A continuum M is a compact, connected metric space. A continuum is said to be tree-like if for every $\varepsilon > 0$, there is an ε-cover of M whose nerve is a simple tree (a connected, one-dimensional, acyclic simplicial complex). In [1] Bing raised the question of whether these continua have the fixed point property. This is one of the most famous unsolved fixed point questions for continua. This paper provides an affirmative answer to Bing's question in the case where the tree-like continuum M has finitely many arc components.

Since tree-like continua are hereditarily unicoherent, it is easily seen that any subcontinuum of a tree-like continuum with finitely many arc components has finitely many arc components (see the proof of Lemma 1.3 below). It follows that any such continuum is hereditarily decomposable (indecomposable continua have uncountably many composants). Continua which are hereditarily decomposable and hereditarily unicoherent are called λ-dendroids. These continua were shown by Cook to be tree-like in [3].

The theorem presented here is thus a special case of the fixed point question for λ-dendroids which was raised by Knaster in [5][1]. Numerous special cases of this question have already been answered. For a survey of these results see [8], Chapter II. In particular Hamilton [4] has shown that all λ-dendroids have the fixed point property for homeomorphisms and Borsuk [2] has shown that λ-dendroids which are arcwise connected (dendroids) have the fixed point property for all continuous maps. The theorem presented here generalizes the latter result.

The paper is in two sections. The first section deals with density properties of arc components in λ-dendroids. Not all of the results in §1 are required in §2, which contains the proof of the fixed point theorem. The other material in §1 is included because the authors feel that it has some independent interest and because it raises a

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question which we have been unable to answer.

1. Density properties of arc components of λ-dendroids.

DEFINITION 1.1. A continuum M is said to be hereditarily decomposable if given any non-degenerate subcontinuum L of M, L can be written as the union of two of its proper subcontinua.

DEFINITION 1.2. A continuum M is said to be hereditarily unicoherent if, given any two subcontinua P and Q of M, $P \cap Q$ is connected.

It is easy to see that if M is a hereditarily decomposable (hereditarily unicoherent) continuum, then every subcontinuum of M is hereditarily decomposable (hereditarily unicoherent). If p and q are two distinct points of a λ-dendroid M, then the fact that M is hereditarily unicoherent implies that there is a unique subcontinuum $l(p, q)$ of M which is irreducible with respect to containing both p and q. The uniqueness of $l(p, q)$ implies that if C is any subcontinuum of M which contains both p and q, then $l(p, q) \subset C$. The fact that M is hereditarily decomposable implies that $l(p, q)$ is an irreducible continuum of type λ, i.e., there is a monotone, continuous function m from $l(p, q)$ onto the closed unit interval $[0, 1]$ such that $m(p) = 0$, $m(q) = 1$ and $m^{-1}(t)$ has void interior in $l(p, q)$ for every $t \in [0, 1]$ (see [7], p. 15, Theorem 10). The sets $m^{-1}(t)$ are called tranches of $l(p, q)$. In what follows it is assumed that the reader has some familiarity with the notion of an irreducible continuum of type λ. For the basic facts concerning these continua the reader is referred to [6], §48 or [7], Ch. 1 (in [7] irreducible continua of type λ are called irreducible continua of type A'). Of the two accounts, [7] is more compact.

LEMMA 1.3. *Let M be a λ-dendroid, L a subcontinuum of M and A an arc component of M which meets L. Then $A \cap L$ is an arc component of L.*

Proof. Let $p \in A \cap L$. If q is in the arc component of L generated by p, then clearly $q \in A$. On the other hand, if $q \in A \cap L$, then $l(p, q)$ is an arc since $p, q \in A$. And since M is hereditarily unicoherent, $l(p, q) \subset L$. Thus q is in the arc component of L generated by p.

THEOREM 1.4. *Let M be a λ-dendroid and let A_1 and A_2 be distinct arc components of M. Then there is a G_δ subset G of M such that $A_1 \subset G$ and $A_2 \cap G = \varnothing$.*

Proof. Let $p_1 \in A_1$, $p_2 \in A_2$ and let $l(p_1, p_2)$ be the unique irreducible

subcontinuum of M from p_1 to p_2.

Case 1. $l(p_1, p_2)$ has exactly two arc components. Then by 1.3 the two arc components of $l(p_1, p_2)$ must be $A_1' = A_1 \cap l(p_1, p_2)$ and $A_2^2 = A_2 \cap l(p_1, p_2)$. $l(p_1, p_2)$ must have at least one nondegenerate tranche since it is not an arc, and it cannot have more than one since it has only two arc components. Call this tranche T. A straightforward argument using the hereditary unicoherence of M will show that if C is any subcontinuum of M which meets both A_1 and A_2, then $T \subset C$.

Let a and b be distinct points of T and let $\{U_1, U_2, U_3, \cdots\}$ and $\{V_1, V_2, V_3, \cdots\}$ be neighborhood bases for M at a and b respectively such that $U_n \cap V_n = \varnothing$ for each $n = 1, 2, 3, \cdots$.

Case (1a). $T \subset A_1$. In this case, for each natural number n let

$$C_n = \mathrm{cl}\left(\{p \in A_2 : l(p, p_2) \cap U_n = \varnothing\}\right).$$

Then for each n, C_n, being the closure of an arcwise connected set, is a subcontinuum of M. Moreover, since each C_n contains p_2 and does not contain a, it must be the case that $C_n \cap A_1 = \varnothing$.

Now let $p \in A_2$. Then since a is not an element of the arc $l(p, p_2)$, there must be an n such that $U_n \cap l(p, p_2) = \varnothing$. Thus $p \in C_n$. So $\bigcup \{C_n : n = 1, 2, \cdots\}$ is an F_σ subset of M which contains A_2 and fails to meet A_1. The complement of this set will be the desired set G.

Case (1b). $T \subset A_2$ (since T has void interior, relative to $l(p_1, p_2)$, one can see that this is the only other possible case). In this case for each natural number n let

$$D_n = \mathrm{cl}\left(\{p \in A_2 : l(p, a) \cap V_n = \varnothing\}\right)$$

and

$$E_n = \mathrm{cl}\left(\{p \in A_2 : l(p, b) \cap U_n = \varnothing\}\right).$$

By a similar argument to that given for C_n in (1a), for each n, D_n, and E_n are subcontinua of M which do not intersect A_1.

Now let $p \in A_2$ and consider the arc $l(p, a)$. If $b \notin l(p, a)$, then there is an n such that $V_n \cap l(p, a) = \varnothing$ and we get $p \in D_n$. If $b \in l(p, a)$, then $a \notin l(p, b)$ ($l(p, b)$ is a subarc of $l(p, a)$ which does not contain a). So there must be an n such that $U_n \cap l(p, b) = \varnothing$ and we get $p \in E_n$. The complement of $\bigcup \{D_n \cup E_n : n = 1, 2, \cdots\}$ will thus be the desired set G.

Case 2. $l(p_1, p_2)$ has more than two arc components. In this

case let C be an arc component of $l(p_1, p_2)$ which contains neither p_1 nor p_2 and let $c \in C$. A straightforward unicoherence argument will show that if F is any subcontinuum of M which meets both A_1 and A_2, then $c \in F$.

Now let $\{W_1, W_2, W_3, \cdots\}$ be a neighborhood base for M at c and let

$$F_n = \mathrm{cl}\left(\{p \in A_2 \colon l(p, p_2) \cap W_n = \varnothing\}\right).$$

An argument similar to that given in (1a) will show that the set $G = M - \bigcup \{F_n \colon n = 1, 2, \cdots\}$ has the desired properties. This concludes the proof.

COROLLARY 1.5. *If M is a λ-dendroid with countably many arc components, then each arc component of M is a G_δ.*

Proof. Let A be an arc component of M and number the other arc components of M; $A_1, A_2, \cdots$. For each natural number n let G_n be a G_δ subset of M which contains A and fails to meet A_n. Then $A = \bigcap \{G_1, G_2, \cdots\}$ is a G_δ.

COROLLARY 1.6. *If M is a λ-dendroid with finitely many arc components, then each arc component of M is an F_σ.*

Proof. Let A be an arc component of M. Then $M - A$ is a finite union of G_δ's (namely the other arc components of M) and is thus a G_δ. A is therefore an F_σ.

COROLLARY 1.7. *If M is a λ-dendroid with finitely many arc components, then some arc component of M has nonvoid interior.*

Proof. This follows immediately from 1.6 and the Baire category theorem.

LEMMA 1.8. *If G_1 and G_2 are disjoint G_δ subsets of a complete metric space X, then $\mathrm{Int}\,(\mathrm{Cl}\,(G_1)) \cap \mathrm{Int}\,(\mathrm{Cl}\,(G_2)) = \varnothing$.*

Proof. $V = \mathrm{Int}\,(\mathrm{Cl}\,(G_1)) \cap \mathrm{Int}\,(\mathrm{Cl}(G_2))$ is open in X and thus admits a complete metric (see e.g. [6], p. 408). Moreover, $G_1 \cap V$ and $G_2 \cap V$ are dense G_δ subsets of V. Therefore $G_1 \cap G_2$ is dense in V (see e.g. [6], Th. 1, p. 417). But $G_1 \cap G_2 = \varnothing$.

COROLLARY 1.9. *If M is a λ-dendroid with countably many arc components and one arc component A of M is dense, then every other arc component of M is nowhere dense.*

Proof. Let A' be another arc component of M. Since A is dense, Int $(\text{Cl}(A)) = M$. Since A and A' are G_δ's (by 1.5), 1.8 implies that Int $(\text{Cl}(A')) = \varnothing$.

COROLLARY 1.10. *If M is a λ-dendroid with countably many arc components, then at most one arc component of M is dense.*

Note that if M is a λ-dendroid with finitely many arc components, then 1.10 follows from 1.7. One might suppose that 1.6 and 1.7 are true of λ-dendroids with countably many arc components, but in fact this is not the case. The authors have produced an example of a λ-dendroid with countably many arc components in which every arc component has void interior. The Baire category theorem implies that at least one of the arc components must fail to be an F_σ. The example is produced roughly as follows: Take the cone over the Cantor set and replace the arcs emanating from the accessible points of the Cantor set by sin $1/x$ curves in such a way that (i) the arcs emanating from inaccessible points of the Cantor set are not disturbed. That is, these arcs will have to be "bent" a little so that they will approximate the new sin $1/x$ curves, but they must nevertheless remain arcs. And (ii) the union of the limiting segments of all of the sin $1/x$ curves is dense in the whole space. A certain amount of delicacy is required in carrying out the above construction. In particular, the diameters of the limiting segments of the sin $1/x$ curves must tend to 0.

Thus 1.6 and 1.7 cannot be strengthened. The authors have been unable to answer the question of whether 1.10 can be strengthened to include all λ-dendroids i.e.,

Question. If M is an arbitrary λ-dendroid, must M have at most one dense arc component? In view of 1.8 it would suffice to show that distinct arc components of M can be enclosed in disjoint G_δ subsets of M. The authors have produced an example of 1-dimensional continuum with exactly two arc components, both of which are dense; but the example is neither hereditarily decomposable nor hereditarily unicoherent (the example is produced by sticking the endpoints of the above example to a "cross-section" of Knaster's indecomposable, chainable continuum with one endpoint).

2. The fixed point theorem.

LEMMA 2.1. *Let M be a hereditarily unicoherent continuum and let $M_1, M_2, \cdots, M_n$ be subcontinua of M. If $M_i \cap M_j \neq \varnothing$ for all $i, j = 1, 2, \cdots, n$, then $M_0 = \bigcap \{M_i : i = 1, 2, \cdots, n\}$ is a nonvoid subcontinuum of M.*

Proof. The proof is by induction on n. If $n = 2$, the result is obviously true. So let $n \geq 3$ and suppose that the lemma is true for all integers $k < n$. Let $A = M_1 \cap M_2 \cap \cdots \cap M_{n-2}$ and let $B = M_2 \cap M_3 \cap \cdots \cap M_{n-1}$. Then by the induction hypothesis and the hereditary unicoherence of M, A and B are nonvoid subcontinua of M. Moreover, $A \cap B \neq \varnothing$ (also, by the induction hypothesis). Therefore, $A \cup B$ is a continuum. Thus $(A \cup B) \cap M_n$ is a continuum. But $(A \cup B) \cap M_n = (A \cap M_n) \cup (B \cap M_n)$ and, once again by the induction hypothesis, $A \cap M_n \neq \varnothing \neq B \cap M_n$. Therefore, since $(A \cap M_n) \cup (B \cap M_n)$ is connected, we must have $M_0 = (A \cap M_n) \cap (B \cap M_n) \neq \varnothing$. That M_0 is a continuum follows directly from the hereditary unicoherence of M.

COROLLARY. *Let M be a hereditarily unicoherent continuum, let $\{M_\alpha : \alpha \in \Gamma\}$ be a collection of subcontinua of M and suppose that for all $\alpha, \beta \in \Gamma$, $M_\alpha \cap M_\beta \neq \varnothing$. Then $M_0 = \bigcap \{M_\alpha : \alpha \in \Gamma\}$ is nonvoid subcontinuum of M.*

Proof. By 2.1 the family $\{M_\alpha : \alpha \in \Gamma\}$ has the finite intersection property. Therefore $M_0 \neq \varnothing$. Now reindex the family $\{M_\alpha : \alpha \in \Gamma\}$ by some initial sequence of ordinals: $\{M_\alpha : \alpha < \gamma\}$ where γ is an ordinal number. For each $\alpha < \gamma$ let $C_\alpha = \bigcap \{M_\beta : \beta < \alpha\}$. We want to show that $C_\gamma = M_0$ is connected. If not, then there is a least ordinal α_0 such that C_{α_0} is not connected. By 2.1 $\alpha_0 \geq \omega$. Also α_0 cannot be a successor ordinal. For if $\alpha_0 = \beta + 1$, then C_β is connected and, since M is hereditarily unicoherent, we get $C_{\alpha_0} = C_\beta \cap M_\beta$ connected. Thus α_0 is a limit ordinal. But then $C_{\alpha_0} = \bigcap \{C_\alpha : \alpha < \alpha_0\}$ is the intersection of a nest of continua and is thus a continuum. This contradiction establishes the corollary.

Let M be a λ-dendroid with finitely many arc components and let $f: M \to M$ be a continuous function. By intersecting a maximal nest of subcontinua M_α of M with the property that $f(M_\alpha) \subset M_\alpha$ one can find a subcontinuum M_0 of M such that $f(M_0) = M_0$ and no proper subcontinuum of M_0 is mapped into itself by f. Moreover, by Lemma 1.3 M_0 will have finitely many arc components. The fixed point theorem for M will be established by a series of lemmas. For the remainder of the section M will denote a fixed λ-dendroid with finitely many arc components and f will denote a continuous mapping of M into itself such that no proper subcontinuum of M is mapped into itself by f. By the preceding discussion, it will suffice to show that such f's have fixed points.

LEMMA 2.2. *f maps some arc component of M into itself.*

Proof. Since the continuous image of an arcwise connected set

is arcwise connected, if A is an arc component of M, then $f(A)$ will be contained in an arc component of M. Moreover, since f is surjective and M has only finitely many arc components, f must permute the arc components of M. Let $\mathscr{A}$ be the set of arc components of M and define $f^*: \mathscr{A} \to \mathscr{A}$ by $f^*(A) = f(A)$ for each $A \in \mathscr{A}$ (since f is surjective, $f(A)$ must always be an entire arc component). Then f^* is a permutation on the finite set $\mathscr{A}$. Now define η to be the family of all subsets N of $\mathscr{A}$ satisfying the following conditions:

(1) $\cup N$ is connected,

(2) $\mathrm{cl}(\cup N) \neq M$ and

(3) N is maximal with respect to possessing properties (1) and (2) simultaneously.

We may assume without loss of generality that no arc component of M is dense for if M contains a dense arc component A, then 1.10 implies that, since $f(A)$ is dense, $f(A) = A$. Thus $\eta \neq \varnothing$.

Claim 1. If $N \in \eta$, then $f^{*-1}(N) \in \eta$.

Proof of claim. Since f^* is a permutation of the finite set $\mathscr{A}$, there must be a natural number n such that $f^{*-1} = f^{*n}$. So $f^{*-1}(N) = f^{*n}(N)$. Therefore, since $\cup N$ is connected, $\cup f^{*-1}(N) = \cup f^{*n}(N) = f^n(\cup N)$ is connected. Thus $f^{*-1}(N)$ satisfies condition (1). Since $\mathrm{cl}(\cup N) \neq M$, there is a nonvoid open set $V \subset M - \cup N$. Since f is surjective $f^{-1}(V)$ is a nonvoid open set in the complement of

$$f^{-1}(\cup N) = \cup f^{*-1}(N) .$$

Thus $\mathrm{cl}(\cup f^{*-1}(N)) \neq M$ and $f^{*-1}(N)$ satisfies condition (2).

Finally, suppose that $D \subset \mathscr{A}$, $f^{*-1}(N) \subsetneqq D$, and $\cup D$ is connected. We will complete the proof of the claim by showing that $\mathrm{cl}(\cup D) = M$. Since $f^{*-1}(N) \subsetneqq D$, $N = f^*(f^{*-1}(N)) \subsetneqq f^*(D)$. Moreover, since $\cup D$ is connected, $\cup f^*(D) = f(\cup D)$ is connected. Thus, since $N \in \eta$, $\mathrm{cl}(f(\cup D)) = \mathrm{cl}(\cup f^*(D)) = M$. Now choose a natural number m such that f^{*m} is the identity permutation. Then $\mathrm{cl}(\cup D) = \mathrm{cl}(\cup f^{*m}(D)) = \mathrm{cl}(f^m(\cup D)) = \mathrm{cl}(f^{m-1}(f(\cup D))) = f^{m-1}(\mathrm{cl}(f(\cup D))) = f^{m-1}(M) = M$. Thus $f^{*-1}(N)$ satisfies condition (3).

Claim 2. If $N_1, N_2 \in \eta$, then $\mathrm{cl}(\cup N_1) \cap \mathrm{cl}(\cup N_2) \neq \varnothing$.

Proof of claim. Just suppose that $N_1, N_2 \in \eta$ and

$$\mathrm{cl}(\cup N_1) \cap \mathrm{cl}(\cup N_2) = \varnothing .$$

Let $F \in \mathscr{A}$ such that $F \notin N_1$ and $\cup N_1 \cup F$ is connected (such a set must exist, since M is connected). Now since $N_1 \in \eta$, $\mathrm{cl}(\cup N_1 \cup F) =$

M. Thus $\mathrm{cl}(\cup N_2) \subset \mathrm{cl}\, F$ and so $(\cup N_2) \cap F$ is connected. Since $N_2 \in \eta$, $\mathrm{cl}(\cup N_2 \cup F) = M$. But $\mathrm{cl}(\cup N_2) \subset \mathrm{cl}\, F$, so $M = \mathrm{cl}(\cup N_2) \cup \mathrm{cl}(F) = \mathrm{cl}\, F$, violating the assumption that no arc component of M is dense. This establishes the second claim.

Now Lemma 2.1 and Claim 2 imply that $M_0 = \cap \{\mathrm{cl}(\cup N): N \in \eta\}$ is a subcontinuum of M. And condition (2) on η implies that M_0 is a proper subcontinuum. Moreover, if N is any element of η, then Claim 1 implies that there is an $N_1 \in \eta$ (namely $f^{*-1}(N)$) such that

$$f(\cup N_1) \subset \cup N .$$

This in turn implies that $\cap \{\mathrm{cl}(f(\cup N)): N \in \eta\} \subset \cap \{\mathrm{cl}(\cup N): N \in \eta\}$. Thus we get

$$f(M_0) = f(\cap \{\mathrm{cl}(\cup N): N \in \eta\}) \subset \cap \{f(\mathrm{cl}(\cup N)): N \in \eta\}$$
$$= \cap \{\mathrm{cl}(f(\cup N)): N \in \eta\} \subset \cap \{\mathrm{cl}(\cup N): N \in \eta\} = M_0 ,$$

contradicting one of the original assumptions about f. This concludes the proof.

THEOREM 2.3. *If M is a λ-dendroid and the collection of arc components of M is finite, then M has the fixed point property.*

Proof. Suppose, to the contrary, that there is a map $f: M \to M$ such that f has no fixed point. As explained at the beginning of this section, we may assume that f maps M onto M and that no proper subcontinuum of M is mapped into itself. From Lemma 2.2, we conclude that there is an arc component A of M such that $f[A] = A$. Then $f[\mathrm{cl}\, A] = \mathrm{cl}\, f[A] = \mathrm{cl}\, A$, so $\mathrm{cl}\, A$ is a subcontinuum of M which is mapped into itself. Thus $\mathrm{cl}\, A = M$.

Choose $p \in A$. Since $p \neq f(p)$ and $f(p) \in A$, the continuum irreducible from p to $f(p)$, $l(p, f(p))$ is an arc. Using the uniform continuity of f, we see that there is a point $b \in l(p, f(p))$, $p \neq b$, such that the arc $l(p, b)$ misses its image under f.

Claim. If $x \in l(p, b)$, then $x \in l(p, f(x))$. Suppose that there is a point $x \in l(p, b)$ such that $x \notin l(p, f(x))$. Since $f[l(p, b)]$ is arcwise connected, there is an arc J, from $f(x)$ to $f(p)$, with $J \subset f[l(p, b)]$. Clearly, $J \cap l(p, b) = \varnothing$, and $x \notin J$. Thus $J \cup l(p, f(x))$ contains an arc H from p to $f(p)$. Since $x \notin J \cup l(p, f(x))$, $x \notin H$. Hence, there arc two arcs from p to $f(p)$, namely one containing x and H. This contradicts the hereditary unicoherence of M and establishes the claim.

Let $\mathscr{L} = \{l(p, q): q \in A$, and if $x \in l(p, q)$, then $x \in l(p, f(x))\}$. Now $\mathscr{L}$ is partially ordered by inclusion and $\{l(p, b)\}$ is a nest in $\mathscr{L}$. By the maximal principle, there is a maximal nest $\mathscr{N}$ in $\mathscr{L}$ which

contains $\{l(p, b)\}$. Suppose that Λ is an index set such that $\mathscr{N} = \{l(p, q_\lambda): \lambda \in \Lambda\}$. Let $L = \cup \{l(p, q_\lambda): \lambda \in \Lambda\}$. Then L is arcwise connected and cl L is a continuum. Note that if $x \in L$, then $x \in l(p, f(x))$.

Moreover, cl L is irreducible from p to some point. For if this is not true, then cl L is the union of two of its proper subcontinua D and E, such that $p \in D \cap E$. Since L is dense in cl L, there are elements λ, $\mu \in \Lambda$ such that $l(p, q_\lambda) \cap (D - E) \neq \varnothing$ and $l(p, q_\mu) \cap (E - D) \neq \varnothing$. Inasmuch as $\mathscr{N}$ is a nest, are we may assume that $l(p, q_\lambda) \subset l(p, q_\mu)$. Since M is hereditarily unicoherent, $l(p, q_\mu) \cap D$ and $l(p, q_\mu) \cap E$ are continua. Thus, $l(p, q_\mu)$ is the union of two of its proper subcontinua, both of which contain p. Thus p is not a point of irreducibility of $l(p, q_\mu)$. This is impossible, so cl L is irreducible from p to some point.

Let $K = \{y: y \in M$ and cl L is irreducible from p to $y\}$. Since M is hereditarily decomposable, it follows from [7, Th. 7, p. 13] that K is a continuum. Furthermore, $K \cap A = \varnothing$. For otherwise, there is a point $t \in K \cap A$, and so $l(p, t)$ is an arc from p to t. This means that $K = \{t\}$ and so cl $L = l(p, t)$. Thus cl $L = L \cup \{t\}$. It is not difficult to see that $t \in l(p, f(t))$, since there are points of L arbitrarily close to t. This means that $l(p, t)$ is a largest element of the nest $\mathscr{N}$. However, the same sort of argument as was used to establish the existence of the arc $l(p, b)$, to start the construction of $\mathscr{N}$, will show that $\mathscr{N}$ cannot have a largest element. Thus cl $L = L \cup K$ and $A \cap$ cl $L = L$.

Claim. $f[K] \cap K \neq \varnothing$.

If this fails, then there are disjoint open sets U and V containing K and $f[K]$ respectively. We may assume that U was chosen small enough that $f[U] \subset V$. Now cl $L - U$ is closed and a subset of A. Thus $f[\text{cl } L - U] \subset f[A] = A$, so $f[\text{cl } L - U] \cap K = \varnothing$. So there is an open set W with $K \subset W \subset U$ and $W \cap f[\text{cl } L - U] = \varnothing$. Since $f[U] \subset V$ and $V \cap W = \varnothing$, it follows that $f[\text{cl } L] \cap W = \varnothing$.

Since $l(p, f(p))$ is an arc in L, we see that $W - l(p, f(p))$ is an open set containing K. Each point of K is a limit point of L, so we may choose $x \in (W - l(p, f(p))) \cap L$. Clearly, $f[l(p, x)]$ is an arcwise connected continuum containing $f(p)$ and $f(x)$. Thus $l(p, f(p)) \cup f[l(p, x)]$ is an arcwise connected continuum containing p and $f(x)$, hence containing the unique arc from p to $f(x)$, $l(p, f(x))$. Because $x \in L$, we must have $x \in l(p, f(x))$, so $x \in l(p, f(p)) \cup f[l(p, x)]$. However, $x \notin l(p, f(p))$, by choice of x, so $x \in f[l(p, x)]$.

Let $z \in l(p, x)$ with $f(z) = x$. Now $l(p, x) \subset L$, so $z \in L$. Thus $x = f(z) \in f[\text{cl } L] \cap W$, which contradicts the concluding statement of the first paragraph and establishes the claim.

Since f permutes the arc components of M, it follows that $f[M - A] = M - A$. Since $K \subset M - A$, there is a component D of $M - A$ such that $K \subset D$. Now $f[D]$ is contained in a component of $M - A$; because $K \cap f[K] \neq \varnothing$, we have $f[D] \cap D \neq \varnothing$, so this component of $M - A$ can only be D itself; that is $f[D] \subset D$. Thus $f[\mathrm{cl}\, D] = \mathrm{cl}\, f[D] = \subset \mathrm{cl}\, D$, so $\mathrm{cl}\, D$ is a subcontinuum of M which is mapped into itself. Since A is a dense arc-component of M, Corollary 1.7 guarantees that $\mathrm{Int}\, A \neq \varnothing$. Now $D \subset M - A$, so $\mathrm{cl}\, D \neq M$. Thus $\mathrm{cl}\, D$ is a proper subcontinuum of M which is mapped into itself. This contradiction concludes the proof.

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