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RINGS WHOSE FAITHFUL LEFT IDEALS ARE COFAITHFUL

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A left module M over a ring R is cofaithful in case there is an embedding of R into a finite product of copies of M. Our main result states that a semiprime ring R is left Goldie, that is, has a semisimple Artinian left quotient ring, if and only if R satisfies (i) every faithful left ideal is cofaithful and (ii) every nonzero left ideal contains a nonzero uniform left ideal. The proof is elementary and does not make use of the Goldie and Lesieur-Croisot theorems. We show that (i) and (ii) are Morita invariant. Moreover, (ii) is invariant under polynomial extensions, and so is (i) for commutative rings. Absolutely torsion-free rings are studied.

The ring Q is a left classical quotient ring for the ring $R \subseteq Q$ if every regular element (nondivisor of zero) of R is invertible in Q and if every element of Q is of the form $b^{-1}a$ where $a, b \in R$ and b is regular; in this case we also say that R is a left order in Q. A ring is said to be left Goldie if it has the ascending chain condition on left annihilators and has finite uniform dimension. (A left R-module has finite uniform dimension if it has no infinite direct sum of nonzero submodules, and it is said to be uniform if it is nonzero and any two nonzero submodules have a nontrivial intersection.) A theorem of Goldie **[8, 9]** and Lesieur and Croisot **[12]** states that a ring is a left order in a semisimple Artinian ring if and only if it is semiprime and left Goldie. It is known that the ascending chain condition on left annihilators is not preserved under an equivalence of categories (Morita invariant); in fact, it does not go up to matrix rings. It is unknown whether being left Goldie is Morita invariant.

In section two we give a proof of the theorem stated in the abstract, and in the prime case we give a proof which shows directly that such a ring is an order in a full matrix ring over a division ring. We also weaken the hypothesis of an important theorem on semiprime PI rings. In the third section we use these techniques to study absolutely torsion-free rings. In particular, we show that an absolutely torsion-free ring is Goldie if and only if it has a uniform left ideal, and that the endomorphism ring of a finitely generated projective module over an absolutely torsion-free ring is absolutely torsion-free.

1. Some general results. All rings will be associative and have an identity element; all modules will be unital. Let R be a ring and S a subset of R. Then the right annihilator of S in R is $\ell_R(S) = \{r \in R \mid Sr = 0\}$ and the left annihilator is $\ell_R(S)$. If X is a subset of a left R-module M, then $\mathrm{Ann}_R(X) = \{r \in R \mid rX = 0\}$. If there is no ambiguity we write $\ell(S)$ instead of $\ell_R(S)$, etc. $Z_R(M)$ will denote the singular submodule of M, the set of elements of M whose annihilator is essential in R.

A module ${}_RM$ is said to be cofaithful if there exist elements $m_1, m_2, \cdots, m_k \in M$ such that $\bigcap_{i=1}^{k} \mathrm{Ann}(m_i) = 0$, or equivalently, if for some direct sum M^k of k copies of M there exists an exact sequence $0 \to R \to M^k$. Every cofaithful module is faithful. On the other hand, every faithful left R-module is cofaithful if and only if R contains an essential Artinian left ideal (see Beachy **[1]**), in which case we say R is essentially left Artinian. A ring R is essentially left Artinian if and only if R has an essential and finitely generated left socle. We study the weaker condition that every faithful left ideal of R is cofaithful. Recall that ${}_RM$ is torsionless if for each $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ with $f(m) \neq 0$.

PROPOSITION (1.1). *The following conditions are equivalent for a ring R.*

(a) *Every faithful left ideal of R is cofaithful.*

(b) *Every ideal of R which is faithful as a left ideal is cofaithful.*

(c) *Every faithful, torsionless left R-module is cofaithful.*

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are immediate.

(b) $\Rightarrow$ (c). Let ${}_RM$ be a faithful torsionless module and A be the sum in R of the homomorphic images of M. If $0 \neq r \in R$, then since M is faithful there exists $m \in M$ such that $rm \neq 0$, and since M is torsionless there exists $f \in \mathrm{Hom}_R(M, R)$ with $rf(m) = f(rm) \neq 0$, which shows that the ideal A is faithful. Thus A is cofaithful and so $\bigcap_{i=1}^{n} \ell_R(a_i) = 0$ for some $a_i \in A$, $1 \leqq i \leqq n$. Since $a_i \in A$, $a_i = \Sigma_j f_{ij}(m_{ij})$ for $m_{ij} \in M$ and $f_{ij} \in \mathrm{Hom}_R(M, R)$, and then $rm_{ij} = 0$ for all i, j implies $ra_i = 0$ for all i, so $\bigcap_{i,j} \mathrm{Ann}_R(m_{ij}) = 0$ and M is cofaithful.

COROLLARY (1.2). *The condition that every faithful left ideal of a ring is cofaithful is Morita invariant.*

Proof. By Beachy **[2]** a module is faithful if and only if it cogenerates every projective module and it is cofaithful if and only if it generates every injective module. A module is torsionless if and only if it is cogenerated by every faithful module. Since the classes of faithful, cofaithful and torsionless modules are all invariant under an equivalence of module categories, the result follows from condition (c) above.

The next two propositions show that our condition implies certain finiteness conditions, although it is much weaker than the descending chain condition for left annihilators. In particular, a commutative, semiprime ring satisfying the condition has finite uniform dimension, and so it must be Goldie.

PROPOSITION (1.3). *Let R be a ring such that every faithful left ideal is cofaithful.*

(a) *R is not a direct product of infinitely many (nontrivial) rings.*

(b) *If R is semiprime, then it contains no infinite direct sum of nonzero ideals.*

Proof. (a) Suppose that R is an infinite direct product of rings. Let A be the set of all elements which are zero in all but finitely many components. Then A is faithful but not cofaithful.

(b) Assume that $A = A_1 \oplus A_2 \oplus \cdots$ is an infinite direct sum of ideals. If R is semiprime, then $A \cap \ell(A) = 0$ and $A \oplus \ell(A)$ is faithful, so by assumption there exist $x_1, \cdots, x_k \in A \oplus \ell(A)$ such that $\bigcap_{i=1}^{k} \ell(x_i) = 0$. But there exists an integer n such that $x_i \in A_1 \oplus \cdots \oplus A_n \oplus \ell(A)$ for all i, and so for any $0 \neq y \in A_{n+1}$ we have $y \in \bigcap_{i=1}^{k} \ell(x_i)$, a contradiction.

PROPOSITION (1.4) (Faith [5]). *A ring R has the descending chain condition on left annihilators if and only if for every subset S of R there exists a finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq S$ such that $\ell_R(x_1, \cdots, x_n) = \ell_R(S)$.*

Proof. If R satisfies the descending chain condition on left annihilators, choose $\{x_1, x_2, \cdots, x_n\}$ so that $\ell(x_1, \cdots, x_n)$ is minimal in the set of all left annihilators of finite subsets of S.

Conversely let $A_1 \supseteq A_2 \supseteq \cdots$ be a descending chain of left annihilators and let $S = \bigcup \ell(A_i)$. Then there exists a subset $\{x_1, x_2, \cdots, x_n\} \subseteq S$ so that $\ell(x_1, x_2, \cdots, x_n) = \ell(S)$. There exists a positive integer k such that $\ell(A_i) \supseteq \{x_1, x_2, \cdots, x_n\}$ for all $i \geqq k$. But for $i \geqq k$, $A_i = \ell(\ell(A_i)) \subseteq \ell(x_1, \cdots, x_n) = \ell(S) \subseteq A_i$, so $A_i = \ell(S)$ and the chain terminates at A_k.

We remark that Handelman and Lawrence [11] have given an example of a prime ring in which every (faithful) left ideal is cofaithful but which does not have the analogous property for right ideals. We next give some examples to show the relationship between this condition and various other finiteness or chain conditions.

A left Noetherian ring need not satisfy our condition, as is shown by the following example due to Small **[16]**. Let R be a simple left Noetherian domain which is not a division ring, let F be the field which is the center of R, and let K be a nonzero left ideal of R. Set $M = R/K$ and let S be the ring of all matrices $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $a \in F$, $b \in M$ and $c \in R$. It is easily seen that S is left Noetherian, and following Small one can show that given any finite subset of M, say $\{m_1, \cdots, m_t\}$, there exists $0 \neq d \in R$ such that $dm_i = 0$ for $i = 1, \cdots, t$. Thus $I = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| \, a \in F, b \in M \right\}$ is a faithful left ideal of S which is not cofaithful.

On the other hand, a left Noetherian ring which is integral over its center has the property that every faithful left ideal is cofaithful since it is a subring of a left Artinian ring (see Blair **[4]**). Also in the positive direction, if R is left Noetherian and $Z(R) = 0$ (e.g. if R is left hereditary) then our condition holds.

The ring $R = \{(n, a) \mid n \in \mathbf{Z}, a \in \mathbf{Z}_{p^\infty}\}$, where $\mathbf{Z}_{p^\infty}$ is Prufer's quasi-cyclic group and multiplication is given by $(n, a)(m, b) = (nm, nb + ma)$, provides an example of a commutative ring with finite uniform dimension for which every faithful ideal is cofaithful (since R is essentially Artinian), but it can be checked that R does not satisfy the chain condition on annihilators.

The next proposition provides many more examples.

PROPOSITION (1.5). *If R has a left classical quotient ring Q which is essentially left Artinian, then R has finite uniform dimension and every faithful left ideal is cofaithful.*

Proof. Let A be an essential Artinian left ideal of Q. If $B_1 \oplus B_2 \oplus \cdots \oplus B_k$ is a direct sum of left ideals of R, then by standard quotient ring techniques $QB_1 \oplus QB_2 \oplus \cdots \oplus QB_k$ is a direct sum of left ideals of Q and so if each $QB_i \neq 0$ then $(QB_1 \cap A) \oplus (QB_2 \cap A) \oplus \cdots \oplus (QB_k \cap A)$ is a direct sum of nonzero left ideals in A. Since A is left Artinian such direct sums must be finite and thus R has finite uniform dimension.

If B is a faithful left ideal of R, then $\bigcap_{b \in B} \ell_Q(b) = 0$ since $\bigcap_{b \in B} \ell_Q(b) \cap R = \bigcap_{b \in B} \ell_R(b) = 0$. Since A is Artinian,

$$\bigcap_{i=1}^{n} (A \cap \ell_Q(b_i)) = 0$$

for some finite subset $b_1, b_2, \cdots, b_n$ of B, and then $\bigcap_{i=1}^{n} \ell_R(b_i) \subseteq \bigcap_{i=1}^{n} \ell_Q(b_i) = 0$ since A is essential.

We say that a ring has enough uniforms if every nonzero left ideal contains a uniform left ideal. If a ring has finite uniform dimension then it has enough uniforms. An infinite direct product of copies of **Z** shows that a ring may have enough uniforms without having finite uniform dimension.

PROPOSITION (1.6). *If R is a ring with enough uniforms, then every nonzero submodule of a free R-module has a uniform submodule.*

Proof. Let F be a free R-module and $M \neq 0$ a submodule. Without loss of generality we may assume M is cyclic, in which case we may also assume F is finitely generated. Let $F = R^n$, and p_n be the projection onto the last summand. If the restriction of p_n to M is a monomorphism then an isomorphic copy of M is contained in R and so M contains a uniform submodule. If p_n restricted to M is not a monomorphism then $M \cap R^{n-1} \neq 0$ and we complete the proof by induction.

COROLLARY (1.7). *The condition that a ring have enough uniforms is Morita invariant.*

In the course of proving that finite uniform dimension goes up to polynomial rings, Shock **[15]** showed that if U is a uniform left ideal of R then $U[x]$ is a uniform left ideal of $R[x]$. With only slight modification Shock's proof shows, in fact, that if ${}_RM$ is a uniform R-module then $M[x] (\simeq R[x] \otimes_R M)$ is a uniform left $R[x]$-module.

PROPOSITION (1.8). *If R has enough uniforms, then $R[x]$ has enough uniforms.*

Proof. Let I be an ideal of $R[x]$, where R has enough uniforms. As an R-submodule of the free R-module $R[x]$, I contains a uniform R-submodule M by Proposition 1.6. Thus there exists $q(x) \in I$ such that $Rq(x)$ is a uniform R-submodule of I. By multiplying $q(x)$ by appropriate elements of R we may assume that the left annihilators of all the nonzero coefficients of $q(x)$ are the same. This "new" $q(x)$ is also an element of $M \subseteq I$, so $Rq(x)$ remains a uniform R-module. Since the left annihilators of the nonzero coefficients of $q(x)$ are all the same we have $R[x]q(x) \simeq R[x] \otimes_R Rq(x)$. By the remarks before the theorem this shows that $R[x]q(x)$ is a uniform left ideal of $R[x]$ contained in I.

PROPOSITION (1.9). *Let R be a commutative ring in which every faithful ideal is cofaithful. Then every faithful ideal of $R[x]$ is cofaithful.*

Proof. Let I be a faithful ideal of the ring $R[x]$ and let I_0 be the ideal of R generated by the coefficients of the polynomials in I. Clearly I_0 is a faithful ideal of R. Thus there exist $a_1, a_2, \cdots, a_t \in I_0$ such that $\ell_R\{a_1, a_2, \cdots, a_t\} = 0$. Let $f_i(x)$ be a polynomial of I in which a_i appears. Let $\deg f_i(x) = n_i$ and set $n_0 = 0$. We show $\ell_{R[x]}(f_1(x), \cdots, f_t(x)) = 0$. If not, and say $h(x)f_i(x) = 0$ for $i = 1, \cdots, t$, then set $m_i = \Sigma_{j=0}^{i-1}(n_i + 1)$ and $g(x) = \Sigma_{i=1}^{t} f_i(x)x^{m_i}$. Now $h(x)g(x) = 0$, and since R is commutative, 6.13 of Nagata **[13]** shows that there exists $0 \neq c \in R$ such that $cg(x) = 0$, and so $ca_i = 0$ for $i = 1, \cdots, t$, a contradiction.

We remark that Theorems 1.8 and 1.9 are true for polynomial rings in a finite number of indeterminants by induction, and then due to the "local" nature of the conditions the results hold for polynomial rings in an arbitrary number of indeterminants.

2. Orders in semisimple Artinian rings.

THEOREM (2.1). *The ring R is semisimple (simple) Artinian if and only if R is semiprime (prime), every nonzero left ideal contains a minimal left ideal, and every faithful left ideal is cofaithful.*

Proof. Assume that R is semiprime, every faithful left ideal is cofaithful, and every nonzero left ideal contains a minimal left ideal, and let S be the sum of all minimal left ideals of R. Then by assumption S is essential in R, and hence faithful since R is semiprime. Thus S must be cofaithful, and so there exists an exact sequence $0 \to R \to S^k$ for some positive integer k. This shows that ${}_RR$ is completely reducible, and therefore semisimple Artinian.

In analogous fashion we are able to characterize orders in semisimple Artinian rings by merely requiring enough uniform left ideals instead of enough minimal left ideals as in Theorem 2.1. We study the prime case first. (Recall that a ring is prime if and only if every nonzero left ideal is faithful.)

THEOREM (2.2). *The ring R is a left order in a simple Artinian ring if and only if R is prime, contains a uniform left ideal, and every nonzero left ideal is cofaithful.*

Proof. If R is an order in a simple Artinian ring then every left ideal is cofaithful and R contains a uniform left ideal by Proposition 1.5.

Conversely, if $Z(R)$, the singular ideal of R, is nonzero, then there is an exact sequence $0 \to R \to Z(R)^k$ for some positive integer k. This implies $Z(R) = R$, a contradiction. Since $Z(R) = 0$, in order to show that R is left Goldie it suffices to show that R has finite uniform dimension. To see this, let U be a uniform left ideal of R. Then R has finite uniform dimension, since for some positive integer k there exists an exact sequence $0 \to R \to U^k$.

We next give a proof of Theorem 2.2 which avoids Goldie's Theorem and simultaneously produces the full matrix ring over a division ring in which the ring is a left order. The proof is inspired by the proof of Faith's Theorem 34 **[6]**. We first observe that if the left uniform dimension of R is n and $0 \to R \to M^m$ is exact for some positive integer m, then there exists an exact sequence $0 \to R \to M^k$ with $k \leqq n$.

THEOREM (2.2 bis). *If the ring R is prime, contains a uniform left ideal, and every nonzero left ideal is cofaithful, then $Q_{cl}(R)$, the left classical quotient ring of R, is an $n \times n$ matrix ring over a division ring.*

Proof. As in the proof of Theorem 2.2, $Z(R) = 0$ and R has finite uniform dimension, say $\dim R = n$. Furthermore, there exists an exact sequence $0 \to R \to U^n$ where U is a uniform left ideal of R. Let V be the quasi-injective hull of U. Since $Z(R) = 0$, $Z(U) = 0$ and U is strongly uniform in the sense of Storrer **[17]**. By Lemma 7.4 of Storrer **[17]**, $D = \mathrm{End}_R(V)$ is a division ring. By Proposition 13 of Faith **[6]**, V is in fact injective. We claim that V has dimension n as a vector space over D. There exists an exact sequence $0 \to R \to V^n$, and if $(v_1, v_2, \cdots, v_n)$ is the image of $1 \in R$ in V^n, we show $\{v_1, \cdots, v_n\}$ is a basis for V over D. Let $v \in V$ and $f: R \to V$ be the map given by $f(r) = rv$. By the injectivity of V^n this extends to a map f': $V^n \to V$ with components $d_i \in D$. Hence $v = f(1) = f'(v_1, \cdots, v_n) = \Sigma_{i=1}^n d_i v_i$. Thus $v_1, \cdots, v_n$ span V. If, on the other hand, $\Sigma_{i=1}^n d_i v_i = 0$, with say $d_j v_j \neq 0$, then $rv_i = 0$ for $i \neq j$ implies $rv_j = 0$ and there is a monomorphism from R into V^{n-1}, so, since V is a uniform R-module, this contradicts the fact that $\dim(R) = n$. If $Q = \mathrm{End}_D(V)$, then Q is the ring of $n \times n$ matrices over D^{opp}, and there is a natural embedding $R \subseteq Q$, since V is faithful. Now V is isomorphic as a Q-module to a minimal left ideal of Q and ${}_QQ \simeq {}_RV^n$, which implies that R is essential in Q. Furthermore, if V_i is the intersection of R and the ith component of Q, then V_i is nonzero for otherwise we have an embedding of R in V^{n-1} and contradict $\dim(R) = n$.

Let A be an essential left ideal of R, and let $A_i = A \cap V_i \neq 0$. Then $A_1 A_i \neq 0$, since R is prime, so there exists $a_i \in A_i$ with $A_1 a_i \neq 0$. Thus we may define a nonzero homomorphism $f\colon A_1 \to A_i \subseteq V$ by $f(a) = aa_i$ for $a \in A_1$. Since U is essential in V and $Z(U) = 0$, we have $Z(V) = 0$. If $\ker(f) \neq 0$, then $\ker(f)$ is an essential submodule of V and if $x \in A_1 \subseteq V$ then $\{r \in R \mid rf(x) = 0\}$ is an essential left ideal of R, which implies $f(x) \in Z(V) = 0$ and $f = 0$, a contradiction. Hence there exists an exact sequence

$$0 \to R \to A_1^n \to \bigoplus_{i=1}^{n} A_i \subseteq A,$$

and so there exists $x \in A$ such that $\ell_R(x) = 0$. Since R is essential in Q, $\ell_Q(x) = 0$, and since Q is left Artinian x must be invertible in Q and hence regular in R. This shows that every essential left ideal of R contains a regular element, so if $q \in Q$, then $(R\colon q) = \{r \in R \mid rq \in R\}$ is an essential left ideal of R since R is essential in Q, and thus $(R\colon q)$ contains a regular element x. Hence $xq = r \in R$ and so $q = x^{-1}r$. Thus R is a left order in Q.

LEMMA (2.3). *Let R be a semiprime ring and U a uniform left ideal of R. Then $P = \ell_R(U)$ is a prime ideal of R, and the image of U in R/P is a uniform left ideal of R/P.*

Proof. Let A and B be left ideals of R such that $AB \subseteq P$. Then $ABU = 0$ and so $BUA = 0$ for otherwise $(BUA)^2 = 0$, while $BUA \neq 0$. Hence $BU \cdot AU = 0$ and so $BU \cap AU = 0$ since R is semiprime. Since U is uniform $BU = 0$ or $AU = 0$ and so $A \subseteq P$ or $B \subseteq P$. Since $P \cap U = 0$, it is easy to see that the image of U in R/P is again uniform.

LEMMA (2.4). *Let R be a semiprime ring in which every faithful left ideal is cofaithful and let S be a left ideal of R. If $A = \ell_R(S)$, then every faithful left ideal of R/A is cofaithful.*

Proof. Let I be the ideal SR. Then $A = \ell_R(I)$, and since R is semiprime, $A = \ell_R(I)$. Let B/A be a faithful left ideal in R/A. If $C = \ell_R(B)$ then $CB = 0$ and so $CB \subseteq A$; hence $C \subseteq A$. Thus $C^2 = 0$, since $C \subseteq A \subseteq B$. Since R is semiprime, $C = 0$ and B is a faithful left ideal of R. By hypothesis there exist $b_1, \cdots, b_t \in B$ such that $rb_i = 0$ for $i = 1, \cdots, t$ implies that $r = 0$. Let $\bar{b}_i$ be the image of b_i in

B/A. Suppose $\bar{r} \in R/A$ is such that $\bar{r}\bar{b}_i = 0$ for $i = 1, \cdots, t$. Then $rb_i \in A$ for $i = 1, \cdots, t$ and $rb_i I = 0$, so $Irb_i = 0$ for $i = 1, \cdots, t$. Hence $Ir = 0$ and $r \in \ell_R(I) = A$. Thus $\bar{r} = 0$ and $\ell_{R/A}(\bar{b}_1, \cdots, \bar{b}_t) = 0$.

THEOREM (2.5). *The ring R is a left order in a semisimple Artinian ring if and only if R is semiprime, has enough uniform left ideals, and every faithful left ideal is cofaithful.*

Proof. Assume R is semiprime, has enough uniforms and every faithful left ideal is cofaithful. Let A be the sum in R of all uniform left ideals and let M be the external direct sum of these left ideals. Since R is semiprime and A is essential, A is a faithful left ideal. Thus M is a faithful torsionless left R-module and by Proposition 1.1 it is cofaithful. Hence there exists an exact sequence $0 \to R \xrightarrow{f} M^k$ for some k. Since $1 \in R, f(1)$ belongs to a finite direct sum of uniform R-modules and so R is isomorphic to a submodule of a finite dimensional module. This shows that R has finite uniform dimension.

Let $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ be a maximal direct sum of uniform left ideals of R. It is clear that U is a faithful left ideal of R and $\bigcap_{i=1}^m P_i = 0$, where $\{P_i\}$ is the set of distinct elements of $\{\ell(U_i)\}$. Thus the canonical map $\phi: R \to R/P_1 \oplus \cdots \oplus R/P_m$ is a monomorphism. By Lemmas 2.3 and 2.4, R/P_i is a prime ring which contains a uniform left ideal and in which every left ideal is cofaithful. By Theorem 2.2, R/P_i is a left order in a simple left Artinian ring S_i. One can show directly that $S = S_1 \oplus \cdots \oplus S_m$ is the left classical quotient ring of $\phi(R)$, or else since R is a subring of the left Artinian ring S, R has the ascending chain condition on left annihilators and we can apply Goldie's theorem.

The Gabriel dimension of a ring is defined by Gordon and Robson **[10]** in terms of localizing Serre subcategories. From Corollary 2.10 of **[10]**, one can easily show that a ring with Gabriel dimension has enough uniforms. Thus we are able to state the following corollary of Theorem 2.5.

COROLLARY (2.6). *A semiprime ring with Gabriel dimension is left Goldie if and only if every faithful left ideal is cofaithful.*

We end this section with a theorem on semiprime rings with a polynomial identity. Our result weakens the hypothesis of theorems of Armendariz-Steinberg, Formanek, Rowen and Small.

THEOREM (2.7). *Let R be a semiprime ring with a polynomial identity and center C, where C satisfies the condition that every faithful ideal is cofaithful. Let S be the set of regular elements of C. Then R is an order in a semisimple Artinian ring and*

(1) $S^{-1}C = F_1 \oplus \cdots \oplus F_k$, *a finite direct product of fields.*

(2) $S^{-1}R = Q_1 \oplus \cdots \oplus Q_k$, *where Q_i is a finite-dimensional central simple algebra with center F_i.*

Proof. Apply Proposition 1.3(b) to the semiprime ring C to see that C satisfies the ascending chain condition on annihilators. The result now follows from Theorem 9 of Formanek [7].

3. Absolutely torsion-free rings. A left exact subfunctor of the identity Id on ${}_R\mathcal{M}$, the category of left R-modules, is called a torsion preradical. For torsion preradicals ρ and σ we write $\rho \leqq \sigma$ if $\rho(M) \subseteq \sigma(M)$ for all $M \in {}_R\mathcal{M}$. Observe that $\sigma(R) = R$ if and only if $\sigma = Id$. For $M \in {}_R\mathcal{M}$, let Rad^M be the smallest torsion preradical σ such that $\sigma(M) = M$. Then for $X \in {}_R\mathcal{M}$, $\mathrm{Rad}^M(X) = \{x \in X \mid x = \sum_{i=1}^n f_i(m_i)$ for $m_i \in M$, $f_i \in \mathrm{Hom}_R(Rm_i, X)\}$, and it follows that $\mathrm{Rad}^M = Id$ if and only if M is cofaithful by Beachy [3].

We recall that a module is said to be prime if for all nonzero submodules $M' \subseteq M$, $AM' = 0$ implies $AM = 0$ for all left ideals A of R. A submodule will be called fully invariant if it is invariant under all endomorphisms. The injective envelope of a module M is denoted $E(M)$.

PROPOSITION (3.1). *The following are equivalent for $M \in {}_R\mathcal{M}$.*

(a) *For all torsion preradicals σ of ${}_R\mathcal{M}$, either $\sigma(M) = 0$ or $\sigma(M) = M$.*

(b) *M is contained in every nonzero fully invariant submodule of $E(M)$.*

(c) *For all $0 \neq x \in M$ and $y \in M$ there exist $r_1, r_2, \cdots, r_n \in R$ such that $\bigcap_{i=1}^n \mathrm{Ann}(r_i x) \subseteq \mathrm{Ann}(y)$.*

(d) *M is prime and if $0 \neq M' \subseteq M$ then for all $y \in M$ there exist $x_1, x_2, \cdots, x_n \in M'$ such that $\bigcap_{i=1}^n \mathrm{Ann}(x_i) \subseteq \mathrm{Ann}(y)$.*

Proof. (a) $\Rightarrow$ (b). If $0 \neq N \subseteq E(M)$ is fully invariant, then $\mathrm{Rad}^N(E(M)) = N$, so $\mathrm{Rad}^N(M) = M \cap \mathrm{Rad}^N(E(M)) = M \cap N \neq 0$, and thus we have $\mathrm{Rad}^N(M) = M$ and so $M \subseteq N$.

(b) $\Rightarrow$ (c). For $0 \neq x \in M$, let N be the sum in $E(M)$ of the homomorphic images of Rx. Then N is fully invariant, so by assumption $M \subseteq N$ and thus $y = \sum_{i=1}^n f_i(r_i x)$ for $r_i \in R$ and $f_i \in \mathrm{Hom}_R(Rx, E(M))$. Therefore $ar_i x = 0$ for all i implies $ay = 0$.

(c) $\Rightarrow$ (d). If $0 \neq M' \subseteq M$, and $AM' = 0$ for some $A \subseteq R$, then let $0 \neq x \in M'$. By assumption for any $y \in M$ there exist $r_1, r_2, \cdots, r_n \in R$ such that $A \subseteq \bigcap_{i=1}^{n} \mathrm{Ann}(r_i x) \subseteq \mathrm{Ann}(y)$, so $AM = 0$ and M is prime. The second condition follows immediately from (c).

(d) $\Rightarrow$ (a). If $0 \neq \sigma(M) = N$ for some torsion preradical σ, then for any $y \in M$ there exist $x_1, x_2, \cdots, x_k \in N$ such that $\bigcap_{i=1}^{k} \mathrm{Ann}(x_i) \subseteq \mathrm{Ann}(y)$. Thus for $x = (x_1, \cdots, x_k) \in N^k$, the mapping $f\colon Rx \to Ry$ defined by $f(ax) = ay$ is a well-defined homomorphism, and since $x \in \sigma(N^k)$ we must have $y = f(x) \in \sigma(M)$. This shows that $\sigma(M) = M$, completing the proof.

Taking $y = 1 \in R$ in condition (d) of Proposition 3.1 shows that ${}_RR$ satisfies the equivalent conditions of the proposition if and only if R is prime and every (faithful) left ideal is cofaithful. Condition (a) is satisfied if and only if $\sigma(R) = 0$ for every torsion preradical σ such that $\sigma \neq Id$; such rings are the absolutely torsion-free rings studied by Rubin **[14]**. Taking $y = 1$ in condition (c) shows that R is absolutely torsion-free if and only if for all $0 \neq r \in R$ there exist $r_1, \cdots, r_n \in R$ such that $sr_ir = 0$ for all i implies $s = 0$, and this gives the condition studied by Handelman and Lawrence **[11]**. It also gives the equivalent condition that every nonzero left ideal is cofaithful, as shown by Viola-Prioli **[18]**, Theorem 1.1.

Many of Rubin's results on absolutely torsion-free rings are easier to prove in the light of Proposition 3.1. A prime left Goldie ring is absolutely torsion-free on the left and right (Rubin **[14]**, Theorem 1.11) since it satisfies the descending chain condition on both left and right annihilators. Since being prime and having every faithful left ideal cofaithful are both Morita invariant, so is being absolutely torsion-free (Rubin **[14]**, Theorem 1.12). Applied to ${}_RR$ condition (b) of Proposition 3.1 states that $E(R)$ has no nontrivial invariant submodules. If $S \supseteq R$ is a subring of the complete ring of quotients of R, then $E({}_SS) = E({}_RR)$ and the condition implies that S is absolutely torsion-free whenever R is (Rubin **[14]**, Theorem 1.15).

PROPOSITION (3.2). *A ring R is left absolutely torsion-free if and only if R is prime, $Z(R) = 0$, and every nonsingular quasi-injective left R-module is injective.*

Proof. Assume that R is left absolutely torsion-free and that $0 \neq {}_RM$ is quasi-injective with $Z(M) = 0$. Then $\mathrm{Rad}^M(M) \neq 0$ implies that $\mathrm{Rad}^M = Id$ by Violi-Prioli **[18]** Theorem 1.1, so M is cofaithful and hence injective.

Conversely, let M be a fully invariant submodule of $E(R)$. Then M is quasi-injective and nonsingular since by assumption $Z(R) = 0$, so M must be injective and thus a direct summand of $E(R)$, say $E(R) = M \oplus N$. But $M \cap R$ is an ideal since M is fully invariant in $E(R)$, so $(M \cap R) \cdot (N \cap R) = 0$ and this implies that $N \cap R = 0$ since R is prime. Thus $N = 0$ since R is essential in $E(R)$, so $M = E(R)$ and it follows from Proposition 3.1 that R is absolutely torsion-free.

Finally, as a consequence of Proposition 3.1 we have the following restatement of Theorem 2.2.

THEOREM (3.3). *A left absolutely torsion-free ring is left Goldie if and only if it has a uniform left ideal.*

We call a module M semicompressible if for all nonzero submodules $N \subseteq M$ there exists an exact sequence $0 \to M \to N^k$ for some positive integer k. (Note that a semicompressible module satisfies the conditions of Proposition 3.1.) The following proposition can be generalized easily to quasi-projective semicompressible modules.

PROPOSITION (3.4). *The endomorphism ring of a projective, semicompressible left R-module is left absolutely torsion-free.*

Proof. Let ${}_RM$ be semicompressible and projective and let $\mathrm{End}_R(M)$ act on the left of M. We will show that $\mathrm{End}_R(M)$ satisfies condition (c) of Proposition 3.1. Let $f, g \in \mathrm{End}_R(M)$, $f \neq 0$, $g \neq 0$. Since $g(M) \neq 0$ and M is semicompressible, there exists a positive integer k and a monomorphism $j : M \to (g(M))^k$. Let $p : M^k \to (g(M))^k$ be the homomorphism with components $p_i = g$. Since M is projective, j lifts to a map $h : M \to M^k$ with components h_i:

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \arrow[d, "j"] & \\ M^k \arrow[r] & (g(M))^k \arrow[r] & 0. \end{tikzcd}$$

Then $phf = fj \neq 0$ since j is monic and $f \neq 0$, so $gh_if = (ph)_if \neq 0$ for some component $(ph)_i$ of ph. Hence $\ell((ph)_if) \subseteq \ell(g)$.

THEOREM (3.5). *The ring of endomorphisms of a finitely generated projective module over a left absolutely torsion-free ring is left absolutely torsion-free.*

Proof. Let R be left absolutely torsion-free. Then ${}_RR$ is semicompressible, so the result will follow from Proposition 3.4 if we can show that any finitely generated free module over R is semicompressible, since a submodule of a semicompressible module is semicompressible. *More generally, we show that if ${}_RM$ is semicompressible, then M^n is also.* For this purpose let $0 \neq N \subseteq M^n$ and let p_n be the projection of M^n onto the last component. If p_n is monic when restricted to N, then since M is semicompressible there exists k such that $0 \to M \to (p_n(N))^k \simeq N^k$ is exact and so $0 \to M^n \to (N^k)^n$ is exact. If p_n is not monic on N then $N \cap M^{n-1} \neq 0$ and the above argument can be applied to $N \cap M^{n-1}$. Continuing we see that there exists an embedding $0 \to M^n \to N^t$ for some t and M^n is semicompressible.

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