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CAUCHY TRANSFORMS AND CHARACTERISTIC FUNCTIONS

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The following problem arises in the study of rational approximation: classify all plane sets E such that $\hat{\mu}(z) \equiv \int d\mu\,(\zeta)/(\zeta - z) = \chi_E(z)$ area almost everywhere for some complex Borel measure μ. A partial solution to this problem for compact sets is given here. The main result is the following.

THEOREM. **Let K be a compact plane set with connected dense interior. Then there is a measure μ such that $\hat{\mu} = \chi_K$ area a.e., if and only if K has finite Painlevé length.**

1. Introduction. Throughout this paper, the word "measure" will mean a complex Borel measure supported on the complex plane **C**.. If μ is a compactly supported measure, we define the *Newtonian potential* of μ by the formula

$$U_{|\mu|}(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|}.$$

It is well known that $U_{|\mu|}$ is finite $dxdy$ a.e. For each z such that $U_{|\mu|}(z) < \infty$ we define the *Cauchy transform* of μ by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}$$

The Cauchy transform is thus defined almost everywhere. We seek compact sets K such that $\chi_K = \hat{\mu}\ dxdy$ a.e., for some μ.

It is easy to see that we may assume that K is connected. For, let $K = K_1 \cup K_2$ with K_1 and K_2 closed and disjoint. Let $\hat{\mu} = \chi_K$ a.e., write μ_i for $\mu|_{K_i}$, $i = 1, 2$ and define a function

$$f = \begin{cases} \hat{\mu}_1 & \text{on } \mathbf{C} - K_1 \\ -\hat{\mu}_2 & \text{on } \mathbf{C} - K_2 \end{cases}.$$

By Liouville's theorem, $f \equiv 0$. It follows easily that $\hat{\mu}_1 = \chi_{K_1}$ and $\hat{\mu}_2 = \chi_{K_2}$.

For a compact $K \subseteq \mathbf{C}$ we denote by $R(K)$ the Banach algebra of continuous functions on K which are uniform limits of rational functions with poles off K. It is well known ([4]) that $\hat{\mu} = 0$ on $\mathbf{C} - K$ if and only if $\mu \in R(K)^{\perp}$.

2. Painlevé Length.

By a *regular* neighborhood of a compact plane set K we mean an open set $V \supseteq K$ such that ∂V consists of finitely many rectifiable curves surrounding K in the usual sense of contour integration. We say that K has finite Painlevé length if there is a number l such that every open $U \supseteq K$ contains a regular neighborhood V of K such that ∂V has length at most l. The infimum of such numbers l is called the *Painlevé length* of K.

The following theorem is well known, but we include a proof for completeness.

2.1. THEOREM. *Let K be a compact connected plane set with Painlevé length $\kappa < \infty$. Then there is a measure μ such that $\hat{\mu} = \chi_K\, dxdy$ a.e.*

Proof. Let $\{U_n\}$ be a decreasing sequence of open sets such that
(i) $K = \bigcap_{n=1}^{\infty} U_n$
(ii) ∂U_j is a rectifiable curve for each j
(iii) Length $\partial U_j < \kappa + \frac{1}{j}$.

Define $\mu_j = 1/2\pi i\, dz$ on ∂U_j for each j. The sequence $\{\mu_n\}$ is bounded and hence a subsequence, again labeled $\{\mu_n\}$, converges weak-star to a limit μ.

For any $\phi \in C_0^{\infty}$, we have

$$\begin{aligned}
&\frac{1}{\pi}\int\int \frac{\partial \phi}{\partial \bar{z}} \hat{\mu}(z)\, dxdy \\
&\quad = \int\left(-\frac{1}{\pi}\int\int \frac{\partial \phi}{\partial \bar{z}} \frac{1}{z-\zeta}\, dxdy\right) d\mu(\zeta) \\
&\quad = \int \phi(\zeta)\, d\mu(\zeta) = \lim_n \int \phi(\zeta)\, d\mu_n(\zeta) \\
&\quad = \lim_n \frac{1}{\pi}\int\int_{U_n} \frac{\partial \phi}{\partial \bar{z}}\, dxdy \\
&\quad = \frac{1}{\pi}\int\int_{K} \frac{\partial \phi}{\partial \bar{z}}\, dxdy
\end{aligned}$$

using the theorems of Green and Fubini. It follows easily that $\hat{\mu} = \chi_K$ a.e.

The converse of this theorem is not true. This is easily seen by taking a closed disc, for example, and attaching a set with zero area but infinite Painlevé length. The converse can also fail when $K = \overline{K^0}$, as the following example shows.

2.2. EXAMPLE. Let $\{x_i\}_{i=1}^{\infty}$ be an enumeration of the rationals in (0,1), let $\{r_i\}_{i=1}^{\infty}$ be any monotone decreasing sequence of positive numbers such that $\Sigma_{i=0}^{\infty} r_i < \infty$, and let $K_0 = \{(x, y): x \in (0, 1), y = x \sin 1/x\} \cup (0, 0)$. We note that K_0 has infinite length.

Let $K = K_0 \cup \bigcup_{n=1}^{\infty} \bar{\Delta}(P_n ; r_n)$, where $P_n = (x_n, x_n \sin 1/x_n)$ and the x_n, r_n are chosen inductively so that

(i) $\bar{\Delta}(P_i ; r_i \cap \bar{\Delta}(P_j ; r_i) = \phi$ for $i \neq j$

(ii) $K_0 \cap \bar{\Delta}(P_j ; r_j)$ is connected for each j

(iii) $\left\{x \in R : \left(x, x \sin \frac{1}{x}\right) \in K_0\right\} - \bigcup_{n=1}^{\infty} \bar{\Delta}(P_n ; r_n)$ contains no interval.

Evidently $K = \overline{K^0}$ and K has infinite Painlevé length. But if we let $\mu = 1/2\pi i\, dz$ on the boundaries of the $\Delta(P_n ; r_n)$, we have $\hat{\mu} = \chi_K$ a.e.

The interior of the compact set in this example is dense, but not connected. In the next section we show that if K^0 is connected and dense in K, and if there is a measure μ such that $\hat{\mu} = \chi_K$ a.e., then K must have finite Painlevé length.

3. Wermer's theorem and some extensions. The following theorem of John Wermer appears as a solution to a problem in [7].

THEOREM. *Let U be the region bounded by a Jordan curve Γ and assume there is a measure μ on Γ such that $\hat{\mu}(z) = 1$ for $z \in U$, $\hat{\mu}(z) = 0$ for $z \notin \Gamma \cup U$. Then Γ is rectifiable.*

We obtain some more general results, using ideas from Ahern and Sarason ([1]), Davie ([2]), and Gamelin and Garnett ([5]). However, many of the points in Wermer's original proof are retained.

The algebra $R(K)$ is called a *Dirichlet algebra* if it has no nonzero real annihilating measures.

Two points p_1 and p_2 of K are said to be in the same *Gleason part*, or simply *part*, of K if whenever $\{f_n\}$ is a sequence in $R(K)$ such that $\|f_n\|_K \leqq 1$ and $|f_n(p_1)| \to 1$, then also $|f_n(p_2)| \to 1$. This is an equivalence relation on K.

A discussion of the properties of Dirichlet algebras and parts may be found in **[4]**.

3.1 THEOREM. *Let K be a compact plane set such that $R(K)$ is a Dirichlet algebra. Assume μ is a measure such that $\hat{\mu} = 1$ on K^0, $\hat{\mu} = 0$ off K. Then the components $\{U_i\}_{i\in I}$ of K^0 are simply connected, ∂U_i is a rectifiable curve for each i, and $\Sigma_{i\in I}$ length $\partial U_i < \infty$. Furthermore $\mu = 1/2\pi i\, d\zeta$ on $\cup_{i\in I}\partial U_i$ with appropriate orientation.*

Proof. Theorem 5.1 of **[5]** implies that the components $\{U_i\}_{i\in I}$ of K^0 are simply connected, and Theorem 11.1 of **[5]** shows that the nontrivial parts of K are precisely the U_i. Glicksberg's decomposition theorem (VI 3.4 of **[4]**) then gives $\mu = \sum_{i\in I} \mu_i$ where μ_i is supported on $\bar{U}_i$ for each i. Theorem VI 3.3 of **[4]** implies that $\mu_i \in R(\bar{U}_i)^\perp$ for each i and it follows that $\hat{\mu}_i = 1$ on U_i, $\mu_i = 0$ off $\bar{U}_i$. It is easy to see that $R(\bar{U}_i)$ is Dirichlet for each i.

We may therefore restrict our attention to one pair (μ_i, U_i), which we relabel (μ, U). It is well known that μ is absolutely continuous with respect to harmonic measure for points in U, since $R(\bar{U})$ is Dirichlet.

By expanding $\hat{\mu}$ in a Laurent series, we obtain $\int_{\partial U} z^k d\mu\,(z) = \delta_{-1,k}$. We can assume $0 \in U$. Let ϕ be the Riemann map of $\Delta = \{|z| < 1\}$ onto U such that $\phi(0) = 0$. Write ρ_0 for harmonic measure at 0 on $\partial\Delta$, and λ_0 the same on ∂U.

LEMMA (Ahern-Sarason **[1]**; Davie **[2]**). *The function ϕ has a measurable extension ϕ^* to a subset E of $\partial\Delta$ of full measure such that ϕ^* is one-to-one on E with a measurable inverse. The operator T: $L^1\{\lambda_0\} \to L^1\{\rho_0\}$ defined by $Tf = f\circ\phi^*$ is an isometric isomorphism which maps $L^\infty\{\lambda_0\}$ isometrically onto $L^\infty\{\rho_0\}$.*

Claim I. The function $1/\phi^*$ is not in the $L^\infty\{\rho_0\}$ closure of the linear span of $\{\phi^{*k} : k \neq -1\}$. To see this, note that $\mu \ll \lambda_0$ implies $d\mu = g d\lambda_0$ for some $g \in L^1\{\lambda_0\}$ so that $Tg \in L^1\{\rho_0\}$. Now suppose there is a sequence $\{Q_j\}_{j=1}^\infty$ of linear combinations of $\{\phi^{*k} : k \neq -1\}$ which converges to $1/\phi^*$ in $L^\infty\{\rho_0\}$. Then also $Q_j Tg \to 1/\phi^*\, Tg$ in $L^1\{\rho_0\}$ and $T^{-1}\{Q_j\}g \to z^{-1}g$ in $L^1\{\lambda_0\}$. But $\int T^{-1}\{Q_j\}g d\lambda_0 = 0$ for all j and $\int z^{-1} g d\lambda_0 = 1$, a contradiction. This establishes the claim and shows that there is an $h \in L^1\{\rho_0\}$ such that $\int \phi^{*k}\bar{h} d\rho_0 = \delta_{-1,k}$.

LEMMA (Ahern-Sarason [1]). *Let* $f \in H^\infty(U)$. *Then there is a sequence* $\{h_n\}_{n=1}^\infty$ *in* $R(\bar{U})$, *with* $\|h_n\|_\infty \leqq \|f\|_\infty$ *for all* n, *such that* $\{h_n(z)\} \to f(z)$ *for all* $z \in U_0$.

Claim II. The equality $\int \zeta \bar{h}(\zeta) d\rho_0(\zeta) = 0$ holds. To prove this, apply the above lemma to ϕ^{-1}. By Mergelyan's theorem ([4]), $R(\bar{U})$ is equal to $P(\bar{U})$, the uniform closure in $C(\bar{U})$ of the polynomials in z. Hence, there is a bounded sequence $P_n(z)$ of polynomials converging pointwise to ϕ^{-1} in U. So $\{P_n(\phi(\zeta))\} \to \zeta$ for all $\zeta \in \Delta$. By Alaoglu's theorem, there is a subsequence, again labeled $\{P_n(\phi^*)\}$ which converges weak-star on $\partial\Delta$ to some Ψ, i.e., converges over L^1. We need only show $\Psi = \zeta$. For fixed k,

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi\left(e^{i\theta}\right) e^{ik\theta} d\theta = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} P_n\left(\phi^*\left(e^{i\theta}\right)\right) e^{ik\theta} d\theta = \delta_{-1,k}.$$

So Ψ and ζ have the same Fourier coefficients, and $\Psi = \zeta$. But now

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\partial\Delta} P_n(\phi^*(\zeta))\bar{h}(\zeta) d\rho_0(\zeta) \\ &= \int_{\partial\Delta} \zeta \bar{h}(\zeta) d\rho_0(\zeta). \\ &= \int_{\partial\Delta} \zeta \bar{h}(\zeta) d\rho_0(\zeta) \end{aligned}$$

which establishes the claim.

Similarly $\int \zeta^k \bar{h}(\zeta) d\rho_0(\zeta) = 0$ for all $k \geqq 0$,and by the F. and M. Riesz theorem, $\bar{h} d\rho_0 = w dz$, $w \in H^1$. Then for any k, $0 < r < 1$,

$$\int_{|z|=r} \phi^k(z) w(z) dz = \int_{|z|=1} \phi^{*k}(z) w(z) dz = \delta_{-1,k}$$

But also $1/2\pi i \int_{|z|=r} \phi^k(z)\, \phi'(z)\, dz = \delta_{-1,k}$, so $(w(z) - \phi'(z)/2\pi i)\, dz$ annihilates all integral powers of ϕ^*, hence all integral powers of z, so that $w(z) = \phi'(z)/2\pi i$, and $\phi' \in H^1$. This implies that ∂U is a rectifiable Jordan curve (see e.g., [3], p. 44). The theorem is now clear.

By similar methods we can prove:

3.2 THEOREM. *Let* K *be a compact plane set such that* $\operatorname{Re}(R(K))$ *has finite defect in* $C_R(\partial K)$. *Then the components* $\{U_i\}_{i\in I}$ *of* K° *are finitely connected and there is a measure* μ *on* ∂K *such that* $\hat{\mu} = 1$ *on* K°, $\hat{\mu} = 0$ *off* K *if and only if the following three conditions hold.*

(i) *For each i, ∂U_i is a cycle composed of rectifiable curves.*

(ii) $\Sigma_{i\in I}$ length $\partial U_i < \infty$

(iii) $\mu = \dfrac{1}{2\pi i} d\zeta$ *on* $\cup_{i\in I} \partial U$ *with appropriate orientation.*

3.4 THEOREM. *Let K be a compact plane set with connected dense interior. Then there is a measure μ with $\hat{\mu} = 1$ on K^0, $\hat{\mu} = 0$ off K if and only if*

(i) *The components of* $\mathbf{C} - K$ *are bounded by rectifiable* curves $\{\gamma_i\}_{i\in I}$ *with finite total length and*

(ii) $\mu = 1/2\pi i\, d\zeta$ *on* $\cup_{i\in I}\gamma_i$ *with appropriate orientation.*

Proof. As before, the sufficiency of the two conditions is obvious. To prove the necessity, let Δ be a large disk containing K, and let $\lambda = 1/2\pi i\, d\zeta|_{\partial\Delta} - \mu$. Then $\hat{\lambda} = 1$ on $(\bar{\Delta} - K^0)^0 = \Delta - K$, and $\hat{\lambda} = 0$ off $\bar{\Delta} = K^0$.

The hypotheses imply that $\bar{\Delta} - K^0$ is finitely connected. In fact, the complement of $\bar{\Delta} - K^0$ has two components, $\mathbf{C} - \bar{\Delta}$ and K^0. Also, the components of $(\bar{\Delta} - K^0)^0 = \Delta - K$ are simply connected. As before, $R(\bar{\Delta} - K^0)$ is a Dirichlet algebra so we can apply Theorem 3.1 to $\bar{\Delta} - K^0$. The conclusions (i) and (ii) follow easily.

3.4 COROLLARY. *Let K be a compact plane set with connected dense interior. Then there is a measure μ with $\hat{\mu} = \chi_K dxdy$* a.e. *if and only if K has finite Painlevé length.*

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