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GENERALIZED AXISYMMETRIC ELLIPTIC FUNCTIONS

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A generalized axisymmetric elliptic function (GASE) $\Psi_\nu: \Omega \subset E^n \to C$ of order $\nu \geqq 0$ solving the partial differential equation

$$(1) \quad \mathscr{L}_\nu(\Psi_\nu) \equiv \frac{\partial^2 \Psi_\nu}{\partial x^2} + \frac{\partial^2 \Psi_\nu}{\partial \rho^2} + \frac{2\nu}{\rho}\frac{\partial \Psi_\nu}{\partial \rho} + a(x)\frac{\partial \Psi_\nu}{\partial x} + c(x)\Psi_\nu = 0$$

with analytic coefficients is subject to Cauchy data: $\Psi_\nu(x, 0) = f(x)$, $(\partial/\partial\rho)(\Psi_\nu(x, 0)) = 0$ along the singular line. These GASE may be generated from associated analytic functions of one complex variable or associated solutions to the corresponding nonsingular equation by certain integral operators. Convexity arguments geometrically characterize the values of GASE from those of the associates and kernel functions of the respective operators.

An extensive theory based on integral operators which characterizes the distribution of singularities of various classes of GASE from the distribution of singularities of their associates was developed by S. Bergman [2], R. P. Gilbert [3, 4], P. Henrici [6] and their colleagues [5]. Our aim is to apply convexity arguments from the analytic theory of polynomials of one complex variable to develop a geometric theory of the value distribution of GASE from the known value distribution of the associates. These results are based on two operators developed by Henrici [3, p. 199]; one which utilizes a kernel function to generate GASE from associated analytic functions of one complex variable and one which generates GASE of positive order from the associated GASE of order zero.

A theory connecting the values of axisymmetric harmonic polynomials (AHP) in E^n with those of associated polynomials of one complex variable was developed by M. Marden [8]. Gegenbauer's integral for ultraspherical polynomials was used to map polynomials of one complex variable onto AHP and then convexity arguments were used to relate their values. Using geometrical methods and R. P. Gilbert's operater A_μ [3, p. 168], the auther [10-12] utilized the conformal mapping properties of the associates to characterize sets of excluded values for generalized axisymmetric potentials (GASP) corresponding to solutions of (1) with $a(x) \equiv c(x) \equiv 0$.

Convexity arguments used in studying GASP were essentially independent of the kernel of the operator A_μ which is non-negative and dependent only on the variable of integration. In general, operators transforming associated analytic functions of one complex variable into GASE have kernels which also depend on the circle in E^n on which the GASE is evaluated. By modifying previous convexity arguments, the influence of this additional dependence is geometrically characterized from the kernels and associates of the respective operators. Methods of refinement of certain types of bounds found in previous results [10-12] are also introduced.

1. Preliminaries. We shall be dealing with the cylindrical coordinates (x, ρ, ϕ),

$$x = x_1 , \quad \rho^2 = x_2^2 + \cdots + x_n^2 , \quad 0 \leqq \phi \leqq 2\pi$$

and spherical coordinates (r, ϕ, θ) where

$$x = r \cos\theta , \quad \rho = r \sin\theta .$$

We shall be considering analytic functions f of one complex variable whose natural domains are taken as open simply connected *axiconvex sets* $\omega \subset \boldsymbol{C}$. That is, $\zeta \in \omega$ if and only if $\zeta t + \bar{\zeta}(1-t) \in \omega$, $0 \leqq t \leqq 1$, and GASE Ψ_ν whose natural domains are *axisymmetric sets* $\Omega \subset E^n$ generated by rotating axiconvex sets about the x-axis.

As is well known, Henrici [6, p. 21] has shown that a family of GASE $\{\Psi_\nu\}_{\nu\geqq 0}$ on Ω may be generated from the *associate* Ψ_0 via the operator

$$\Psi_\nu(x, \rho) = \alpha_\nu \int_0^\pi \Psi_0(x, \rho\cos t)(\sin t)^{2\nu-1} dt \tag{2}$$

with normalization

$$\alpha_\nu^{-1} = \int_0^\pi (\sin t)^{2\nu-1} dt = \sqrt{\pi}\,\Gamma(\nu)/\Gamma(\nu + 1/2)$$

where Ψ_ν is extended as an even function in ρ.

In our study it will be convenient to refer to the *circles*

$$X = \{(x, \rho, \phi) \mid 0 \leqq \phi \leqq 2\pi\}$$

and

$$X_* = \{(x_*, \rho_*, \phi) \mid 0 \leqq \phi \leqq 2\pi\}$$

where

$$X \prec X_* \Longleftrightarrow x = x_* \quad \text{and} \quad \rho \leqq \rho_* \cdot$$

Whenever X_* is in Ω, $\Psi_0(\{X \mid X \prec X_*\})$ is a curve in $\boldsymbol{C}$ and $\mathscr{C}\{\Psi_0(X_*)\}$

is its closed convex hull. A refinement of $\Gamma = \mathscr{C}\{\Psi_0(X_*)\}$ is provided by

$$\mathscr{S}\{\Psi_0(X_*)\} = \Gamma - \partial_0[\Gamma]$$

where

$$\partial_0[\Gamma] = \{\zeta \in \partial\Gamma \mid \exists \xi, \eta \in \partial\Gamma \ni \zeta = \xi t + (1-t)\eta, 0 < t < 1\}$$

is the set of extreme points on the boundary of Γ. We now turn to

2. Distribution of values of GASE.

THEOREM 1. *Let $\{\Psi_\nu\}_{\nu \geqq 0}$ be a family of* GASE *with domain Ω. Then for each circle X_* in Ω*

$$\Psi_\nu(X) \in \mathscr{S}\{\Psi_0(X_*)\} \tag{3}$$

for all circles $X \prec X_*$ and for all orders $\nu \geqq 0$. In particular, the only possible zeros of Ψ_ν occur on those circles X_* for which $\mathscr{S}\{\Psi_0(X_*)$ contains the origin.

Proof. Transposing terms in (2) leads to

$$\int_0^\pi [\Psi_0(x_*, \rho_* \cos t) - \Psi_\nu(X)](\sin t)^{2\nu-1} dt = 0 \quad X \prec X_* . \tag{4}$$

Let us assume that for some order $\nu > 0$, $\Psi_\nu(X) \in \mathscr{S}\{\Psi_0(X_*)\}$. Under these conditions, the following inequality is satisfied for $t \in (0, \pi) \equiv I$,

$$\arg [\Psi_0(x_*, \rho_* \cos t) - \Psi_\nu(X)] < \pi . \tag{3}$$

In addition, the integrand of (2) contains $(\sin t)^{2\nu-1}$, a positive factor for $t \in I$, which when combined with (3) produces

$$\arg \{[\Psi_0(x_*, \rho_* \cos t) - \Psi_\nu(X)](\sin t)^{2\nu-1}\} < \pi \tag{6}$$

for $t \in I$. In view of this inequality, we may follow the reasoning found in [9] to conclude that the integrand of (2) considered as the limit of a sum of vectors which terminate in a convex sector with vertex at $\Psi_\nu(X)$, cannot vanish, a contradiction to (2). Observing that the inclusion $\mathscr{S}\{\Psi_0(X)\} \subset \mathscr{S}\{\Psi_0(X_*)\}$ holds for all circles $X \prec X_*$ completes the proof.

As consequences of Theorem 1, let us now derive the following corollaries.

COROLLARY 1.1. *For each α, $\{\Psi_\nu^\alpha\}_{\nu \geqq 0}$ is a family of* GASE *with domain Ω. If the associates of these families assume values on Ω which lie in mutually exclusive closed convex sets, then for all circles X and*

X' in Ω and all orders μ and ν,

$$\Psi_\mu^\alpha(X) \neq \Psi_\nu^\beta(X') \tag{7}$$

for all distinct α and β.

Proof. This follows from the fact that for X in Ω

$$\mathscr{S}\{\Psi_\gamma(X)\} \subset \mathscr{C}\{\Psi_\gamma(X)\}\ , \quad \gamma = \alpha, \beta\ ,$$

and that from Theorem 1

$$\mathscr{C}\{\Psi_0^\alpha(X)\} \cap \mathscr{C}\{\Psi_0^\beta(X')\} = \varnothing$$

when α and β are distinct.

In the special case in which the associate of a family of GASE reduces to a finite linear combination of GASE of order zero, a result analogous to those of Marden and Walsh [7, p. 74] which consider the null sets of linear combinations of polynomials of one complex variable is found in

COROLLARY 1.2. *If the associate of a family of* GASE $\{\Psi_\nu\}_{\nu \geqq 0}$ *with domain Ω is represented as $\Psi_0 = \sum_{i=1}^n a_i \Psi_0^i$ where Ψ_0^i are* GASE *on Ω for $1 \leqq i \leqq n$, then*

$$\Psi_\nu(X) = \sum_{i=1}^n a_i \Psi_\nu^i(X)\ , \quad X \in \Omega \tag{8}$$

where $\Psi_\nu^i(X) \in \mathscr{C}\{\Psi_0^i(\Omega_0)$, $1 \leqq i \leqq n$, for all orders ν and circles X in compact axiconvex subsets Ω_0 of Ω.

Proof. We identify each GASE Ψ_ν^i with its associate Ψ_0^i and

$$\mathscr{C}\left\{\sum_{i=1}^n a_i \Psi_0^i(\Omega_0)\right\} = \sum_{i=1}^n a_1 \mathscr{C}\{\Psi_0^i(\Omega_0)\}\ .$$

These results may be recast to provide an analytical description of the *γ-circles* of a GASE, that is, of circles X on which Ψ_ν assumes the value γ. Let us consider

COROLLARY 1.3. *Let the family of* GASE $\{\Psi_\nu\}_{\nu \geqq 0}$ *be defined on Ω. Then the only possible γ-circles of Ψ_ν on Ω are circles X_* which are γ-circles of Ψ_0 or which include circles $X \prec X_*$ on the locus*

$$\arg\left(\frac{\Psi_0(X_*) - \gamma}{\Psi_0(X) - \gamma}\right) = \pm\pi\ . \tag{9}$$

Proof. If there exists a circle X_* in Ω for which $\Psi_0(X_*) = \gamma$, then γ is in $\mathscr{C}\{\Psi_0(X_*)\}$ and consequently X_* is a possible zero of

$\Psi_\nu - \gamma$. On the other hand, if there exists a circle $X \prec X_*$ on the locus

$$\arg(\Psi_0(X_*) - \gamma) = \pm\pi + \arg(\Psi_0(X) - \gamma) ,$$

then the image of the disk $\{X | X \prec X_*\}$ under $\Psi_0 - \gamma$ is a curve whose closed convex hull includes the origin. As this exhausts the cases for which $\mathscr{C}\{\Psi_0(X)\}$ meets the origin, the only possible zeros of $\Psi_\nu - \gamma$ satisfy the above locus.

We next establish that locally the distribution of values of GASE can be made independent of the associate.

COROLLARY 1.4. *Let $\{\Psi_\mu^\alpha\}_{\mu \geqq 0}$ and $\{\Psi_\nu^\beta\}_{\nu \geqq 0}$ be families of* GASE *which are generated from distinct associates on Ω. If for some orders μ and ν, the values assumed by Ψ_μ^α and by Ψ_ν^β on a segment J of the axis of symmetry are separated by a line, there exists an axiconvex set $\Omega_J \subset \Omega$ on which*

$$\Psi_\mu^\alpha(\Omega_J) \cap \Psi_\nu^\beta(\Omega_J) = \varnothing$$

for all orders μ and ν.

Proof. From (2) we find that $\Psi_\mu^\alpha(X) = \Psi_0^\alpha(X)$ and $\Psi_\nu^\beta(X) = \Psi_0^\beta(X)$ for all positive orders whenever X is a point of J. Since a line $\mathscr{L}$ separates the curves $\Psi_\mu^\alpha(J)$ and $\Psi_\nu^\beta(J)$, there are axiconvex sets Ω^α and Ω^β in Ω for which $\mathscr{S}\{\Psi_0^\alpha(X)\} \cap \mathscr{S}\{\Psi_0^\beta(X')\} = \varnothing$ for all circles X in Ω^α and X' in Ω^β. Defining $\Omega_J = \Omega^\alpha \cap \Omega^\beta$ completes the proof.

Results having more immediate application can be deduced from a second operator introduced by Henrici [3, p. 201] which generates a family $\{F_\nu\}_{\nu \geqq 0}$ of GASE from an associated analytic function f and the Riemann function k_ν for (1). This operator takes the form

$$F_\nu(X) = c_\nu \int_0^\pi k_\nu(X, t) f(\sigma) dt \tag{10}$$

where c_ν is a normalizing constant and $\sigma = x + i\rho\cos t$, $0 \leqq t \leqq 2\pi$.

An extension of the previous method allows a characterization of the values of F_ν in terms of the product of the integral of the Riemann function and a function depending on f. The advantage of this approach lies in the fact that since the integral of k_ν is independent of f, the qualitative properties of F_ν may be determined from the conformal mapping properties of f.

To accomplish this, let us restrict our attention to axisymmetric sets $\Omega_\nu \subset \Omega$ on which the Riemann function satisfies

(11) $$\arg\{k_\nu(X, t)\} \leqq \pi - \gamma_\nu$$

for $X \in \Omega_\nu$ and $t \in I$. Whenever (7) holds and the associate f is analytic on the corresponding $\omega_\nu \subset C$, the family $\{F_\nu\}_{\nu \geqq 0}$ is said to be γ_ν*-convex* on Ω_ν. The angle γ_ν and the set Ω_ν may be independent of the order ν as is the case for GASP where $k_\nu(X, t)$ reduces to $(\sin t)^{2\nu-1}$:

The point set for which $f(\omega_\nu)$ subtends an angle of at least γ_ν is designated by

$$\mathscr{S}[f(\omega_\nu), \gamma_\nu] .$$

We now turn to

THEOREM 2. *Let* $\{F_\nu\}_{\nu \geqq 0}$ *be a family of* GASE *which is* γ_ν*-convex on* Ω_ν. *Then on* Ω_ν, *each* F_ν *may be represented as*

(12) $$F_\nu(X) = c_\nu \eta(X) \int_0^\pi k_\nu(X, t)dt$$

where $\eta(X) \in \mathscr{S}[f(\omega_\nu), \gamma_\nu]$. *If* $\mathscr{S}[f(\omega_\nu), \gamma_\nu]$ *does not meet the origin,* F_ν *has no zeros on* Ω_ν.

Proof. For each circle X in Ω_ν, (7) holds. Hence, $\int_0^\pi k_\nu(X, t)dt \neq 0$ which permits the function η to be defined by (8). As in Theorem 1, a contradiction is reached if (8) is rewritten as

(13) $$\int_0^\pi k_\nu(X, t)[f(\sigma) - \eta(X)]dt = 0$$

and we assume that $\eta(X) \notin \mathscr{S}[f(\omega_\nu), \gamma_\nu]$ since then

(14) $$\arg\{k_\nu(X, t)[f(\sigma) - \eta(X)]\} < \pi$$

for $t \in I$.

Null circles of GASE with polynomial associates are connected with the zeros of their associates through

COROLLARY 2.1. *Let* $\{F_\nu\}_{\nu \geqq 0}$ *be a family of* GASE *which is* γ_ν *convex on* Ω_ν *have* f, *a polynomial of degree* n, *as associate. If the circle* X *is a zero of* F_ν, f *has at least one zero in the circle*

(15) $$|\zeta - x| \leqq \rho \cot(\gamma_\nu/n) , \quad \zeta \in C .$$

Proof. Let us factor the integrand of (9) as

(16) $$f(\sigma) - \eta(X) = \alpha_0 \prod_{j=1}^{n} (\sigma - \alpha_j) .$$

Assuming that $|\alpha_j - x| > \rho \cot(\gamma_\nu/n)$, $1 \leqq j \leqq n$, leads us to the inequality

$$\arg\{\alpha_0^{-1}(f(\sigma) - \eta(X))\} < (\gamma_\nu/n)n \tag{17}$$

for $t \in I$. Since k_ν is γ_ν-convex on Ω_ν, we arrive at (14) which leads to a contradiction of (12) by the usual argument.

We next find that a partial analog of the Riemann mapping theorem is included in

COROLLARY 2.2. *Let Ω be an axisymmetric subset of E^n with axis of symmetry J. Then for any point X_0 on J, there is a* GASE *F_n with domain Ω such that $F_n(J)$ is a simple curve thru the origin, $(\partial/\partial x)(F_n(X_0)) > 0$ and*

$$|F_n(X)| < 1$$

for all X in Ω.

Proof. Let $\omega \subset C$ be the axiconvex set associated with Ω and I be its axis of symmetry. The Riemann mapping theorem assets the existence of a one-one analytic function f with domain ω such that if $(x_0, 0) \in I$, $f(x_0, 0) = 0$, $f_x(x_0, 0) > 0$ and $|f(\zeta)| < 1$ for all ζ in ω.

GASP are generated by (10) from $k_n(X, t) = (\sin t)^{2n-1}$. Since the values attained by F_n on J and f on I agree and F_n has no singularities on Ω [3, p. 179], we establish from (12) that for X in Ω,

$$\begin{aligned} |F_n(X)| &\leqq \alpha_n^{-1}\int_0^\pi |f(\sigma)|(\sin t)^{2n-1}dt \\ &< \alpha_n^{-1}\int_0^\pi (\sin t)^{2n-1}dt = 1 \,. \end{aligned}$$

These results are applied to

3. Generalized axisymmetric Helmholtz functions.

The generalized axisymmetric Helmholtz functions (GASH) arise as solutions to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \rho^2} + \frac{2\nu}{\rho}\frac{\partial u}{\partial \rho} + k^2 u = 0,\ k, \nu > 0 \,.$$

It is well known [6, p. 26] that GASH are generated thru the operator

$$F_\nu(X) = c_\nu \int_0^\pi J_{\nu-1}(k\rho \sin t) f(k\sigma)(\sin t)^\nu dt \tag{18}$$

where

$$c_\nu = (k\rho/2)^{1-\nu}\Gamma(\nu + 1/2)/\Gamma(1/2) \,.$$

This operator establishes a correspondence between the Neumann series expansion of analytic functions f regular about the origin [3, p. 214]

$$f(k\sigma) = (k\sigma)^{-\nu} \sum_{n=0}^{\infty} a_n J_{\nu+n}(k\sigma) \tag{18}$$

and the GASH represented by the Bessel-Gegenbauer series

$$F_\nu(x, \rho) \equiv \tilde{F}_\nu(r, \theta) = d_\nu \sum_{n=0}^{\infty} \frac{a_n n!}{\Gamma(2\nu + n)} J_{\nu+n}(kr) C_n^\nu(\cos\theta) \tag{19}$$

where $d_\nu = \Gamma(2\nu)(kr)^{-\nu}$.

If $\xi_{\nu-1}$ denotes the smallest positive zero of the Bessel function $J_{\nu-1}$ of order $\nu - 1$, $\arg\{J_{\nu-1}(\lambda)\}$ is constant for $0 < \lambda < \xi_{\nu-1}$. Hence, Theorem 2 permits us to deduce.

THEOREM 3. *Let* $\{F_\nu\}_{\nu\geqq 0}$ *be a family of* GASE *with domain* Ω *and associate* f. *Then on all circles common to* Ω *and the cylinder* $\rho \leqq \xi_{\nu-1}k^{-1}$, F_ν *may be represented as*

$$F_\nu(x, \rho) = 2^\mu \Gamma(\mu + 1)\eta(x, \rho)\frac{J_\mu(k\rho)}{(k\rho)^\mu} \tag{20}$$

where $\mu = \nu - 1/2$ *and* $\eta(x, \rho) \in \mathscr{C}\{f(k\omega)\}$. *If* $\mathscr{C}\{f(k\omega)\}$ *does not contain the origin,* F_ν *has no zeros on* Ω *for* $\rho \leqq \xi_{\nu-1}k^{-1}$.

Proof. Sonine's first integral allows the evaluation

$$\int_0^\pi J_{\nu-1}(k\rho \sin t)(\sin t)^\nu dt = \Gamma(1/2)(2/k\rho)^{1/2} J_{\nu-1/2}(k\rho) . \tag{21}$$

By (12), we are permitted to define the function η in the cylinder $\rho \leqq \xi_{\nu-1}k^{-1}$ by

$$\lambda_\nu \int_0^\pi f(k\sigma) J_{\nu-1}(k\rho \sin t)(\sin t)^\nu dt = c_\nu \eta(x, \rho) J_{\nu-1}(k\rho)/(k\rho)^{\nu-1/2} , \tag{22}$$

where

$$\lambda_\nu = (k\rho/2)^{1-\nu}\Gamma(\nu + 1/2)/\Gamma(1/2)$$

since

$$\arg\{J_{\nu-1}(k\rho \sin t)(\sin t)^\nu\} = 0 , \quad t \in I$$

implies that (21) is nonvanishing. By rearranging terms in (22), we find that if $c_\nu = 2^{\nu-1/2}\Gamma(\nu + 1/2)$, (21) becomes

$$\int_0^\pi [f(k\sigma) - \eta(x, \rho)] J_{\nu-1}(k\rho \sin t)(\sin t)^\nu dt = 0 . \tag{23}$$

The previous argument applies and the proof is complete.

As a further application of this method, let us consider the following

EXAMPLE. For $\nu > 0$, let the GASH F_ν be defined on E^n by

$$F_\nu(X) = e_\nu \sum_{n=0}^{\infty} \frac{(\nu + n)n!\, i^n}{\Gamma(2\nu + n)} C_n^\nu(k) C_n^\nu(\cos\theta) J_{\nu+n}(k\rho) \tag{24}$$

where

$$e_\nu = 2^\nu \Gamma(\nu)\Gamma(2\nu) r^{-\nu} .$$

By (18), the associate of F_ν has the expansion

$$f(ik\sigma) = 2^\nu \Gamma(\nu)(k\sigma)^{-\nu} \sum_{n=0}^{\infty} i^n(\nu + n) C_n^\nu(k) J_{\nu+n}(k\sigma) \tag{25}$$

which by [1, p. 64] we identify as the Neumann series $f(ik\sigma) = \exp(ik\sigma)$. Since $f(ik\sigma) = f(ik(\sigma + 2\pi jk^{-1}))$ for $j = 0, \pm 1, \pm 2, \cdots$, the image of the union $\omega(k)$ of the rectangles

$$\omega_j(k) = \{\xi + i\eta \mid |\eta| < \xi_{\nu-1}k^{-1}, |\xi^*| < \pi k^{-1}, \xi = \xi^* + 2\pi jk^{-1}\}$$

under f lies in the annular region

$$\mathscr{A}(k) = \{\zeta \mid |\arg\zeta| < \pi k^{-1}, \exp(-\xi_{\nu-1}k^{-1}) < |\zeta| < \exp(\xi_{\nu-1}k^{-1})\}$$

provided $k > 1$. In the event that $k \leqq 1$, $\omega(k)$ reduces to the infinite strip $|\eta| < \xi_{\nu-1}k^{-1}$ and $\mathscr{A}(k)$ the annulus $\exp(-\xi_{\nu-1}k^{-1}) < |\zeta| < \exp(\xi_{\nu-1}k^{-1})$. It is easily seen from (18) and the periodicity of f that

$$F_\nu(x, \rho) = F_\nu(x + 2\pi jk^{-1}, \rho) , \qquad j = 0, \pm 1, \pm 2, \cdots . \tag{26}$$

Theorem 3 permits us to conclude that on $\Omega(k)$, the union of the truncated cylinders $\Omega_j(k)$ generated by rotating $\omega_j(k)$ into E^n, the GASH (24) reduces to

$$F_\nu(x, \rho) = 2^\mu \Gamma(\mu + 1)\eta(x, \rho) J_\mu(k\rho)/(k\rho)^\mu$$

where η assumes its values in $\mathscr{A}(k)$.

A natural question arising from Corollary 2.1 is that of recovering the distribution of values of the associate from that of the GASH. Such a connection is revealed thru the inverse operator [3, p. 216]

$$f(k\sigma) = l_\nu \int_{-1}^{+1} k'(\sigma, r, \xi) F_\nu(r\xi, r(1 - \xi^2)^{1/2}) d\xi \tag{27}$$

with $l_\nu = \Gamma(2\nu)(kr)^{-1}$ where the kernel k' is modified so as to avoid singularities. The GASH F_ν and consequently f are entire functions. Since $\arg(d\xi)$ is constant for $\xi \in (1, 1)$ it follows that

THEOREM 4. *Let f and F_ν be entire functions. If for $\sigma \in \boldsymbol{C}$, $r \in (0, R]$ and $\xi \in (-1, +1)$, $\arg\{k'(\sigma, r, \xi)\} \leqq \pi - \gamma$, then f may be represented by*

$$f(k\sigma) = l_\nu \eta(r) \int_{-1}^{+1} k(\sigma, r, \xi) d\xi \tag{28}$$

where $\eta(r) \in \mathscr{S}[\bigcup_{|\xi| \leqq 1} F_\nu(r\xi, r\,|\,\xi\,|), \gamma]$.

4. Analytic continuation of solutions. Gilbert [4] discusses methods of analytic continuation of his operators by allowing the coordinates (x, ρ, ϕ) to attain complex values thru deformation of the contour of integration so as to avoid singularities of the integrand. The methods used here may be adopted to that approach provided that the deformed contour $\mathscr{L}_*$ is sufficiently smooth so that the variation if $\arg(d\xi)$ over $\mathscr{L}_*$, $\mathscr{V}\{\arg(d\xi)\}_{\mathscr{L}_*}$, is small or there are factors in the integrand which compensate for $\mathscr{V}\{\arg(d\xi)\}_{\mathscr{L}_*}$.

This situation arises in the case of GASP u_ν on Ω generated by Gilbert's operator [3, p. 168]

$$u_\nu(x, \rho) = \alpha_\nu^{-1} \int_{\mathscr{L}} f(\tau)(\zeta - \zeta^{-1})^{2\nu-1}\zeta^{-1} d\zeta \tag{29}$$

with $\tau = x + (i\rho/2)(\zeta - \zeta^{-1})$ and $\mathscr{L} = \{\zeta = \exp(i\theta) \mid 0 \leqq \theta \leqq \pi\}$. On $\mathscr{L}$, $\mathscr{V}\{\arg(\zeta^{-1}d\zeta\}_{\mathscr{L}}$ and $\mathscr{V}\{\arg(\zeta - \zeta^{-1})\}_{\mathscr{L}}$ are constant. If u_ν is continued beyond Ω by fixing the endpoints of $\mathscr{L}$ and continuously deformed into $\mathscr{L}_*$, an ellipse with eccentricity $\varepsilon \ll 1$, continuity guarantees that $\mathscr{V}\{\arg(\zeta^{-1}d\zeta)\}_{\mathscr{L}_*}$ and $\mathscr{V}\{\arg(\zeta - \zeta^{-1})\}_{\mathscr{L}_*}$ are small so that previous reasoning can be applied to obtain a result analogous to Theorem 2 on the continued domain Ω_*. See [11].

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