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ALMOST PERIODIC HOMEOMORPHISMS OF E^2 ARE PERIODIC

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In this paper we show that every almost periodic homeomorphism of the plane onto itself must be periodic. This improves well-known results.

1. Introduction. In [3] Foland showed that every almost periodic homeomorphism of a *disk* onto itself is topologically either a reflection in a diameter or a rotation. Hemmingsen [7] studies homeomorphisms on compact subsets of E^2, with equicontinuous families of iterates, and shows that if such a compact set has an interior point of infinite order, then the compact set is a disk or annulus. If it is a disk, then the homeomorphism is a rotation or reflection. Kerékjártó [8, pp. 224–226] showed that every periodic homeomorphism of a disk onto itself is a conjugate of either a rotation or a reflection. It was brought to my attention by S. Kinoshita that Kerékjártó in [9] obtains a characterization of those homeomorphisms of S^2 onto itself which are regular; that is, homeomorphisms h such that $\{h^n\}_{n\in I}$ forms an equicontinuous family. It is known [4] that almost periodic homeomorphisms on compact metric spaces satisfy this property, so that our theorem for E^2 would follow from the theorem for S^2.

However, our proof of the main theorem uses Bing's ε-growth technique [6] to obtain an invariant disk, and thus *re*-does a portion of [2], [7], and [9] in a particularly nice way.

Montgomery began a study of almost periodic transformation groups in [13], with the main results for E^3. One very nice theorem states that if G is a one-parameter almost periodic transformation group (a.p.t.g.) of E^3 whose minimal closed invariant sets are one-dimensional, and whose orbits are uniformly bounded, then G is the identity. Our theorem may be regarded as something of an analogue to this theorem for E^2. That is, our theorem shows that if $G = \{h^n\}_{n\in I}$ is an a.p.t.g. of E^2, $h \neq e$, then the orbits are not uniformly bounded.

2. Preliminaries. The definitions used here of the following are as in [4] and [6]: *Relatively dense* subsets of the integers; homeomorphisms *almost periodic at a point, pointwise almost periodic* (p.a.p.), and *almost periodic* (a.p.) on the space; *invariant set*; and *minimal set* are defined in [4]. *Property S*, *ε-growth*, and *ε-sequential growth* are defined in [6]. The *orbit* of x in the space X is the set

$\{h^n(x) \mid n \in I\}$, and is denoted by $0(x)$.

We will use the following known results.

PROPOSITION 2.1. [6, pg. 212]. *Let K be a subset of a metric space X. If K has property S, then K is locally connected.*

PROPOSITION 2.2. [6, pg. 215]. *If K is a subset of a metric space X and K has property S, then $\bar{K}$ has property S. Thus, if K has property S, then $\bar{K}$ is locally connected.*

PROPOSITION 2.3. [6, pg. 216]. *Let X be a metric space with property S, H and K subsets of X, and $\varepsilon > 0$. If K is an ε-sequential growth of H, then K has property S and is open in X.*

NOTE. The double arrow in $f: A \twoheadrightarrow B$ denotes an *onto* function.

3. Obtaining invariant disks. In this section we use the concept of an ε-sequential growth to enable us to obtain E^2 as the union of an increasing tower of invariant disks for any a.p. homeomorphism of the plane onto itself.

LEMMA 3.1. *Let X be a compact metric space and let $\{f_n\}$ be an equicontinuous collection of functions on X. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that* diam $(f_n(\delta\text{-set})) < \varepsilon$, *for all $n \in I$.*

Proof. Let $\varepsilon > 0$. For each $x \in X$, there exists $\gamma > 0$ such that diam $(f_n(\gamma\text{-nbd of } x)) < \varepsilon$ for all $n \in I$, since $\{f_n\}$ is equicontinuous. Choose such a neighborhood for each $x \in X$. This forms a cover of X and therefore some finite subcollection covers X. Let δ be a Lebesgue number for this subcover. Then diam $(f_n(\delta\text{-set})) < \varepsilon$ for all $n \in I$.

LEMMA 3.2. *Let h be a homeomorphism of S^2 onto itself such that $h(p) = p$ where p is the north pole of S^2, and let X be a locally connected continuum in S^2, containing p, such that $h(X) = X$. Let $\varepsilon =$* diam S^2, *and by uniform continuity of h, let $\delta > 0$ such that* diam $(h(\delta\text{-set})) < \varepsilon/2$. *Then if* diam $X < \delta$ *and U is the component of $S^2 - X$ containing the south pole, we have $h(U) = U$.*

Proof. We first show that each component of $S^2 - X$ must go onto some component of $S^2 - X$. Let V be a component of $S^2 - X$, and suppose there exist points x and $y \in V$ such that $h(x) \in W_1$, $h(y) \in W_2$, where $W_1 \neq W_2$ are components of $S^2 - X$. Let A be an arc from x to y in V. Since A misses X, and $h(X) = X$, $h(A)$ misses X. But $h(A)$ is connected and contains points of different components of $S^2 - X$, and therefore must contain a point of X. This is a contradiction. Therefore $h(V)$ is a subset of a component of $S^2 - X$. The same argument applied to h^{-1}, shows h^{-1} (componet) $\subseteqq$ some component of $S^2 - X$, so that $h(V)$ is a component of $S^2 - X$. We next show that $h(U) = U$. Suppose $h(U) \neq U$. Then there is a component $W(\neq U)$ of $S^2 - X$, such that $h(W) = U$. Now diam $W < \delta$, and therefore diam $h(W) < \varepsilon/2$. Therefore $h(W) \neq U$. This is a contradiction. Thus $h(U) = U$.

LEMMA 3.3. *Let h be an almost periodic homeomorphism of E^2 onto E^2 and let $\varphi: E^2 \to S^2$ be the inverse of the stereographic projection. Let p be the north pole of S^2. Let $g: S^2 \twoheadrightarrow S^2$ be defined by* $g(x) = \begin{cases} \varphi h \varphi^{-1}(x), & \text{for } x \in S^2 - \{p\} \\ p & \text{, for } x = p \end{cases}$. *Then g is an* a.p. *homeomorphism of S^2 onto S^2.*

Proof. Let $\varepsilon > 0$. We must show that there exists a relatively dense subset A of I such that $d(x, g^n(x)) < \varepsilon$ for all $x \in S^2$ and all $n \in A$. Now we know there exists a relatively dense subset A of I such that $d(x, h^n(x)) < \varepsilon$ for all $x \in E^2$ and all $n \in A$. Also, it follows from pg. 20 of [1] that φ has the property that $d(y, y') \geqq d(\varphi(y), \varphi(y'))$ for all $y, y' \in E^2$. Now since $d(y, h^n(y)) < \varepsilon$ for all $y \in E^2$ and all $n \in A$, $d(\varphi^{-1}(x), h^n\varphi^{-1}(x)) < \varepsilon$ for all $x \neq p \in S^2$, all $n \in A$. Thus $d(\varphi\varphi^{-1}(x), \varphi h^n \varphi^{-1}(x)) < \varepsilon$ and $d(x, \varphi h^n \varphi^{-1}(x)) < \varepsilon$ for all $x \in S^2$, all $n \in A$. It follows that $d(x, g(x)) < \varepsilon$ for all $x \in S^2$, all $n \in A$, and g is a.p.

THEOREM 3.1. *Let h be an* a.p. *homeomorphism on S^2 such that h keeps the north pole p fixed. Then for each $\eta > 0$, there exists an η-disk E which is invariant under h* (in fact $h(E) = E$), *and contains p in its interior.*

Proof. Let γ be the diameter of S^2. Then there exists $\delta > 0$ such that diam $(h(\delta\text{-set})) < \gamma/2$, by uniform continuity of h. Let $0 < \varepsilon < \min\{\eta, \delta, \gamma\}$, and let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers such that $\sum \varepsilon_i < \varepsilon \leqq \eta$. We will obtain E as an ε-sequential growth of the set $\{p\}$.

Let $D_1 = \{p\}$. The set $\{h^n\}_{n \in I}$ is equicontinuous [4, pg. 341], and $\varepsilon_1 > 0$. Thus by Lemma 3.1, there exists $\delta_1 > 0$ such that diam $(h^n(\delta_1\text{-set})) < \varepsilon_1$ for all $n \in I$. Let $\mathscr{U}_1 = \{U_{11}\}$ be a cover of D_1 by an open connected set of S^2 such that $\mu(\mathscr{U}_1) < \min\{\delta_1, \varepsilon_1\}$. Let $D_2 = \bigcup_{n \in I} h^n(U_{11})$ and note that D_2 is invariant. We show that D_2 is an ε_1-growth of D_1. We must show parts (i) and (ii) of the definition of ε-growth.

Proof of (i). If $x \in D_2 - D_1$, then there exists an integer n such that $x \in h^n(U_{11})$. But $h^n(U_{11})$ is connected and diam $(h^n(U_{11})) < \varepsilon_1$. Also, $h^n(U_{11})$ contains p and so meets D_1.

Proof of (ii). U_{11} is an open set containing the compact set D_1. $S^2 - U_{11}$ is compact, and disjoint from D_1 which is compact. Thus $d(D_1, S^2 - U_{11}) = 2\alpha_1$ for some $\alpha_1 > 0$, and it follows that the α_1-nbd. of D_1 is a subset of D_2. Thus (i) and (ii) hold and D_2 is an ε_1-growth of D_1.

We now wish to obtain an ε_2-growth of D_2. We note that since D_2 is invariant, so is $\bar{D}_2$. Now for $\varepsilon_2 > 0$, there exists $\delta_2 > 0$ such that diam $(h^n(\delta_2\text{-set})) < \varepsilon_2$ for all n. Again this is possible by Lemma 3.1. Let $\mathscr{U}_2$: $U_{2,1}, U_{2,2}, \cdots, U_{2,k_2}$ be a finite cover of $\bar{D}_2$ by open connected subsets of S^2 of diameter $< \min\{\delta_2, \varepsilon_2\}$ and let

$$D_3 = \bigcup_{n \in I} h^n\left(\bigcup_{i=1}^{k_2} U_{2,i}\right).$$

Then D_3 is invariant.

We show that D_3 is an ε_2-growth of D_2. We prove parts (i) and (ii) of the definitions of ε-growth.

Proof of (i). Let $x \in D_3 - D_2$. Then $x \in h^n(U_{2,i})$ for some pair n, i. But $h^n(U_{2,i})$ is connected, meets D_2, and has diameter $< \varepsilon_2$.

Proof of (ii). $\bar{D}_2$ and $S^2 - \bigcup_{n \in I} h^n(\bigcup_{i=1}^{k_2} U_{2,i})$ are disjoint compact subsets of S^2 and thus are a positive distance apart, say $2\alpha_2$. Then the α_2-nbd. of $\bar{D}_2$, and therefore the α_2-nbd. of D_2, is a subset of D_3.

Thus (i) and (ii) hold, and D_3 is an ε_2-growth of D_2.

It is clear that we may continue the process inductively, obtaining at the ith stage, a connected open set D_i which is an ε_{i-1}-growth of D_{i-1}. Let $E' = \bigcup_{i=1}^{\infty} D_i$. Then by Proposition 2.3, E' is open and has property S. Thus $\bar{E}'$ is a locally connected continuum, by Proposition 2.2. Further $\bar{E}'$ is invariant. We show that $\bar{E}'$ has no cut points. Note that E' has no cut points since it is open (and connected). Thus any cut point of $\bar{E}'$ would be in $\bar{E}' - E'$, so that there would exist a component of $\bar{E}'$ containing points of $\bar{E}' - E'$ only. But these are all limit points of E'. This is a contradiction, and it follows that $\bar{E}'$ has no cut points.

Thus $\bar{E}'$ is a locally connected continuum with no cut points, and from Theorem 9 of [11] it follows that the boundary of each of its complementary domains is a simple closed curve. Now one of its complementary domains, say F, contains the open southern hemisphere, and therefore has diameter $\geqq \gamma$, while each of the other complementary domains has diameter less than ε, since diam $\bar{E}' < \varepsilon$. Thus by Lemma 3.2, F is invariant, and $h(F) = F$. Let $E = S^2 - F$.

Then diam $E < \varepsilon$, $h(E) = E$, and E is a disk, by the Jordan-Schoenflies theorem [6, pg. 257], since it's a continuum not separating S^2 and has a simple closed curve as its boundary. Clearly E contains p in its interior. Then E is the desired 2-cell.

COROLLARY 3.1.1. *Let h be an* a.p. *homeomorphism of E^2 onto itself. Then E^2 is the union of an increasing sequence of disks $\{B_i\}_{i=1}^{\infty}$ such that*

(1) $B_1 \subsetneqq B_2^0 \subsetneqq B_2 \subsetneqq B_3^0 \subsetneqq B_3 \subsetneqq \cdots \subsetneqq B_n^0 \subsetneqq B_n \subsetneqq \cdots$ *and*

(2) $h(B_n) = B_n$ *for all* n.

Proof. Let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers. By Theorem 3.1, there exist disks K_i' on S^2 such that (1) diam $K_i' < \varepsilon_i$, (2) $h(K_i') = K_i'$ and (3) K_i' contains p, the north pole of S^2. Let $K_1 = K_1'$, $K_2 =$ first K_i' such that $K_i' \subsetneqq (K_1)^0$, $K_3 =$ first K_i' such that $K_i' \subsetneqq (K_2)^0$, etc. Let $\varphi: S^2 \to E^2$ be the stereographic projection. Then $\{B_i\} = \{\varphi(K_i)\}$ is the desired sequence.

4. The main theorem.

In this section we prove the main theorem of this paper.

LEMMA 4.1. *Let B_1 and B_2 be 2-cells in E^2 such that $B_1 \subsetneqq B_2^0$. Let h be a homeomorphism of B_2 onto itself such that*

(1) $h(B_1) = B_1$,

(2) $h = \varphi^{-1} r \varphi$ *for some rotation r on the disk D_2 with center at the origin and radius 2, where $\varphi: B_2 \twoheadrightarrow D_2$ is a homeomorphism, and*

(3) $\varphi(\mathrm{Bd}\, B_1)$ *is a circle centered at the origin. Then there exists a homeomorphism $g: B_2 \twoheadrightarrow D_2$ such that*

(1) $g(\mathrm{Bd}\, B_1)$ *is the unit circle, and*

(2) $h = g^{-1} r g$.

Proof. We first make a definition. We call a homeomorphism $f: D_2 \twoheadrightarrow D_2$ *radial* iff f takes each radius onto itself, and is such that circles centered at the origin go onto circles centered at the origin.

Now let $\Psi: D_2 \twoheadrightarrow D_2$ be a radial homeomorphism of D_2 onto itself such that $\Psi(\varphi(\mathrm{Bd}\, B_1))$ is the unit circle. Then $\Psi\varphi$ is a homeomorphism of B_2 onto D_2 such that $\Psi\varphi(\mathrm{Bd}\, B_1)$ is the unit circle. Further, for any rotation r, since $\Psi^{-1} r \Psi = r$, $\varphi^{-1} r \varphi = \varphi^{-1}(\Psi^{-1} r \Psi)\varphi = \varphi^{-1}\Psi^{-1} r \Psi\varphi = (\Psi\varphi)^{-1} r (\Psi\varphi)$. Thus we let $g = \Psi\varphi$ and g is the desired homeomorphism.

LEMMA 4.2. *Let B_1 and B_2 be 2-cells in E^2 such that $B_1 \subsetneqq B_2^0$. Let $h: B_2 \twoheadrightarrow B_2$ be a homeomorphism such that*

(1) $h(B_1) = B_1$

(2) *there exists a homeomorphism* $\varphi_1: B_1 \twoheadrightarrow$ *unit disk such that* $h|B_1 = \varphi_1^{-1}r_1\varphi_1$, *for some rotation* $r_1: E^2 \twoheadrightarrow E^2$, *and*

(3) *there exists a homeomorphism* $\varphi_2: B_2 \twoheadrightarrow D_2$, *where* D_2 *is the disk of radius* 2 *about the origin, such that*

(a) $h = \varphi_2^{-1}r_2\varphi_2$, *for some rotation* r_2 *of* E^2 *onto itself, and*

(b) $\varphi_2(\mathrm{Bd}\, B_1) =$ *unit circle.*

Then there exists a homeomorphism $g: B_2 \twoheadrightarrow D_2$ *such that*

(1) $g|B_1 = \varphi_1$,

(2) $g(\mathrm{Bd}\, B_1) =$ *unit circle, and*

(3) $h = g^{-1}r_1g$.

Proof. Let the annulus between $\mathrm{Bd}\, D_1$ and $\mathrm{Bd}\, D_2$ be decomposed into the continuous collection $\mathscr{A}$ of arcs which are the intersections of the radii of D_2 with the annulus. Note that $\varphi_1\varphi_2^{-1}: D_1 \twoheadrightarrow D_1$ is a homeomorphism that takes $\varphi_2(x)$ to $\varphi_1(x)$ for each $x \in B_1$. We extend $\varphi_1\varphi_2^{-1}$ to a homeomorphism $\Psi: D_2 \twoheadrightarrow D_2$ by taking each element $A \in \mathscr{A}$ with endpoint $\varphi_2(x) \in \mathrm{Bd}\, D_1$ to the element $A' \in \mathscr{A}$ with endpoint $\varphi_1(x)$ in $\mathrm{Bd}\, D_1$, in such a way that distance along the segments A and A' are preserved. Thus Ψ is a homeomorphism of D_2 onto itself such that $\Psi | D_1 = \varphi_1\varphi_2^{-1}$.

Now let $g = \Psi\varphi_2$. We show that g is the required homeomorphism. Since $\Psi = \varphi_1\varphi_2^{-1}$ on D_1, $\Psi\varphi_2 = (\varphi_1\varphi_2^{-1})\varphi_2 = \varphi_1$ on B_1, so g is an extension of φ_1. Also $g(\mathrm{Bd}\, B_1) =$ unit circle. It remains to show that $h = g^{-1}r_1g$ on $B_2 - B_1$.

It is sufficient to show that $r_2 = r_1$ on $\mathrm{Bd}\, D_1$. Now $\varphi_1^{-1}r_1\varphi_1 = \varphi_2^{-1}r_2\varphi_2$ on $\mathrm{Bd}\, B_1$, so $r_2 = (\varphi_1\varphi_2^{-1})^{-1}r_1(\varphi_1\varphi_2^{-1})$ is a conjugate of a rotation. But it follows from [12] that the rotations are characterized by numbers in $1-1$ correspondence with $0 \leqq x < 1$, and any conjugate $f^{-1}rf$ of a rotation is characterized (even though not necessarily a rotation) by the same number as the number for the rotation r. Thus the characterizing number for a conjugate of r_1 is the same as for r_1. It follows that $r_1 = r_2$, and $h = g^{-1}r_1g$ on B_2.

THEOREM 4.1. *Let h be an almost periodic homeomorphism of E^2 onto itself. Then h is periodic.*

Proof. Let $\{B_i\}$ be an increasing tower of 2-cells of E^2 such that $B_1 \subsetneqq B_2^0 \subsetneqq B_2 \subsetneqq B_3^0 \subsetneqq B_3 \subsetneqq \cdots \subsetneqq B_n^0 \subsetneqq B_n \subsetneqq \cdots$, $\cup B_i = E^2$, and $h(B_i) = B_i$. This sequence exists by Corollary 3.1.1.

Case (i). *h is orientation preserving.* Since B_i is invariant, $h|B_i$ is a.p. on B_i and orientation preserving, and therefore is a conjugate of a rotation on the 2-cell D_i centered at the origin and of radius i. (This follows from [3].) That is, there exists a rotation $r_i\colon D_i \twoheadrightarrow D_i$ and a homeomorphism $\varphi_i\colon B_i \twoheadrightarrow D_i$ such that $h|B_i = \varphi_i^{-1} r_i \varphi_i$.

We will show that each r_i must be rational. Suppose by way of contradiction that r_i is an irrational rotation, for some i (the first such i). Then since $r_i = \varphi_i(h|B_i)\varphi_i^{-1}$ and B_{i-1} is invariant, $\varphi_i(B_{i-1})$ is invariant under r_i and in fact $\mathrm{Bd}\,(\varphi_i(B_{i-1}))$ is invariant under r_i. Let x be any point in $\mathrm{Bd}\,(\varphi_i(B_{i-1}))$. Since r_i is an invariant rotation $\overline{0(x)}$ under r_i will contain the circle C_x of radius $|x|$. Thus $C_x \subseteqq \varphi_i(\mathrm{Bd}\,(B_{i-1}))$. But $\varphi_i(\mathrm{Bd}\,(B_{i-1}))$ is a simple closed curve, and it follows that $C_x = \varphi_i(\mathrm{Bd}\,(B_{i-1}))$. By Lemma 4.1, we may assume that $\varphi_i(B_{i-1}) =$ radius (i − 1) disk, and by Lemma 4.2, we may assume that $\varphi_i|B_{i-1} = \varphi_{i-1}$, (that is φ_i is an extension of φ_{i-1}) and further that $r_i = r_{i-1}$. Thus since r_i is the first irrational rotation, $i = 1$.

Clearly this process may be continued inductively, obtaining $r_i = r_{i-1}$, for all i. But then h would be the conjugate of an irrational rotation on E^2. However, such a rotation is not a.p. since $d(x, h^n(x)) \longrightarrow \infty$ as $x \longrightarrow \infty$ (for fixed n). This is a contradiction. It follows that each r_i is a rational rotation.

Now since each r_i is a rational rotation, it is of finite order, say n_i. But since $h|B_i$ is a conjugate of a periodic homeomorphism of order n_i on the disk D_i, each point of B_i (except the "center") has the same order, namely n_i. Thus each point of B_{i-1} has order n_i under $h|B_i$ and therefore under $h|B_{i-1}$. We may backtrack inductively until $i = 2$, so that each of $\{h|B_1, h|B_2, \cdots, h|B_i\}$ makes each point of B_i (except the "center") a point of order n_i; that is, the orbit consists of n_i points. It follows that for any $j > i$, the points of B_j must all have order n_j and therefore $n_j = n_i$. Thus h must be periodic on $\cup B_i = E^2$. But a periodic orientation preserving homeomorphism on E^2 is a conjugate of a rotation on E^2 [8, 2, 14]. Thus h is a conjugate of a rotation.

Case (ii). *h is orientation reversing.* Since B_i is invariant, $h|B_i$ is a.p. on B_i and orientation reversing, and hence a conjugate of a reflection [3]. Thus the fixed point set of $h|B_i$ is a "diameter" of B_i, and every other point of B_i has order 2 (its orbit consists of 2 points). Thus $h|B_i$ is of order 2, also. By induction $\{h|B_1, h|B_2, h|B_3, \cdots, h|B_i\}$ are each of order 2. For any $j > i$, the order of $h|B_j =$ order of $h|B_i$, by the same argument. Thus h is of period 2 on $\cup B_i = E^2$, and also is orientation reversing. It follows that h is a conjugate of a reflection [8, 2].

REMARK. It is clear that we have also proved that if h is an almost periodic homeomorphism of S^2 onto itself which is orientation preserving, and therefore keeps at least one point fixed [1, pg. 237], then h is a conjugate of a rotation. How are arbitrary almost periodic homeomorphisms of S^2 onto itself characterized? There are many nonconjugate fixed point free, almost periodic homeomorphisms of S^2 onto itself. For example, let f be a reflection of S^2 thru the equator, and let r be any rotation of S^2 thru the axis containing the north and south poles. Then rf is fixed point free, no two are conjugate, and each of these is almost periodic. (Note that if r is the $180°$ rotation, then rf is the antipodal map.) Are conjugates of these maps the only fixed point free homeomorphisms on S^2? Gerhard Ritter has just informed me that he will answer this question in the affirmative, in a forthcoming paper.

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