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SOME PROPERTIES OF THE NASH BLOWING-UP

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Intuitively, in the Nash blowing-up process each singular point of an algebraic (or analytic) variety is replaced by the limiting positions of tangent spaces (at non-singular points). The following properties of this process are shown: 1) It is, locally, a monoidal transform; 2) in characteristic zero, the process is trivial if and only if the variety is non-singular. Examples show that this is not true in characteristic $p > 0$; that, in general, the transform of a hypersurface is not locally a hypersurface; and that this process does not give, in general, minimal resolutions.

Introduction. In this paper, the term algebraic variety (over a field k) means reduced, separated algebraic scheme over k; the term analytic variety means reduced, separated analytic space over $\mathbf{C}$, the complex numbers. Let k be an algebraically closed field (resp. $k = \mathbf{C}$), X a reduced closed subscheme of a Zariski open $U \subset \mathbf{A}^n$ (resp. a reduced closed complex subspace of an open $U \subset \mathbf{C}^n$) of pure dimension r, defined by $\{f_1, \cdots, f_m\} \subset \Gamma(U, \mathcal{O}_U)$. By the Nash blowing-up of X we mean the pair (X^*, p) obtained by the following process. Let $S(X)$ be the set of singular points of X, X_0 its complement in X, $\eta: X_0 \to X \times G_r^n$ (G_r^n is the grassmanian of r-planes in n-space) the morphism determined by $\eta(x) = (x, T_{X,x})$ for each closed point $x \in X_0$ (here $T_{X,x}$ is the tangent space of X at x, which can be identified with an r-plane in n-space), X^* the closure of $\eta(X_0)$ in $X \times G_r^n$ (resp. the closure in the metric topology), $p: X^* \to X$ induced by the first projection. In the complex case it is not obvious that X^* is an analytic variety; see [7], Theorem 16.4 for a proof (or see Theorem 1 of this note).

It is possible to prove that (X^*, p) is (up to unique X-isomorphism) independent of the immersion (as a locally closed subset) of X in an affine space, hence the process globalizes.

Sketch of proof. Working (to simplify) in the algebraic case with closed points only, and calling $G_r(T) = \{r\text{-linear planes in } T\}$ for any vector space T, one verifies that $Z = \bigcup_{x \in X} x \times G_r(T_{X,x})$ is a subvariety of $X \times G_r^n$, and X^* is contained in Z. If X' is a locally closed in $\mathbf{A}^{m'}$, we have (using notations as above, but with primes): $X'^* \subset X' \times G_r^{m'}$. Assume $q: X \to X'$ is an isomorphism. Then,

$$(x, L) \to (q(x), dq(L)),$$

for $(x, L) \in Z$, defines an isomorphism $Z \to Z'$. This clearly induces an isomorphism $X^* \to X'^*$, commuting with the projections.

A natural question, which apparently has not been seriously studied, is to determine the desingularization properties of this process. In this note we present some very basic results in this direction: (a) in characteristic zero, $p: X^* \to X$ is an isomorphism if and only if X is nonsingular, (b) in positive characteristic, (a) is false. We also verify (which allows to show (b) in a very clear way) that, locally, a Nash blowing-up is a monoidal transform, with center a suitable ideal. The proof of (a) presented here is analytic, and uses results of J. Stutz on branched coverings (cf. **[4]** and **[5]**). It would be interesting to have an algebraic proof which probably would throw more light on the main question: if, in characteristic zero, this process desingularizes (cf. Remark 3).

We finish with some examples, which indicate other features of the process (see §3).

1. Monoidal transforms. In this section k is, in the algebraic case, an algebraically closed field; in the analytic case $k = \mathbf{C}$. Our arguments hold in either case.

Recall that given a reduced subscheme X of $\mathbf{A}_k^n$ (resp. a reduced subspace X of an open U in $\mathbf{C}^n$) and $\{g_0, \cdots, g_s\} \subset \Gamma(X, \mathcal{O}_X)$, the monoidal transform of X with center $I = (g_0, \cdots, g_s)$ can be constructed by taking the closure (in $X \times \mathbf{P}^s$) of $\varphi(Y)$, where $Y = X \setminus V(I)$ ($V(I) =$ locus of I) and $\varphi: Y \to X \times \mathbf{P}^s$ is defined by $\varphi(x) = (x, (g_0(x), \cdots, g_s(x))) \in X \times \mathbf{P}^s$, for any closed point x (see **[1]**, Remark 2).

REMARK 1. We shall use the following notations:

(1) We have two closed embeddings of G_r^n in $\mathbf{P}^N$, $N = \binom{n}{r} - 1$:

(i) the map Λ, which sends the point corresponding to the r-plane L, of parametric equations $x_i = \sum_{d=1}^r b_d^i t_d$, $i = 1, \cdots, n$, to the point of $\mathbf{P}^N$ of homogeneous coordinates $(\Delta_{i_1 \cdots i_r})$, $1 \leqq i_1 < \cdots < i_r \leqq n$, where $\Delta_{i_1 \cdots i_r}$ is the $r \times r$ subdeterminant of $\| b_d^i \|$ formed by the columns $i_1, \cdots, i_r$.

(ii) the map ψ, which sends the point corresponding to the r-plane L, defined by the equations $\sum_{j=1}^n a_j^i x_j = 0$, $i = 1, \cdots, n - r$, to the point of $\mathbf{P}^N$ of homogeneous coordinates $(\Delta_{j_1 \cdots j_{n-r}})$, where $\Delta_{j_1 \cdots j_{n-r}}$ is the $(n - r) \times (n - r)$ subdeterminant of $\| a_j^i \|$ defined by the columns $j_1, \cdots, j_{n-r}$. In the terminology of **[2]**, Ch. VII, Λ corresponds to the use of Grassman coordinates and ψ to the dual Grassman coordinates; cf. Theorem I, p. 294 of **[2]** for their relations.

(2) Let X, η be as in the introduction. We shall write

$$\psi_0 = (\mathrm{id} \times \psi)\eta : X_0 \to X \times \mathbf{P}^N$$
$$\Lambda_0 = (\mathrm{id} \times \Lambda)\eta : X_0 \to X \times \mathbf{P}^N.$$

Clearly, there are natural isomorphisms:

$$X^* \approx \mathrm{cl}(\psi_0(X_0)) \approx \mathrm{cl}(\Lambda_0(X_0)),$$

where cl denotes closure in the corresponding ambient space.

(3) Given integers $n \geqq r > 0$, $m \geqq n - r$, and an $(m \times n)$-matrix $\| a_{ij} \| = A$, let S (resp. S') denote the set of increasing sequences of $(n-r)$-positive integers less than $m+1$ (resp. $n+1$); if $\alpha = (i_1, \cdots, i_{n-r}) \in S$, $\beta = (j_1, \cdots, j_{n-r}) \in S'$, then $\Delta_{\alpha\beta}$ is the subdeterminant of A obtained by considering the rows $i_1, \cdots, i_{n-r}$ and the columns $j_1, \cdots, j_{n-r}$.

THEOREM 1. *A Nash blowing-up is locally a monoidal tranform* (with center a suitable ideal).

Proof. We may assume X is affine (resp. an analytic set in $U \subset \mathbf{C}^n$); write X as union of its irreducible components, $X = X_1 \cup \cdots \cup X_d$ (in the analytic case, shrink U if necessary). Using the notations of the introduction, let $M = \| \partial f_i / \partial x_j \|$, $i = 1, \cdots, m$, $j = 1, \cdots, n$. Clearly, for each $i = 1, \cdots, d$, there is $(\alpha_i, \beta_i) \in S \times S'$ such that $\Delta_{\alpha_i\beta_i}$ does not vanish on X_i; hence $W'_i = X_i \setminus V(\Delta_{\alpha_i\beta_i})$ is a nonempty open of X_i. For each $i = 1, \cdots, d$, fix $h_i \in \Gamma(X, \mathcal{O}_X)$ such that $h_i = 0$ on $\bigcup_{j \neq i} X_j$, $h_i \neq 0$ on X_i. Consider the ideal I generated by $\{g_\beta\}$, $\beta \in S'$, where $g_\beta = \Sigma_{i=1}^d h_i \Delta_{\alpha_i\beta}$. We claim that the monoidal transform with center I agrees with the Nash blowing-up of X.

In fact, first note that $V(I) \supset S(X)$. Call $W = X \setminus V(I)$, then (since all points in $X_0 \setminus W$ are non-singular) $X^* \approx$ closure of $\psi_0(W)$ in $\mathbf{P}^N$. Hence, to show our contention we must check that the maps $W \to \mathbf{P}^N$ given by $p_2\psi_0 = \psi_1$ (p_2 is the second projection) and by $\{g_\beta\}$ agree. It is enough to check this at an arbitrary $z \in W_i = W'_i \setminus V(h_i)$ $(i = 1, \cdots, d)$. But for $z \in W_i$, as points of $\mathbf{P}^N$,

$$(g_\beta(z)) = \left(\sum_{j=1}^{d} h_j(z) \Delta_{\alpha_j\beta}(z) \right) = (h_i(z)\Delta_{\alpha_i\beta}(z)) = (\Delta_{\alpha_i\beta}(z)) = \psi(L'),$$

where L' is the point of G_r^n corresponding to the r-plane L, defined by $\{\Sigma(\partial f_{k_l}(z)/\partial x_j)x_j = 0\}$, $(k_l) = \alpha_i$, which is the tangent space to X at z (since $\Delta_{\alpha_i\beta_i} \neq 0$). As clearly $\psi(L') = \psi_1(z)$, the assertion is proved.

REMARK 2. If X (of dimension r) is defined by $n-r$ equations, then the proof is simpler. In fact, we may take I to be the Jacobian ideal, formed by the $(n-r)\times(n-r)$ minors of $\|\partial f_i/\partial x_j\|$, $i=1,\cdots,n-r$, $j=1,\cdots,n$. In general this is not true, as the example of two planes in $\mathbf{A}^4$, meeting at one point, shows.

In this example also it can be seen that, in general, the support of the ideal of Theorem 1 is not the singular locus of X.

EXAMPLE 1. Let $\mathrm{ch}(k)=2$, consider the plane curve $y^2+x^3=0$. By Remark 1, its Nash blowing-up is the monoidal transform with center $I=(2y,3x^2)=(x^2)$. This is a principal ideal, hence $p: X^*\to X$ is an isomorphism.

EXAMPLE 2. If $\mathrm{ch}(k)=q>2$, the Nash blowing-up of the plane curve $y^2-x^q=0$ is trivial. The verification is as in Example 1.

2. Proof of the main theorem.

THEOREM 2. *Let k be an algebraically closed field of characteristic zero* (resp. $k=\mathbf{C}$), *X a pure r-dimensional algebraic* (resp. analytic) *variety over k, (X^*,p) the Nash blowing-up of X. Then, p is an isomorphism if and only if X is nonsingular.*

Proof. By descent theory we may assume, in the algebraic case, that $k=\mathbf{C}$. Moreover, it is clear (e.g., from Theorem 1) that $(X^h)^*=(X^*)^h$ where X^h denotes the analytic variety associated to an algebraic variety X. Hence, it suffices to prove the theorem in the analytic case.

One implication is obvious. Let us show that if X has singularities the morphism p is not an isomorphism. Let $S=S(X)$ be the singular set of X. We distinguish two cases.

Case a. S has a component of codimension 1.

Let W be a component of codimension 1. We claim that there is a point $x_0\in W$, such that X can be embedded, locally about x_0, in a polydisk $U\subset\mathbf{C}^n$, in such a way that (with $x_1,\cdots,x_n$ coordinates in $\mathbf{C}^n$, and writing, to simplify, $X\subset U$, $W\subset U$, etc.):

(i) x_0 corresponds to the origin

(ii) W is defined by $x_r=\cdots=x_n=0$.

(iii) Let $X_1,\cdots,X_m$ be the irreducible components of X. Then, there are analytic functions $f_{ij}(x_1,\cdots,x_r)$, defined on

$$D=\{x\in\mathbf{C}^r/(x_1,\cdots,x_r,0,\cdots,0)\in U\}$$

such that

$$x \to (x_1, \cdots, x_{r-1}, x_r^{s_j}, f_{r+1,j}(x), \cdots, f_{n,j}(x))$$

defines a homeomorphism $D \to X_j$, $j = 1, \cdots, m$.

(iv) The integer s_j of (iii) is the multiplicity of X_j at any $x \in W$, and

$$f_{ij} = \sum_{k=s_j}^{\infty} a_{ij}^{(k)}(x_1, \cdots, x_{r-1})x_r^k.$$

This is a consequence of the following results. Let $W_0 = \{x \in W: S$ is nonsingular at x, $\dim C_4(X, x) = r$, $\dim C_5(X, x) = r + 1\}$. Hence, $C_i(X, x)$, $i = 4, 5$, are the indicated Whitney tangent cones to X at x (see **[6]**, §3 for the definitions). In **[4]**, Proposition 3.6, it is proved that W_0 is a dense open set in W. Propositions 2.5 of **[5]** and 4.2, 4.6 of **[4]** imply that for any $x_0 \in W_0$, there is a local embedding of the type described above.

From now on, we shall assume that X is contained in such an open $U \subset \mathbf{C}^n$. (The result that we are proving is clearly local on X.)

Note that now, keeping the notations of Remark 1, the map Λ_0: $X_0 \to X \times \mathbf{P}^N$ can be described as follows: calling $\varphi_{ij} = x_i$, $1 \leqq i \leqq r - 1$, $\varphi_{rj} = x_r^{s_j}$, $\varphi_{r+k,j} = f_{r+k,j}$, $k = 1, \cdots, n - r$, then, for $j = 1, \cdots, m$:

$$\Lambda_0:(\varphi_{ij}(x))_{i=1,\cdots,n} \to ((\varphi_{ij}(x))_{1 \leqq i \leqq n}, (\Delta_{i_1}^{(j)}, \cdots, i_r)) \in X \times \mathbf{P}^N,$$

where $0 < i_1 < i_2 < \cdots < i_r \leqq n$, and $\Delta_{i_1}^{(j)}, \cdots, i_r$ is the subdeterminant of $\| \partial\varphi_{ij}/\partial x_k \|$, $i = 1, \cdots, n$, $k = 1, \cdots, r$ formed of the rows $i_1, \cdots, i_r$. Note that $\Delta_{1,2,\cdots,r}^{(j)} = s_j x_r^{s_j - 1}$.

We may assume $p: X^* \to X$ to be bijective (otherwise, the theorem is trivial). Then, if $A = \{x \in \mathbf{P}^N / z_{1,\cdots,r} \neq 0\}$, by using condition (iv) of the parametrization we see that $\mathrm{cl}(\Lambda_0(X_0)) \subset U \times A$. Let us identify, by using $\mathrm{id} \times \Lambda$, the varieties X^* and $\mathrm{cl}(\Lambda_0(X_0))$. Then, the irreducible components X_j^* of $X^* \subset U \times A$ are parametrized by:

$$(*) \qquad (\varphi_{ij}(x), \cdots, \varphi_{nj}(x), (s_j^{-1} x_r^{-s_j+1} \Delta_{i_1}^{(j)}, \cdots, i_r)),$$

$1 \leqq i_1 < \cdots < i_r \leqq n$ (except $(1, 2, \cdots, r)$), $j = 1, \cdots, m$. By condition (iv), $(s_j x_r^{s_j - 1})^{-1} \Delta_{i_1}^{(j)}, \cdots, i_r$ are analytic functions.

Now there are two possibilities: (1) A component of X is singular at x_0; (2) All components of X are nonsingular at x_0.

To show that p is not an isomorphism, it clearly suffices to show that if $X^{(q)}$ is the qth iterated blowing-up of X (i.e., $X^{(0)} = X$, $X^{(1)} = X^*, \cdots, X^{(q)} = (X^{q-1})^*$), then the induced canonical morphism

$p_q: X^{(q)} \to X$ is not an isomorphism. We shall see that in either situation (1) or (2) above, this is the case.

Consider (1) first; let X_j ($j = 1, \cdots, M \leqq m$) be the components of X which are singular at x_0. After changing coordinates (if necessary), we may assume that we have a parametrization of the components of X satisfying (i) to (iv), and also:

$$f_{r+1,1} = \sum_{k=d}^{\infty} a^{(k)}_{r+1,1}(x_1, \cdots, x_{r-1}) x_r^k$$

where $a^{(d)}_{r+1,1} \neq 0$, and d is not a multiple of $s = s_1$. Write $(i) = (i_1, \cdots, i_r)$, and, for $j = 1, \cdots, m$,

$$(s x_r^{s-1})^{-1} \Delta^{(j)}_{(i)} = \psi'^{(j)}_{(i)} = \sum b^{(j)}_{(i),k}(x_1 \cdots x_{r-1}) x_r^k;$$

let $\psi^{(j)}_{(i)} = \psi'^{(j)}_{(i)} - b^{(1)}_{(i),0}$. Consider $X^* \subset U \times A$ (we maintain the identification of X^* with cl $\Lambda_0(X_0)$). After an obvious change of coordinates we may assume that $p^{-1}(W)$ is defined, in $U \times A$, by $z_i = 0$, $i \geqq r$, and the parametrization of $X^*_j = p^{-1}(X_j)$ induced by $(*)$ is:

$$(x_1, \cdots, x_{r-1}, x_r^s, (\psi^{(j)}_{(i)})), \qquad j = 1, \cdots, m.$$

If for some $j = 1, \cdots, M$, there is an (i) such that $b^{(j)}_{(i),k} \neq 0$ with $k < s_j$, then the multiplicity of X^*_j, at some point near x_0, is less than s, and hence p is not an isomorphism. If not, then $(*)$ induces a parametrization of $X^* \subset (U \times A)$, satisfying (i) to (iv). We can repeat the process. We claim that after at most μ blowing-ups, with $\mu = [d/s]$, either p_μ is not bijective, or the multiplicity s of $p_\mu^{-1}(X_1)$, at some point near $p_\mu^{-1}(x_0)$, drops. In fact, were $p_0 = p, p_1, \cdots, p_\mu$ bijective, then one of the entries of the induced parametrization of $p_\mu^{-1}(X_1)$ is of the form

$$(**) \qquad \sum_{k=d}^{\infty} \gamma_{k,\mu}\, a^{(k)}_{r+1,1}(x_1, \cdots, x_{r-1}) x_r^{k-\mu s},$$

where $\gamma_{k,\mu} = s^{-\mu} \prod_{v=1}^{\mu} k - (v-1)s$. Since $a^{(m)}_{q_1} \neq 0$, and $(d, s) < n$, then $0 < d - \mu s < s$ for $\mu = [d/s]$, and hence the multiplicity of such $X_1^{(\mu)}$, at some point near $p_\mu^{-1}(x_0)$, is less than s.

Consider the case (2). Since we assume that p is bijective, then for all $x \in W$, $T_{X_i,x} = T_{X_j,x}$, $i = 1, \cdots, m$. We also may assume, after a further change of coordinates, that, aside from (i) to (iv), we have: X_m is the r-plane $x_{r+1} = \cdots = x_r = 0$ (i.e., $f_{k,m}(x) = 0$, $k = r+1, \cdots, n$). As before, we see that if the iterated blowing-ups

$$p_u: X^{(u)} \to X^{(u-1)}, \qquad 1 \leqq u \leqq q$$

were bijective, then we can obtain a parametrization of the components of $X^{(q)}$, such that $X_m^{(q)} = p_q^{-1}(X_m)$ and $X_1^{(q)} = p_q^{-1}(X_1)$ are given, respectively, by $(x_1, \cdots, x_r, 0, \cdots, 0)$ and $(x_1, \cdots, x_r, \psi_1, \cdots, \psi_L)$ (for some integer L) where, for some i_0,

$$\psi_{i_0} = \sum_{k=d}^{\infty} \frac{k!}{(k-q)!} a_{r+1,1}^{(k)}(x_1, \cdots, x_{r-1}) x_r^{k-q}.$$

If $a_{r+1,1}^{(k)} = 0$ for $k < d$, and nonzero for $k = d$, then for $q = d - 1$, ψ_{i_0} has the form

$$\psi_{i_0} = a(x_1, \cdots, x_{r-1}) x_r + \cdots, \qquad a \neq 0.$$

Clearly, if $a(z_1, \cdots, z_n) \neq 0$, then $T_{X_1^{(q)},z} \neq T_{X_m^{(q)},z}$, and the next Nash blowing-up has at least two points lying over z. Hence, p cannot be an isomorphism.

Case b. S has codimension > 1 at each of its points.

The only nontrivial case is the following: assume that for all $x \in S$, for all $\{x_i\} \to x$, $i = 1, 2, \cdots, x_i$ nonsingular, such that $\{T_{X,x_i}\}$ converges (in G_r^n), $\lim T_{X,x_i}$ is a fixed space T_x (otherwise, $p^{-1}(x)$ has more than one point). Assume this is the case. Pick any $x_0 \in S$, and embed locally X in a polydisk U in $\mathbf{C}^n$ (as before, we just write $X \subset U$, $S \subset U$, etc.). Then, $C_4(X, x_0)$ has dimension r. In fact, this cone is the set of limit poisitions of lines, tangent to nonsingular points of X. By **[7]** (Part I, Preliminaries), the function $d: G_r^n \times \mathbf{P}^1 \to \mathbf{R}$ where, for an r-plane $L \in G_r^n$ and a line $\ell \in \mathbf{P}^1$, $d(L, \ell) =$ distance between L and ℓ (intuitively, the sine of the angle between L and ℓ) is continuous. From this, it follows that $C_4(X, x_0) \subset T_{x_0} = T$, hence $\dim C_4(X, x_0) \leqq r$. Since the inequality $\dim C_4(X, x_0) \geqq r$ always holds, we get $\dim C_4(X, x_0) = r$. By Proposition 2.6 of **[4]**, this equality implies that, after shrinking U if necessary, the projection π on T, along a s(n − r)-plane transversal to T satisfies: $B(\pi) = \{x \in X / \pi$ ramifies at $x\} = S$. Hence, $\dim B(\pi) < n - 1$. This inequality implies, by the statement 1.8 of **[4]**, that (after further shrinking of U, if necessary) all the irreducible components X_i of X are nonsingular, and $S(X) = B(\pi) = \bigcup_{i \neq j} (X_i \cap X_j)$. Thus, there is more than a component at x_0 (since x_0 is a singular point of X), and $T_{X_i,x_0} = T$ for all i. By changing coordinates (if necessary), we may assume $X_1 = T$, and by the implicit function theorem we may parametrize simultaneously the components X_j:

$$(x_1, \cdots, x_r, \varphi_{r+1,j}(x'), \cdots, \varphi_{n,j}(x')), \qquad x' = (x_1, \cdots, x_r).$$

Exactly as in part (2) of case (a), we see that after a finite number of Nash blowing-ups, the components of X get separated, and hence p cannot be an isomorphism.

The proof of the theorem is complete.

We have the following corollary, which seems to be well known:

COROLLARY 1. *If C is an algebraic curve, defined over an algebraically closed field a characteristic zero* (resp. an analytic curve), *a finite sequence of Nash blowing-ups desingularizes the curve.*

Proof. In general, the Nash blowing-up $p: C^* \to C$ is a finite birrational (resp. bimeromorphic) morphism. Then it is clear that after a finite number of Nash blowing-ups, we reach the normalization. Since this is nonsingular, the result follows.

As we saw, this is false in positive characteristic.

REMARK 3. In [7], J. Lipman proves, in a purely algebraic way, that for an algebraic variety X, the monoidal transform with center the sheaf of Jacobian ideals is trivial if and only if X is smooth. Since, by Remark 2, for complete intersections this transform agrees with the Nash blowing-up, it gives an algebraic proof of Theorem 2 in this case.

3. Some remarks and examples. In general, the Nash blowing-up of a hypersurface is not locally a hypersurface, as the following example shows.

EXAMPLE 3. Let $\mathrm{char}(k) = 0$. Consider the plane curve X of parametric equations:

$$x = t^4, \quad y = \varphi(t) = t^{11} + t^{13}.$$

Let $p: X^* \to X$ be the Nash blowing-up, $x_0 \in X$ the origin. Then $p^{-1}(x_0)$ has only one point x_1, and a neighborhood X_1 of x_1 in X^* is naturally contained in $\mathbf{A}^2 \times \mathbf{A}^1 \subset \mathbf{A}^2 \times G_1^2$, and has a parametrization

$$x = t^4, \qquad y = t^{11} + t^{13}, \qquad u = \frac{11}{4} t^7 + \frac{13}{4} t$$

(cf. proof of Theorem 2). We claim that the embedding dimension of X_1 *at* x_1 is 3. In fact, if $\mathrm{emb.\,dim}_{x_1} X_1 = 2$, it would follow

$$y = t^{11} + t^{13} = g\left(t^4, \frac{11}{4} t^7 + \frac{13}{4} t^9\right)$$

for some $g(x, u) \in k[[x, u]]$. An elementary computation shows that this is impossible.

Even if it were true that the Nash blowing-up process desingularizes, in general one would not get "minimal" resolutions. Consider this example:

EXAMPLE 4. Let $\mathrm{ch}(k) \neq 2$. Consider the surface X (in $\mathbf{A}^3$) defined by $y^2 - x^2z = 0$. It is well known that the normalization X' of this surface is nonsingular. Moreover, this normalization can be obtained by applying the monoidal transform with center the z-axis. Thus, one can get a desingularization $\pi: X' \to X$, with π finite. However, the Nash blowing-up $p: X^* \to X$ is not finite, in fact $p^{-1}(0)$ is a projective line. But X^* is nonsingular; in fact, using Remark 1, and the fact that (for any ideal I), the monoidal transforms with center the ideal I and I^2 coincide, it is easy to see that the Nash blowing-up of X can be obtained by composing $\pi: X' \to X$ and the quadratic transform of X' with center the point $\pi^{-1}(0)$.

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