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GENERALIZED RIGHT ALTERNATIVE RINGS

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We show that weakening the hypotheses of right alternative rings to the three identities

(1) $(ab, c, d) + (a, b, [c, d]) = a(b, c, d) + (a, c, d)b$
(2) $(a, a, a) = 0$
(3) $([a, b], b, b) = 0$

for all a, b, c, d in the ring will not lead to any new simple rings. In fact, the ideal generated by each associator of the form (a, b, b) is a nilpotent ideal of index at most three. Our proofs require characteristic $\neq 2$, $\neq 3$.

Introduction. We shall call a ring a GRA ring (for generalized right alternative ring) if it satisfies the following three identities:

$$(1) \quad 0 \equiv \bar{A}(a,b,c,d) = (ab,c,d) + (a,b,[c,d]) - a(b,c,d) - (a,c,d)b$$

$$(2) \quad 0 \equiv (a,a,a)$$

$$(3) \quad 0 \equiv ([b,a],a,a).$$

On all rings that we study in this paper, we assume that for $n = 2$ or $n = 3$, the map $x \rightarrow nx$ is one-to-one and onto. This is equivalent to weakly characteristic $\neq 2$, $\neq 3$ (see [1]). All three conditions are consequences of the right alternative law $(a, x, x) \equiv 0$ and characteristic $\neq 2$, and thus GRA rings are generalizations of right alternative rings. Similar conditions have been studied by E. Kleinfeld, H. F. Smith, I. R. Hentzel, and G. M. Piacentini, usually through an idempotent decomposition. The results given here generalize much of their work, mainly by dispensing with the assumption of an idempotent. Our work shows the relationship of these rings to right alternative rings; this is a simpler and more direct approach than that which has been done before.

When we are dealing with a GRA ring R, we shall let I be the additive subgroup generated by all associators of the form (a,b,b) for all $a,b \in R$. I is a measure of how far R is from being a right alternative ring. We show that I is an ideal of R, that I is commutative, and that I is the sum of ideals of R whose cube is zero. This means that if R is simple, or even nil-semi-simple, then R is right alternative. Since all the hypotheses on R are consequences of the right alternative law, showing that R is right alternative is as strong a result as one could hope for.

The three hypotheses chosen are individually expressive of well-known ring structure. Given (1), then (2) holds $\Leftrightarrow$ the ring is power-associative. Given (1) and (2), then (3) holds $\Leftrightarrow$ the ring under the symmetric product $a \circ b = ab + ba$ is a Jordan ring.

Basic identities and definitions. An ideal I of R is called trivial if $I \neq 0$ and $I^2 = 0$. A ring R is called semi-prime if R has no trivial ideals. R is called simple if $R^2 \neq 0$ and R has no ideals except 0 and R itself. The associator (a,b,c) is defined by $(a,b,c) = (ab)c - a(bc)$. The commutator $[a,b]$ is defined by $[a,b] = ab - ba$.

To simplify the notation, dot and juxtaposition will be used to indicate multiplication. When both appear, juxtaposition indicates that product is taken first. Thus $ab \cdot c = (ab)c$.

In expressions where elements are supposed to appear, we often place a set of elements. This means we are considering the additive group spanned by all the elements generated as the arguments of the expression vary through the indicated sets. Thus (R,x,x) means the additive subgroup generated by $\{(r,x,x) | r \in R\}$.

The following identities are used:

$$(4) \quad 0 \equiv \bar{B}(a, b) = (a, b, b) + (b, a, b) + (b, b, a).$$

$$(5) \quad 0 \equiv \bar{C}(a, b, c) = [a, (b, c, c)] + [c, (b, a, c)] + [c, (b, c, a)].$$

$$(6) \quad 0 \equiv \bar{D}(a, b, c, d) = (a, b, cd) - (a, bd, c) - (a, b, d)c + (a, d, c)b.$$

$$(7) \quad 0 \equiv \bar{E}(a, b, c, d) = (ab, c, d) - (a, bc, d) + (a, b, cd) - a(b, c, d) - (a, b, c)d.$$

Proof. Property (4) is a linearization of property (2). To show (5), it will suffice to show $0 \equiv [a, (b,a,a)]$. This follows since $0 \equiv ([b,a],a,a) + \bar{A}(a,b,a,a) - \bar{A}(b,a,a,a) = -[a,(b,a,a)]$. Property (7) is the Teichmüller equality which holds in any non-associative ring. Property (6) follows since $\bar{A}(a,b,d,c) + \bar{D}(a,b,c,d) = \bar{E}(a,b,d,c)$.

Main section. For comparison with other papers discussed in the final section of this paper, we will need a form of Lemma 1 and Lemma 2 that does not require (3).

LEMMA 1. *Let R be a nonassociative ring satisfying* (1) *and* (2). *Then* $0 \equiv [a,(b,a,a)] + 4(b,a,a)a - 2(b,a,a^2)$.

Proof.

$$
\begin{aligned}
0 &= \bar{A}(a,a,a,b) + \bar{A}(a,a,b,a) - \bar{A}(b,a,a,a) + \bar{D}(a,b,a,a) \\
&\quad - \bar{D}(a,a,b,a) - \bar{D}(b,a,a,a) + a \cdot \bar{B}(b,a) + 2\bar{B}(b,a) \cdot a + \bar{B}(ba, a) \\
&= a(b,a,a) + 3(b,a,a)a - 2(b,a,a^2) + \{(b,a,a^2) + (b,a^2,a) + (a,b,a^2) \\
&\quad + (a^2,b,a) + (a,a^2,b) + (a^2,a,b)\}.
\end{aligned}
$$

The expression in braces is zero by the linearized form of (4); the remainder is the conclusion of the lemma.

LEMMA 2. *Let R be a nonassociative ring satisfying* (1) *and* (2). *Then*

$$
\begin{aligned}
12(b,x,x)a = &-3\{(b,a,x^2) + (b,x^2,a)\} \\
&+ 5\{(b,x,ax) + (b,ax,x)\} \\
&+ \{(b,x,xa) + (b,xa,x\} \\
&- [a,(b,x,x)] - x,(b,a,x)] - [x,(b,x,a)].
\end{aligned}
$$

Proof. Linearize Lemma 1 to obtain

$$
\begin{aligned}
0 \equiv \bar{F}(b,a,x) &= 2(b,a,x^2) + 2(b,x,ax) + 2(b,x,xa) \\
&- [a,(b,x,x)] - [x,(b,a,x)] - [x,(b,x,a)] \\
&- 4(b,x,x)a - 4(b,a,x)x - 4(b,x,a)x.
\end{aligned}
\tag{8}
$$

The proof follows since

$$
\begin{aligned}
0 &= \bar{F}(b,a,x) + 4\bar{D}(b,x,a,x) - 4\bar{D}(b,a,x,x) \\
&\quad - \bar{D}(b,x,a,x) - \bar{D}(b,x,x,a) - \bar{D}(b,a,x,x) = -3(b,a,x^2) - 3(b,x^2,a) \\
&\quad + 5(b,x,ax) + 5(b,ax,x) + (b,x,xa) + (b,xa,x) - [a,(b,x,x)] \\
&\quad - [x,(b,a,x)] - [x,(b,x,a)] - 12(b,x,x)a.
\end{aligned}
$$

The remainder of this section will deal with GRA rings.

LEMMA 3. *If R is a GRA ring, then for each fixed $b \in R$, $P_b =$ the additive subgroup spanned by $\{(b,x,x) \mid x \in R\}$ is a right ideal of R.*

The proof is immediate from Lemma 2 and $0 = \bar{C}(a,b,x)$.

LEMMA 4. *If R is a GRA ring, then*

$$(a,x,x)(b,y,y)+(a,y,y)(b,x,x)=0$$

for all $a,b,x,y \in R$.

Proof. We call a map $D:R \to R$ a derivation if $(ab)D = a(bD)+(aD)b$. From (1) it is clear that $aD_x = (a,x,x)$ is a derivation on R. From Lemma 3, $aD_{x,y} = ((a,x,x),y,y)$ is also a derivation on R since $((a,x,x),y,y) = ((a,x,x)y)y - (a,x,x)y^2 \in P_a$. This means

$$(ab)D_{x,y} = a(bD_{x,y}) + (aD_{x,y})b.$$

In contrast to this, if we expand differently,

$$\begin{aligned}(ab)D_{x,y} &= ((ab)D_x)D_y = ((aD_x)b + a(bD_x))D_y = (aD_xD_y)b \\ &+ (aD_x)(bD_y) + (aD_y)(bD_x) + a(bD_xD_y) = (aD_{x,y})b + (aD_x)bD_y) \\ &+ (aD_y)(bD_x) + a(bD_{x,y}).\end{aligned}$$

Comparing this with the previous sentence gives $(aD_x)(bD_y)+(aD_y)(bD_x)=0$; this is the identity of Lemma 4.

LEMMA 5. *If R is a GRA ring, then $(a,x,x)(b,y,y)+(b,x,x)(a,y,y)=0$.*

Proof. $(a,x,x)(a,y,y) = -(a,x,x)\{(y,a,y)+(y,y,a)\}$ by (4) $= \{(a,a,y)+(a,y,a)\}(y,x,x)$ by Lemma 4 $= -(y,a,a)(y,x,x)$ by (4). We have established $(a,x,x)(a,y,y) = -(y,a,a)(y,x,x)$. Iterating this three times gives $(a,x,x)(a,y,y) = -(y,a,a)(y,x,x) = (x,y,y)(x,a,a) = -(a,x,x)(a,y,y)$. Thus $(a,x,x)(a,y,y)=0$. We then linearize this in the element a to get the identity of Lemma 5.

THEOREM 1. *Let R be a GRA ring. The following properties hold for R.*

(a) *I is an ideal of R.*
(b) *Every element of I is a sum of elements of $\cup_{a\in R}P_a$.*
(c) $[I,I]=0$.
(d) $(P_a)^2=0$ *for all $a \in R$.*
(e) $[I^2,R]=0$.

Proof. Property (b) is clear. Since P_a is a right ideal for each $a \in R$, I is also a right ideal; equation (1) then shows that I is a two-ideal of R. This shows (a). We now show (c): $(a,y,y)(b,x,x) = -(a,x,x)(b,y,y)$ from Lemma 4 $I = (b,x,x)(a,y,y)$ from Lemma 5. Lemma 5 also shows (d). We now show (e): from (2) we have $[a^2,a] = 0$; linearizing this gives $[a^2,b] + [ab + ba,a] = 0$. Now, if i and j are elements of I, and b is an element of R, $[ij + ji,b] + [ib + bi,j] + [jb + bj,i] = 0$. From parts (a) and (c) we have $2[ij,b] = 0$; therefore $[I^2, R] = 0$.

Theorem 1 shows that I is commutative. It follows that for each $a \in R$, P_a is an ideal of the subring I. It also follows that I is nil. Actually, we have shown I is a Baer-lower-radical ring. We will go on and show a much stronger condition on nilpotence, but we will state the above result as a theorem now.

THEOREM 2. (a) *If R is a simple GRA ring, then R is right alternative.*

(b) *If R is a nil-semi-simple GRA ring, then R is right alternative.*

LEMMA 6. *Let R be a GRA ring. Then*

(a) $(a,(b,c,c),d) = (a,c,c)d \cdot b - (a,c,c)b \cdot d$.
(b) $(a,(b,c,c),b) = 0$.
(c) $I[R, I] = [R, I]\, I = 0$.

Proof of (a). By (1) and Lemma 3, the map $xD = (x,c,c)d$ is a derivation. Thus $(ab)D = (aD)b + a(bD)$. As in Lemma 4, D_c is also a derivation; so $(ab)D_c = (aD_c)b + a(bD_c)$. Combining these two expressions gives

$$(ab, c, c)d = (a, c, c)d \cdot b + a \cdot (b, c, c)d = (a, c, c)b \cdot d + a(b, c, c) \cdot d.$$

Therefore $(a,(b,c,c),d) = (a,c,c)\, d \cdot b - (a,c,c)\, b \cdot d$.

The statement (b) follows from (a). We now prove (c). Let $d \in I$. Then $(a,(b,c,c),d) = (a,c,c)\, d \cdot b - (a,c,c)\, b \cdot d$ by (a) $=$ $(d,(a,c,c),b)$ by parts (a) and (c) of Theorem 1 $= -(d,(b,c,c),a)$ by part (b) of this proof. Now, continuing, $0 = (a,(b,c,c),d) + (d,(b,c,c),a) = a(b,c,c) \cdot d - a \cdot (b,c,c)\, d + d\, (b,c,c) \cdot a - d \cdot (b,c,c)\, a = d\, [a,(b,c,c)]$ by parts (a), (c), and (e) of Theorem 1.

Since $d \in I$, we have shown that $I\, [R, I] = 0$. By part (c) of Theorem 1, we have $[R, I]\, I = 0$ as well. This finishes the proof of Lemma 6.

LEMMA 7. *Let R be a GRA ring. Then,*

(a) *$K = \{([R, I],x,x) \mid x \in R\}$ is an ideal and $IK = KI = 0$.*

(b) *$A(I)=\{a \in I \mid (a,x,x)+aI+Ia \subseteq K$ for all $x \in R\}$ is an ideal, $I \cdot A(I)+A(I) \cdot I \subseteq K$, and $[R,I] \subseteq A(I)$.*

Proof of (a). K is understood to be the subgroup spanned by the indicated elements. K is a right ideal of R from Lemma 3. From (1) and Lemma 6.c, $b([R,I],x,x) \subseteq ([b,[R,I]],x,x)+([R,I],x,x)b \subseteq K$. Thus K is a left ideal of R. By (1) and Theorem 1.c, if $i \in I$, $([r,i],x,x)=[r,(i,x,x)]$. Thus $IK+KI \subseteq I[R,I]+[R,I]I=0$ by Lemma 6.c.

Proof of (b). First notice that $A(I)$ is an ideal. This requires (1), part (a), Lemma 6.c, and Theorem 1.c. Clearly, $I \cdot A(I)+A(I) \cdot I \subseteq K$. Lemma 6.c says $[R,I] \subseteq A(I)$.

THEOREM 3. *Let $\langle(a,b,b)\rangle$ be the ideal of R generated by the single associator (a,b,b). Then $\langle(a,b,b)\rangle^3=0$.*

Proof. By Lemma 3 and Lemma 7, $\langle(a,b,b)\rangle \subseteq P_a+A(I)$. Using Theorem 1.d, $\langle(a,b,b)\rangle^2 \subseteq P_aP_a+I \cdot A(I)+A(I) \cdot I+A(I)A(I) \subseteq K$. Therefore, $\langle(a,b,b)\rangle^3 \subseteq IK+KI=0$.

COROLLARY. *If R is a semi-prime GRA ring, then R is right alternative.*

Proof. From Lemma 7.a, $K^2=0 \Rightarrow K=0$. From the proof of Theorem 3, $\langle(a,b,b)\rangle^2 \subseteq K=0$; thus $\langle(a,b,b)\rangle=0$. Since $(a,b,b)=0$ for all $a,b \in R$, R is right alternative.

Example of a GRA ring. If A is an associative and commutative ring with an element $1/2$, and M is any module over A, then $S=A \times M$ can be made into a GRA ring by the following definition of addition and multiplication. Addition is coordinatewise. Multiplication is given by $(a,m)(a',m')=(aa',1/2am'+1/2a'm)$. If we identify M with $\{0\} \times M$, then M is a two-sided ideal of S, and $[S,M]=0$. If $1/2+1/2=e$ (e is the identity of A), then e is an idempotent of S, and $-4(m,e,e)=m$ for all $m \in M$. The ring S gives us a counterexample to various questions we might raise. For example:

1. There exist GRA rings which are not right alternative.
2. In a GRA ring R, I need not be in the nucleus.
3. In a GRA ring R, $((R,x,x),x,x)$ need not be zero.
4. Based on the example S, one might attempt to show that in any GRA ring R, $[R,I]=0$. This we have not been able to show, but we have shown that for any element $b \in R$, $\langle((R,b,b),b,b)\rangle$ is an ideal of R

which commutes elementwise with R. This corresponds closely to M since $M = \langle((S,e,e),e,e)\rangle$.

Related work. Hypotheses similar to (1), (2), and (3) have been studied. In **[6]** and **[3]**, the three identities listed below were assumed.

(a) $(ab,c,d) + (a,b,[c,d]) = a(b,c,d) + (a,c,d)\,b.$
(b) $(a,b,cd) + ([a,b],c,d) = c(a,b,d) + (a,b,c)\,d.$
(c) $(a,a,a) = 0.$

In **[4]**, the condition (b) was replaced by flexibility;
(b′) $(a,b,a) = 0$ for all $a,b \in R$.

LEMMA 8. (Kleinfeld). *If R is a ring satisfying (a), (b), and (c), or (a), (b'), and (c), then R is a GRA ring.*

Proof. We must show $([b,a],a,a) = 0$ for all $a, b \in R$. From (a) and (c) we have $([b,a],a,a) = -[a,(b,a,a)]$. The proof of Lemma (1) required only (a) and (c). Therefore,

$$\begin{aligned} 0 &= [a,(b,a,a)] + 4(b,a,a)a - 2(b,a,a^2) \\ &= 3([b,a],a,a) - 2\{(b,a,a^2) + ([b,a],a,a) - a(b,a,a) - (b,a,a)a\} \end{aligned}$$

If (b) holds, the expression in braces is 0, and hence $([b,a],a,a) = 0$. If (b′) holds, using (a) and (b′), we get $(b,a,a^2) - ([b,a],a,a) - a(b,a,a) - (b,a,a)a = 0$. As above, we get

$$0 = -([b,a],a,a) - 2\{(b,a,a^2) - ([b,a],a,a) - a(b,a,a) - (b,a,a)a\}.$$

Therefore, (a), (b′), and (c) imply (3).

THEOREM 4. *If R is a semi-prime ring satisfying (a), (b), and (c), or (a), (b'), and (c), then R is alternative.*

Proof. By the corollary to Theorem 3, R is right alternative. If (b′) holds, then R is alternative. If (b) holds, by the mirror form of the corollary to Theorem 3, R is left alternative. Thus R is alternative in this case as well.

Theorem 4 is an impressive generalization of **[3]**, **[4]**, **[5]**, and **[6]**.

I. R. Hentzel and G. M. Piacentini have studied rings satisfying only conditions (1) and (2). They have shown that when such rings are simple and possess an idempotent, then they must be right alternative. In view of this result, it seems that perhaps equation (3) is not necessary. This seems even more plausible since only equations (1) and (2) imply the result

$$([[a,b],c],c,c) \equiv 0.$$

The proof of Lemma 3 requires that $([a,x],x,x) = 0$ for all a,x in the ring. Without this, $D_{a,b}$ is not a derivation. The fact that $D_{a,b}$ was a derivation was the basis of all our results.

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