

## SUBCONTINUITY FOR MULTIFUNCTIONS

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**The concept of subcontinuity is extended to multifunctions. This notion is then used to obtain a number of results on multifunctions with closed graphs and to develop criteria under which a multifunction is upper semicontinuous.**

1. **Introduction.** In [1] R. V. Fuller introduced the concept of subcontinuous function and used it to obtain conditions implying continuity as well as some comparisons to compactness preserving functions and some results on functions with closed graphs. In [1] Fuller stated that he thought that his results would hold for multifunctions. The purpose of this paper is to show that such is the case. Indeed the definition of subcontinuity can be extended and conditions implying upper semi-continuity for multifunctions are derived, including a generalization of a result of Muenzenberger and Smithson [4].

By a *multifunction* we mean a correspondence  $F: X \rightarrow Y$  on a set  $X$  into a set  $Y$  such that  $F(x)$  is a nonempty subset of  $Y$  for each  $x \in X$ . If  $F$  is a multifunction, then the *graph of  $F$*  is the subset  $\{(x, y): x \in X, y \in F(x)\}$  of  $X \times Y$ . We denote the graph of  $F$  by  $G(F)$ . Then a multifunction  $F$  has a *closed graph* if  $G(F)$  is a closed subset of  $X \times Y$ . Whereas  $F$  is a *closed multifunction* if  $F(A)$  is closed in  $Y$  for all closed sets  $A \subset X$ . Further,  $F$  is *upper semi-continuous* (u.s.c.) if and only if for each closed set  $B \subset Y$ ,  $F^{-1}(B) = \{x: F(x) \cap B \neq \emptyset\}$  is closed. This is equivalent to saying that  $F^{-1}$  (where  $F^{-1}(y) = \{x: y \in F(x)\}$ ) is a closed multifunction on  $F(X) = \bigcup \{F(x): x \in X\}$ .

If  $A \subset X$ , then  $A^-$  denotes the closure of  $A$ . Other notation used is that of Kelley [3].

2. **Subcontinuous multifunctions.** Using Fuller's definition as a model we obtain the following definition.

**DEFINITION.** A multifunction  $F: X \rightarrow Y$  is *subcontinuous* if and only if whenever  $\{x_\alpha, \alpha \in D\}$  is a convergent net in  $X$  and  $\{y_\alpha, \alpha \in D\}$  is a net in  $F(X)$  with  $y_\alpha \in F(x_\alpha)$  then  $\{y_\alpha, \alpha \in D\}$  has a convergent subnet.

Next we say that  $F$  is *inversely subcontinuous* in case  $F^{-1}$  is subcontinuous. Note that since  $F$  and  $F^{-1}$  are both multifunctions we did not need a separate definition in terms of nets for inverse subcontinuity. Also, if  $F$  is single valued then the present definitions

are equivalent to Fuller's definitions. Finally, since the concepts of subcontinuity and inverse subcontinuity are dual, a result about one also has a dual result for the other. In most cases we shall give only the result for subcontinuous functions.

Immediate consequences of the definition of subcontinuous multifunction  $F$  are: If  $K \subset X$  is compact and if  $F(K)$  is closed, then  $F(K)$  is compact. That is, in this case  $F$  preserves compact sets. Consequently a point closed (i.e.  $F(x)$  closed for each  $x \in X$ ) subcontinuous multifunction is point compact (i.e.  $F(x)$  is compact for each  $x \in X$ ).

**THEOREM 2.1.** *If  $F: X \rightarrow Y$  is a subcontinuous multifunction on the space  $X$  into the completely regular space  $Y$  and if  $K \subset X$  is compact, then  $F(K)^-$  is compact.*

*Proof.* Suppose  $K \subset X$  is compact and let  $\{y_\alpha, \alpha \in D\}$  be a net in  $F(K)$ . Let  $\mathcal{B}$  be a base for the uniformity on  $Y$  and partial order  $\mathcal{B}$  by inclusion downward. Then give  $D \times \mathcal{B}$  the product partial order. Then for each  $a = (\alpha, B) \in D \times \mathcal{B}$  pick an element  $z_a$  in  $B[y_\alpha] \cap F(K)$ . Next let  $x_a$  be an element of  $K$  such that  $z_a \in F(x_a)$ . Now since  $K$  is compact some subnet  $\{x_{a_n}\}$  of  $\{x_a, a \in D \times \mathcal{B}\}$  converges and consequently some subnet of  $\{z_{a_n}\}$  say  $\{z_{a_m}\}$  converges, since  $F$  is subcontinuous, to  $z_0$ . Now  $a_m = (\alpha_m, B_m)$  and  $z_{a_m} \in B_m[y_{\alpha_m}]$ . Next let  $U$  be a neighborhood of  $z_0$  and let  $B$  be a symmetric member of  $\mathcal{B}$  such that  $B \circ B[z_0] \subset U$ . There is an  $a_{m'}$  such that  $a_{m'} < a_m$  (i.e.  $\alpha_{m'} < \alpha_m$  and  $B_m \subset B_{m'}$ ) implies that  $z_{a_{m'}} \in B[z_0]$  and since  $z_m$  is a subnet of  $z_a$ ,  $B_m \subset B$ . Now  $(z_0, z_{a_m}) \in B$  and  $z_{a_m} \in B_m[y_{\alpha_m}]$  so  $(y_{\alpha_m}, z_{a_m}) \in B_m \subset B$  and since  $B$  is symmetric  $(z_0, y_{\alpha_m}) \in B \circ B$ . Thus  $y_{\alpha_m} \in B \circ B[z_0] \subset U$ . Hence, a subnet of  $\{y_\alpha, \alpha \in D\}$  converges.

Next we characterize compact preserving multifunctions in terms of subcontinuity.

**THEOREM 2.2.** *Let  $F: X \rightarrow Y$  be a multifunction. Then  $F$  is compact preserving if and only if  $F|K \rightarrow F(K)$  is subcontinuous for each compact set  $K \subset X$ .*

*Proof.* Suppose  $F$  is compact preserving and let  $K \subset X$  be compact. Then  $F(K)$  is compact and consequently  $F|K \rightarrow F(K)$  is subcontinuous.

Next let  $K \subset X$  be compact and suppose  $F|K \rightarrow F(K)$  is subcontinuous. But then each set in  $F(K)$  must have a convergent subnet and thus  $F(K)$  is compact. Hence,  $F$  is compact preserving.

3. Closed graphs and upper semicontinuity. A number of

examples exist which show that a function with a closed graph need not be continuous. However, it is easy to show that a function with a closed graph into a compact space is continuous (see remark below). By using subcontinuity we can get a significant generalization of this.

**THEOREM 3.1.** *Let  $F: X \rightarrow Y$  be a subcontinuous multifunction which has a closed graph. Then  $F$  is upper semicontinuous.*

*Proof.* Let  $A \subset Y$  be closed and let  $x_0 \in F^{-1}(A)^-$ . Further, let  $\{x_\alpha, \alpha \in D\}$  be a net in  $F^{-1}(A)$  which converges to  $x_0$ . Let  $\{y_\alpha, \alpha \in D\}$  be net in  $A$  such that  $y_\alpha \in F(x_\alpha)$  for each  $\alpha$ . Then, since  $F$  is subcontinuous, there is a convergent subnet  $y_{\alpha_n} \rightarrow y_0 \in A$ . If  $y_0 \notin F(x_0)$ , then  $(x_0, y_0) \notin G(F)$  and since  $G(F)$  is closed, there exist open sets  $U \subset X$ ,  $V \subset Y$ , such that  $(x_0, y_0) \in U \times V \subset X \times Y \setminus G(F)$ . But  $x_\alpha \rightarrow x_0$  and  $y_{\alpha_n} \rightarrow y_0$ . Thus there is an  $\alpha_n$  such that  $x_{\alpha_n} \in U$  and  $y_{\alpha_n} \in V$  which is a contradiction. Thus  $y_0 \in F(x_0)$  and  $x_0 \in F^{-1}(A)$ . Hence,  $F$  is u.s.c.

We can deduce another result from the proof of Theorem 3.1. This result strengthens known results on upper semicontinuous point closed multifunctions.

**PROPOSITION 3.2.** *Let  $F: X \rightarrow Y$  be a subcontinuous multifunction with a closed graph. Let  $x_\alpha \rightarrow x_0$  and let  $\{y_\alpha\}$  be a net such that  $y_\alpha \in F(x_\alpha)$  for each  $\alpha$ . If  $y_\alpha \rightarrow y_0$ , then  $y_0 \in F(x_0)$ .*

If we assume that  $Y$  is regular in Theorem 3.1, then the converse is also true.

**THEOREM 3.3.** *If  $F: X \rightarrow Y$  is a point closed upper semicontinuous multifunction into a regular space, then  $F$  has a closed graph.*

*Proof.* Suppose  $(x, y) \notin G(F)$ . Then  $y \notin F(x)$ . Thus there are disjoint open sets  $U, V \subset Y$  such that  $F(x) \subset U$  and  $y \in V$ . Further, there is an open set  $W \subset X$  with  $x \in W$  such that  $F(W) \subset U$  ( $W$  exists because  $F$  is u.s.c.). Thus  $(x, y) \in W \times V$  and  $W \times V \cap G(F) = \emptyset$ . Hence,  $G(f)$  is closed.

**REMARK.** In [1] Fuller had a result for single valued functions which is analogous to Theorem 3.3. In this result Fuller assumed that  $Y$  was Hausdorff. But in the case of multifunctions we need to assume that  $Y$  is regular.

As we have seen a multifunction with a closed graph is in a sense almost u.s.c. In this regard there is an exercise in Kelley [3] which asserts that for such a function  $F(K)$  is closed and  $F^{-1}(K')$  is

closed whenever  $K, K'$  are compact (a proof of this appears in [4]). Next we develop some conditions under which a multifunction has a closed graph. For this we need the following definition.

DEFINITION. A multifunction  $F: X \rightarrow Y$  is called *locally closed* if and only if for each  $p \in X$  and for each neighborhood  $U$  of  $p$  there is a neighborhood  $V$  of  $p$  such that  $V \subset U$  and  $F(V)$  is closed.

A closed function or multifunction on a regular space is locally closed. But a locally closed function need not be closed.

THEOREM 3.4. *If  $F: X \rightarrow Y$  is locally closed and if  $F^{-1}(y)$  is closed for each  $y \in Y$ , then  $F$  has a closed graph.*

*Proof.* If  $y \notin F(x)$ , then  $x \notin F^{-1}(y)$  and so there is a neighborhood  $U$  of  $x$  such that  $U \cap F^{-1}(y) = \emptyset$  and  $F(U)$  is closed. Then there is an open set  $V$  with  $y \in V$  such that  $F(U) \cap V = \emptyset$ . Then  $U \times V$  is a neighborhood of  $(x, y)$  which misses  $G(F)$ . Hence,  $F$  has a closed graph.

COROLLARY 3.5. *If  $F: X \rightarrow Y$  is a closed multifunction on a regular space  $X$  and if  $F^{-1}(y)$  is closed for each  $y \in Y$ , then  $F$  has a closed graph.*

REMARK. Corollary 3.5 is a restatement of Theorem 3.3.

Since a closed multifunction on a regular space is locally closed, and since a multifunction with compact range is subcontinuous; Theorems 3.1 and 3.4 given the following result from [5] which was a generalization of a result of Halfar [2].

COROLLARY 3.6. *If  $F: X \rightarrow Y$  is a closed multifunction on a regular space  $X$  into a compact space  $Y$  for which  $F^{-1}(y)$  is closed for all  $y \in Y$ , then  $F$  is upper semicontinuous.*

Another immediate consequence of the definition of locally closed is that a locally closed multifunction with closed point inverses is point closed.

By putting together some of the above we can prove.

THEOREM 3.7. *If  $X$  is a regular, locally compact space, and if  $F$  is a multifunction on  $F$ , then the following are equivalent.*

- (i)  $F$  has a closed graph
- (ii) If  $K \subset X$  is compact, then  $F(K)$  and  $F^{-1}(y)$  are closed for each  $y \in Y$ .

(iii)  $F$  is locally closed and  $F^{-1}(y)$  is closed for each  $y \in Y$ .

*Proof.* That (iii) implies (i) is Theorem 3.4 and (i) implies (ii) was proved in [4]. Finally, since  $X$  is regular and locally compact,  $X$  has a compact bases for each point and hence, (ii) implies (iii).

4. Compact preserving multifunctions and  $k_3$ -spaces. In [1] Fuller defines what he calls a  $k_3$ -space. This definition is only slightly different from the definition of a  $k$ -space as given by Kelley [3]. In fact when the space is Hausdorff they are identical. We shall use the definition given by Fuller.

DEFINITION. A topological space  $X$  is a  $k_3$ -space provided that a set is open in  $X$  if and only if its intersection with each compact subset of  $X$  is relatively open.

Our next theorem gives an extension of the main result in [4].

THEOREM 4.1. Let  $F: X \rightarrow Y$  be a multifunction on a  $k_3$ -space. If  $F$  is compact preserving and if  $F$  has a closed graph, then  $F$  is upper semicontinuous.

*Proof.* Let  $A \subset Y$ . If  $F^{-1}(A)$  is not closed, then there is a compact set  $K \subset X$  such that  $K \cap (X \setminus F^{-1}(A))$  is not open in  $K$ . Now  $F(K)$  is compact and thus  $A \cap F(K)$  is compact. Hence,  $F^{-1}(A \cap F(K)) = C$  is closed. Next we show that  $F^{-1}(A) \cap K = C \cap K$ . For this

$$\begin{aligned} C \cap K &= F^{-1}(A \cap F(K)) \cap K = F^{-1}(A) \cap F^{-1}(F(K)) \cap K \\ &= F^{-1}(A) \cap K. \end{aligned}$$

Thus  $F^{-1}(A) \cap K$  is relatively closed and so  $(X \setminus F^{-1}(A)) \cap K$  is open in  $K$ . Thus  $F^{-1}(A)$  is closed.

Next we prove another result for  $k_3$ -spaces which is a variation on one of Fuller's results.

THEOREM 4.2. Let  $F: X \rightarrow Y$  be a multifunction on a Hausdorff  $k_3$ -space  $X$  into a Hausdorff space  $Y$ . If  $F$  is compact preserving and if  $F^{-1}(y)$  is closed for all  $y \in Y$ , then  $F$  is upper semicontinuous.

*Proof.* Let  $A \subset Y$  be closed, then in order to show that  $F^{-1}(A)$  is closed it is sufficient to show that  $F^{-1}(A) \cap K$  is closed for each compact set  $K \subset X$ . For this note that  $F^{-1}(A) \cap K = (F|K)^{-1}(A)$ , and thus it will suffice to show that  $F|K$  is u.s.c. Now since  $K$  is compact  $T_2$ , it is regular and since  $F|K$  is closed ( $F|K$  is compact

preserving and  $Y$  is  $T_2$ ),  $F$  is locally closed with  $F^{-1}(y)$  closed for each  $y$ . Then  $F|K$  has a closed graph by Theorem 3.4 and thus Theorem 4.1 shows that  $F|K$  is u.s.c. Hence,  $F^{-1}(A) \cap K$  is closed and the theorem follows.

#### REFERENCES

1. R. V. Fuller, *Relations among continuous and various noncontinuous functions*, Pacific J. Math., **25** (1968), 495-509.
2. E. Hefner, *Compact mappings*, Proc. Amer. Math. Soc., **8** (1957), 828-830.
3. J. L. Kelley, *General Topology*, D. van Nostrand Co., Princeton, 1955.
4. T. B. Muenzenberger and R. E. Smithson, *Continuity and closed functions*, to appear.
5. R. E. Smithson, *On criteria for continuity*, Nieuw archief voor Wiskunde, **14** (1966), 89-92.

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