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CARATHÉODORY AND HELLY-NUMBERS OF CONVEX-PRODUCT-STRUCTURES

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Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be the Carathéodory-numbers of the convexity-structures $\mathscr{C}_1$ for X_1, respectivily $\mathscr{C}_2$ for X_2. It is shown that the Carathéodory-number $\mathfrak{c}$ of the convex-product-structure $\mathscr{C}_1 \oplus \mathscr{C}_2$ for $X_1 \times X_2$ satisfies the inequality $\mathfrak{c}_1 + \mathfrak{c}_2 - 2 \leqq \mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$; $\mathfrak{c}_1, \mathfrak{c}_2 \geqq 2$.

The upper bound for $\mathfrak{c}$ can be improved by one, resp. two, if a certain number, namely the so-called exchange-number, of one resp. each of the structures $\mathscr{C}_1$ and $\mathscr{C}_2$ is less than or equal to the Carathéodory-number of that structure.

A new definition of the Helly-number is given and Levi's theorem is proved with this new definition. Finally it is shown that the Helly-number of a convex-product-structure is the greater of the Helly-numbers of $\mathscr{C}_1$ and $\mathscr{C}_2$.

1. Preliminary remarks and definitions. Existing notations and definitions have been taken from [3], [4] and, in particular, from [8]. Let $\mathscr{C}$ be a collection of subsets of a set X; by $\bigcap \mathscr{C}$ and $\bigcup \mathscr{C}$ we denote the intersection and the union respectivily, of the elements of $\mathscr{C}$. $\mathscr{C}$ is called a *convexity-structure* for X iff $\emptyset \in \mathscr{C}$, $X \in \mathscr{C}$ and $\bigcap \mathscr{F} \in \mathscr{C}$ for each subcollection $\mathscr{F} \subset \mathscr{C}$; the pair $(X, \mathscr{C})$ is called a *convexity-space*. The *$\mathscr{C}$-hull* of a set $S \subset X$, denoted by $\mathscr{C}(S)$, is defined by $\mathscr{C}(S) = \bigcap \{C \mid C \in \mathscr{C} \wedge S \subset C\}$. We shall write $\mathscr{C}(a_1, \cdots, a_n)$ instead of $\mathscr{C}(\{a_1, \cdots, a_n\})$, and $\mathscr{C}(p \cup (A \backslash a))$ instead of $\mathscr{C}(\{p\} \cup (A \backslash \{a\}))$.

Let X_i be a nonempty set and let $\mathscr{C}_i$ be a convexity-structure for X_i; $i = 1, 2$. Then $\mathscr{C}_1 \oplus \mathscr{C}_2 = \{A \times B \mid A \in \mathscr{C}_1 \wedge B \in \mathscr{C}_2\}$ is a convexity-structure for the Cartesian-product $X_1 \times X_2$. The pair $(X_1 \times X_2, \mathscr{C}_1 \oplus \mathscr{C}_2)$ is a called the *convex-product-space*, also called the *Eckhoff-space*. Note that the $\mathscr{C}_1 \oplus \mathscr{C}_2$-hull of $E \subset X_1 \times X_2$ is given by $(\mathscr{C}_1 \oplus \mathscr{C}_2)(E) = \mathscr{C}_1(\pi_1 E) \times \mathscr{C}_2(\pi_2 E)$, where π_1 is the projection of $X_1 \times X_2$ on X_i; $i = 1, 2$. Also note that if $e_1, e_2, e_3 \in X_1 \times X_2$ with $e_1 \neq e_2$ and $\pi_i(e_i) = \pi_i(e_3)$ for $i = 1, 2$, then $e_3 \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(e_1, e_2)$.

2. The Carathéodory-number and the exchange-number. A convexity-structure $\mathscr{C}$ for X is said to possess the *Carathéodory-number* $\mathfrak{c}$ if $\mathfrak{c}$ is the smallest nonnegative integer such that $\mathscr{C}(S) = \bigcup \{\mathscr{C}(T) \mid T \subset S \wedge |T| \leqq \mathfrak{c}\}$, for all $S \subset X$. The following lemma is an immediate consequence of this definition.

LEMMA 2.1. *Let* $\mathscr{C}$ *be a convexity-structure for* X *with Carathéodory-number* $\mathfrak{c}$ *and let* $\mathfrak{k} \in N$ $(N = 1, 2, 3, \cdots)$. *Then the following holds*:

(i) $(\exists A)[A \subset X \wedge |A| = \mathfrak{c} \wedge \mathscr{C}(A) \not\subset \bigcup \{\mathscr{C}(A \backslash a) \mid a \in A\}]$;

(ii) $(\exists A)[A \subset X \wedge |A| = \mathfrak{k} \wedge \mathscr{C}(A) \not\subset \bigcup \{\mathscr{C}(A \backslash a) \mid a \in A\}] \Rightarrow \mathfrak{c} \geqq \mathfrak{k}$.

DEFINITION 2.1. The *exchange-number* of a convexity-structure $\mathscr{C}$ for X is the smallest positive integer $\mathfrak{e}$, such that

$$(\forall A)(\forall p)[A \subset X \wedge p \in X \wedge \mathfrak{e} \leqq |A| < \infty \Longrightarrow \mathscr{C}(A) \subset \bigcup \{\mathscr{C}(p \cup (A \backslash a)) \mid a \in A\}] .$$

Of course, if the exchange-number $\mathfrak{e}$ of the convexity-structure $\mathscr{C}$ for X exists then $\mathfrak{e} \geqq 1$; if $\mathscr{C}$ is a T_1-convexity-structure (see [4]) then $\mathfrak{e} \geqq 2$; if $A \subset X$, $|A| \geqq \mathfrak{e}$ and $p \in \mathscr{C}(A)$ then $\mathscr{C}(A) = \bigcup \{\mathscr{C}(p \cup (A \backslash a)) | a \in A\}$, see [5], axiom C3; if the Carathéodory-number $\mathfrak{c}$ of $\mathscr{C}$ exists too, then $\mathfrak{e} \leqq \mathfrak{c} + 1$, which follows directly from Lemma 2.1(ii).

EXAMPLE 2.1. Take $X = \boldsymbol{R}^{\mathfrak{n}} (\mathfrak{n} \in N)$ and $\mathscr{C} =$ conv, the usual convexity-structure for the Euclidean-space $\boldsymbol{R}^{\mathfrak{n}}$. The classical theorem of Carathéodory implies that $\mathfrak{c} = \mathfrak{n} + 1$; see [2]. In [6] J. R. Reay proved that $\mathfrak{e} = \mathfrak{n} + 1$.

EXAMPLE 2.2. Let $\mathscr{C} = \{X\} \cup \{A \mid A \subset X \wedge |A| \leqq \mathfrak{k}\}$, $\mathfrak{k} \in N$. Then $\mathfrak{c} = \mathfrak{k} + 1$ and $\mathfrak{e} = 2$.

EXAMPLE 2.3. Take $M \subset X$, $|M| = \mathfrak{m}$ $(\mathfrak{m} \in N)$, and define $\mathscr{C} = \{X\} \cup \{A \mid A \subset X \wedge M \not\subset A\}$. Because $\mathscr{C}(A) = X$ if $M \subset A$, and $\mathscr{C}(A) = A$ if $M \not\subset A$, it follows that $\mathfrak{c} = \mathfrak{m}$ and $\mathfrak{e} = \mathfrak{m} + 1$.

EXAMPLE 2.4. The convexity-structure $\mathscr{C} = \{X\} \cup \{A \mid A \subset X \wedge |A| < \infty\}$, with $|X| = \infty$, has no Carathéodory-number, but the exchange-number is 2.

EXAMPLE 2.5. The convex-product-structure conv $\oplus$ conv for $\boldsymbol{R} \times \boldsymbol{R}$ has Carathéodory-number 2 and exchange-number 3.

THEOREM 2.1. *Let* $\mathscr{C}_i$ *be a convexity-structure for* X_i, $X_i \neq \emptyset$; *let* $\mathfrak{c}_i$ *and* $\mathfrak{e}_i$ *be the Carathéodory-number respectivily the exchange-number of* $\mathscr{C}_i$; $i = 1, 2$. *The Carathéodory-number* $\mathfrak{c}$ *of* $\mathscr{C}_1 \oplus \mathscr{C}_2$ *exists and the following assertions hold*:

I. *If* $\min(\mathfrak{c}_1, \mathfrak{c}_2) = 1$ *then*

a. $\mathfrak{c}_1 + \mathfrak{c}_2 - 1 \leqq \mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$

b. $(\exists i)[i \in \{1, 2\} \wedge \mathfrak{e}_i \leqq \mathfrak{c}_i] \Rightarrow \mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2 - 1$.

II. *If* $\min(\mathfrak{c}_1, \mathfrak{c}_2) \geqq 2$ *then*

a. $\mathfrak{c}_1 + \mathfrak{c}_2 - 2 \leqq \mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$

b. $(\exists i)[i \in \{1, 2\} \wedge \mathfrak{e}_i \leqq \mathfrak{c}_i] \Rightarrow \mathfrak{c}_1 + \mathfrak{c}_2 - 2 \leqq \mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2 - 1$

c. $(\forall i)[i \in \{1, 2\} \wedge \mathfrak{e}_i \leqq \mathfrak{c}_i] \Rightarrow \mathfrak{c} = \mathfrak{c}_2 + \mathfrak{c}_2 - 2$.

Proof. First we show that the Carathéodory-number $\mathfrak{c}$ of $\mathscr{C}_1 \oplus \mathscr{C}_2$ exists and that $\mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$. Let $\emptyset \neq E \subset X_1 \times X_2$ and $(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(E) = \mathscr{C}_1(\pi_1 E) \times \mathscr{C}_2(\pi_2 E)$. Hence there exists a set $S_i \subset \pi_i E$, $|S_i| \leqq \mathfrak{c}_i$ such that $a_i \in \mathscr{C}_i(S_i)$; $i = 1, 2$. $S_i \subset \pi_i E$ implies, there exists a set $F_i \subset E$ such that $\pi_i F_i = S_i$ and $|F_i| = |S_i|$; $i = 1, 2$. So $(a_1, a_2) \in \mathscr{C}_1(S_1) \times \mathscr{C}_2(S_2) = \mathscr{C}_1(\pi_1 F_1) \times \mathscr{C}_2(\pi_2 F_2) \subset \mathscr{C}_1(\pi_1(F_1 \cup F_2)) \times \mathscr{C}_2(\pi_2(F_1 \cup F_2)) = (\mathscr{C}_1 \oplus \mathscr{C}_2)(F_1 \cup F_2)$. Obviously $|F_1 \cup F_2| \leqq \mathfrak{c}_1 + \mathfrak{c}_2$. Because $(\mathscr{C}_1 \oplus \mathscr{C}_2)(F) \subset (\mathscr{C}_1 \oplus \mathscr{C}_2)(E)$ for each $F \subset E$, we have $(\mathscr{C}_1 \oplus \mathscr{C}_2)(E) = \bigcup \{(\mathscr{C}_1 \oplus \mathscr{C}_2)(F) \mid F \subset E \wedge |F| \leqq \mathfrak{c}_1 + \mathfrak{c}_2\}$; hence $\mathfrak{c}$ exists and $\mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$.

To determine the lower bound for c we choose, according to Lemma 2.1.(i), a set $A_i \subset X_i$ such that $|A_i| = \mathfrak{c}_i$ and $\mathscr{C}_i(A_i) \not\subset \bigcup \{\mathscr{C}_i(A_i \backslash a) \mid a \in A_i\}$; $i = 1, 2$. Take $b_1 \in A_1$ and $b_2 \in A_2$, and consider the set $G = (A_1 \times \{b_2\}) \cup (\{b_1\} \times A_2) \subset X_1 \times X_2$ Obviously $|G| = \mathfrak{c}_1 + \mathfrak{c}_2 - 1$. There are two cases (take $\mathfrak{c}_1 \leqq \mathfrak{c}_2$):

1. Let $\mathfrak{c}_1 = 1$. Then $A_1 = \{b_1\}$ and $G = \{b_1\} \times A_2$. So we have $(\mathscr{C}_1 \oplus \mathscr{C}_2)(G) = \mathscr{C}_1(b_1) \times \mathscr{C}_2(A_2) \not\subset \mathscr{C}_1(b_1) \times \bigcup \{\mathscr{C}_2(A_2 \backslash b) \mid b \in A_2\} = \bigcup \{(\mathscr{C}_1 \oplus \mathscr{C}_2)(G \backslash e) \mid e \in G\}$, and it follows from Lemma 2.1(ii) that $\mathfrak{c} \geqq \mathfrak{c}_1 + \mathfrak{c}_2 - 1$.

2. Let $\mathfrak{c}_1 \geqq 2$. Then also $c_2 \geqq 2$. So there exists an element $d_i \in A_i$ with $d_i \neq b_i$; $i = 1, 2$. Note that $\pi_1((b_1, b_2)) = \pi_1((b_1, d_2))$ and $\pi_2((b_1, b_2)) = \pi_2((d_1, b_2))$. The last remark of § 1 gives us that $(b_1, b_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)((d_1, b_2), (b_1, d_2)) \subset (\mathscr{C}_1 \oplus \mathscr{C}_2)(G \backslash (b_1, b_2))$. Define $F = G \backslash \{(b_1, b_2)\}$. Clearly $|F| = \mathfrak{c}_1 + \mathfrak{c}_2 - 2$ and $\pi_i F = A_i$; $i = 1, 2$. Moreover $(\mathscr{C}_1 \oplus \mathscr{C}_2)(F) \not\subset \bigcup \{(\mathscr{C}_1 \oplus \mathscr{C}_2)(F \backslash e) \mid e \in F\}$. From Lemma 2.1(ii) it follows that $\mathfrak{c} \geqq \mathfrak{c}_1 + \mathfrak{c}_2 - 2$.

We now prove I.b and II.b. Let us assume that $\mathfrak{e}_1 \leqq \mathfrak{c}_1$. Take $\mathfrak{c}_1, \mathfrak{c}_2 \geqq 1$, $\emptyset \neq E \subset X_1 \times X_2$ and $(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(E)$. We show that there exists a set F such that

$$(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(F),\ F \subset E \text{ and } |F| \leqq \mathfrak{c}_1 + \mathfrak{c}_2 - 1\,.$$

The first part of the proof of this theorem implies that there exists a set F_i such that $a_i \in \mathscr{C}_i(\pi_i F_i)$, $F_i \subset E$ and $|\pi_i F_i| = |F_i| \leqq \mathfrak{c}_i$; $i = 1, 2$. We may assume that $|F_i| = \mathfrak{c}_i$ $\forall i \in \{1, 2\}$, because if $|F_i| \leqq \mathfrak{c}_i - 1$ for some $i \in \{1, 2\}$ then $|F_1 \cup F_2| \leqq \mathfrak{c}_1 + \mathfrak{c}_2 - 1$, and we are done. If $F_1 \cap F_2 \neq \emptyset$, then define $F = F_1 \cup F_2$. So $|F| \leqq \mathfrak{c}_1 + \mathfrak{c}_2 - 1$ and $(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(F)$. If $F_1 \cap F_2 = \emptyset$, we distinguish two cases:

1. $(\exists i)[i \in \{1, 2\} \wedge \pi_i F_1 \cap \pi_i F_2 \neq \emptyset]$. Take $i = 1$. Hence $\pi_1 F_1 \cap \pi_1 F_2 \neq \emptyset$. Now there exist elements $e_1 \in F_1$ and $e_2 \in F_2$ such that $\pi_1(e_1) = \pi_1(e_2)$. Note that $e_1 \neq e_2$. Define $F = (F_1 \backslash \{e_1\}) \cup F_2$. Clearly $|F| \leqq c_1 + c_2 - 1$ and $\pi_i F_i \subset \pi_i F$; $i = 1, 2$. So $(a_1, a_2) \in \mathscr{C}_1(\pi_1 F_1) \times \mathscr{C}_2(\pi_2 F_2) \subset \mathscr{C}_1(\pi_1 F) \times \mathscr{C}_2(\pi_2 F) = (\mathscr{C}_1 \oplus \mathscr{C}_2)(F)$.

2. $(\forall i)[i \in \{1, 2\} \Rightarrow \pi_i F_1 \cap \pi_i F_2 = \emptyset]$. Take $e \in F_2$. Then $\pi_1(e) \notin \pi_1 F_1$. Because $e_1 \leqq c_1$, there exists an element $e_1 \in F_1$ such that $a_1 \in \mathscr{C}_1(\pi_1(e) \cup \pi_1 F_1 \backslash \pi_1(e_1)) \subset \mathscr{C}_1(\pi_1(e \cup F_1 \backslash e_1))$. Define $F = (F_1 \backslash \{e_1\}) \cup F_2$. Obviously $|F| \leqq c_1 + c_2 - 1$, $a_1 \in \mathscr{C}_1(\pi_1 F)$ and $a_2 \in \mathscr{C}_2(\pi_2 F_2) \subset \mathscr{C}_2(\pi_2 F)$. Hence $(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(F)$.

Finally we prove II.c. Take again $\emptyset \neq E \subset X_1 \times X_2$ and $(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(E)$. We shall show that there exists a set F such that

$$(a_1, a_2) \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(F), \; F \subset E, \text{ and } |F| \leqq c_1 + c_2 - 2\,.$$

In the proof of II.b we found a set $G_i \subset E$ such that $|\pi_i G_i| = |G_i| \leqq c_i$, $a_i \in \mathscr{C}(\pi_i G_i)$ and $|G_1 \cup G_2| \leqq c_1 + c_2 - 1$; $i = 1, 2$. As in the proof of II.b we may assume that $|G_i| = c_i$; $i = 1, 2$. If $|G_1 \cup G_2| > 1$ then define $F = G_1 \cup G_2$, hence $a_i \in \mathscr{C}_i(\pi_i F)$ and $|F| \leqq c_1 + c_2 - 2$, so we are done. The case that $|G_1 \cap G_2| = 1$ still remains. Assume $G_1 \cap G_2 = \{e\}$, and $|G_i| \geqq 2$ for $i = 1, 2$. Throughout the remainder of the proof we take $i, j \in \{1, 2\}$ with $i + j = 3$. Let $e_i \in G_j$, $e_i \neq e$. There are two cases:

1. If $\pi_i(e_i) \notin \pi_i G_i$ then, because $e_i \leqq c_i$, there exists an element $u_i \in G_i$ such that $u_i \neq e_i$ and $a_i \in \mathscr{C}_i(\pi_i(e_i) \cup (\pi_i G_i \backslash \pi_i(u_i))) \subset \mathscr{C}_i(\pi_i(e_i \cup (G_i \backslash u_i)))$.

2. If $\pi_i(e_i) \in \pi_i G_i$ then there exists an element $v_i \in G_i$ such that $v_i \neq e_i$ and $\pi_i(e_i) = \pi_i(v_i)$; hence $\pi_i G_i = \pi_i(e_i \cup (G_i \backslash v_i))$, so $a_i \in \mathscr{C}_i(\pi_i G_i) = \mathscr{C}_i(\pi_i(e_i \cup (G_i \backslash v_i)))$.

We may conclude that in both cases there exists an element $d_i \in G_i$ such that $d_i \neq e_i$ and $a_i \in \mathscr{C}_i(\pi_i(e_i \cup (G_i \backslash d_i)))$.

If $d_1 = e = d_2$ then define $F = G_1 \cup G_1 \backslash \{e\}$. Hence $|F| \leqq c_1 + c_2 - 2$ and, because $e_i \in G_j$, $e_i \neq e$ we have $(a_1, a_2) \in \mathscr{C}_1(\pi_1(e_1 \cup (G_1 \backslash e))) \times \mathscr{C}_2(\pi_2(e_2 \cup (G_2 \backslash e))) \subset (\mathscr{C}_1 \oplus \mathscr{C}_2)(G_1 \cup G_2 \backslash e) = (\mathscr{C}_1 \oplus \mathscr{C}_2)(F)$. If $d_i \neq e$ for some $i \in \{1, 2\}$ then, taking e.g. $i = 1$, we define $F = G_1 \cup G_2 \backslash \{d_1\}$. Clearly $|F| \leqq c_1 + c_2 - 2$ and because $e_1 \in G_2$ we have $(a_1, a_2) \in \mathscr{C}_1(\pi_1(e_1 \cup (G_1 \backslash d_1))) \times \mathscr{C}_2(\pi_2 G_2) \subset (\mathscr{C}_1 \oplus \mathscr{C}_2)(G_1 \cup G_2 \backslash d_1) = (\mathscr{C}_1 \oplus \mathscr{C}_2)(F)$.

It follows that $c \leqq c_1 + c_2 - 2$ and because $c \geqq c_1 + c_2 - 2$, as we have seen already, we may conclude that $c = c_1 + c_2 - 2$. This completes the proof of Theorem 2.1.

EXAMPLE 2.6. Take $X_1 = \boldsymbol{R}^{\mathfrak{m}}$ and $X_2 = \boldsymbol{R}^{\mathfrak{n}}$ ($\mathfrak{m}, \mathfrak{n} \in \boldsymbol{N}$), and $\mathscr{C}_1 = \text{conv} = \mathscr{C}_2$ (see Example 2.1). Because $\mathfrak{c}_1 = \mathfrak{e}_1 = \mathfrak{m} + 1$ and $\mathfrak{c}_2 = \mathfrak{e}_2 = \mathfrak{n} + 1$ it follows from Theorem 2.1. (II.c) that the Carathéodory-number of the convex-product-structure $\text{conv} \oplus \text{conv}$ for $\boldsymbol{R}^{\mathfrak{m}+\mathfrak{n}}$ is $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2 - 2 = \mathfrak{m} + \mathfrak{n}$.

EXAMPLE 2.7. Take $\mathscr{C}_i = \{X_i\} \cup \{A \mid A \subset X_i \wedge |A| \leqq \mathfrak{k}_i\}$, $\mathfrak{k}_i \geqq 1$, then because $\mathfrak{e}_i = 2 \leqq \mathfrak{k}_i + 1 = \mathfrak{c}_i$, $\forall i \in \{1, 2\}$, it follows from Theorem 2.1(II.c) that $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2 - 2 = \mathfrak{k}_1 + \mathfrak{k}_2$.

EXAMPLE 2.8. Take $M_i \subset X_i$, $|M_i| = \mathfrak{m}_i$ and define $\mathscr{C}_i = \{X_i\} \cup \{A \mid A \subset X_i \wedge M_i \not\subset A\}$; $i = 1, 2$. Because $\mathfrak{e}_i = \mathfrak{m}_i + 1 > \mathfrak{m}_i = \mathfrak{c}_i$ it follows from Theorem 2.1(I.II.a) that $\mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2 = \mathfrak{m}_1 + \mathfrak{m}_2$. We shall show now that even $\mathfrak{c} = \mathfrak{m}_1 + \mathfrak{m}_2$. Consider a set $E \subset X_1 \times X_2$ such that $E = E_1 \cup E_2$, with $|E_i| = \mathfrak{m}_i$, $|\pi_i E_i| = 1$, $\pi_i E_j = M_i$ and $\pi_i E_1 \cap \pi_i E_2 = \varnothing$; $i, j \in \{1, 2\}$ and $i + j = 3$. It is easy to see that $(\mathscr{C}_1 \oplus \mathscr{C}_2)(E) = X_1 \times X_2$. However, in general, $\bigcup \{(\mathscr{C}_1 \oplus \mathscr{C}_2)(E \backslash e) \mid e \in E\} = (\pi_1 E \times X_2) \cup (X_1 \times \pi_2 E) \neq X_1 \times X_2$. Hence $\mathfrak{c} = \mathfrak{m}_1 + \mathfrak{m}_2$.

EXAMPLE 2.9. Take $M \subset X_1$, $|M| = \mathfrak{m}$ and $\mathscr{C}_1 = \{X_1\} \cup \{A \mid A \subset X_1 \wedge M \not\subset A\}$. Take $X_2 = \boldsymbol{R}^{\mathfrak{n}}$ and $\mathscr{C}_2 = \text{conv}$. We know that $\mathfrak{e}_1 = \mathfrak{m} + 1 > \mathfrak{m} = \mathfrak{c}_1$, and $\mathfrak{e}_2 = \mathfrak{n} + 1 = \mathfrak{c}_2$. From Theorem 2.1(II.b) it follows that $\mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2 - 1 = \mathfrak{m} + \mathfrak{n}$. As in the previous example we can show that $\mathfrak{c} = \mathfrak{m} + \mathfrak{n}$. In order to prove this we have to look for a set $E \subset X_1 \times X_2$, $|E| = \mathfrak{m} + \mathfrak{n}$, such that the convex hull of E is not the union of the convex hulls of proper subsets of E. Take $E = E_1 \cup E_2$ with $|E_1| = \mathfrak{m}$, $|E_2| = \mathfrak{n}$, $|\pi_i E_j| = 1$, $\pi_i E_1 \cap \pi_i E_2 = \varnothing$, $\pi_1 E_1 = M$ and $\mathscr{C}_2(\pi_2 E) \neq \bigcup \{\mathscr{C}_2(\pi_2 E \backslash a) \mid a \in \pi_2 E\}$; $i, j \in \{1, 2\}$, $i + j = 3$. Note that $|\pi_2 E| = \mathfrak{n} + 1$. Now we have $(\mathscr{C}_1 \oplus \mathscr{C}_2)(E) = X_1 \times \mathscr{C}_2(\pi_2 E)$. But, in general, $\bigcup \{(\mathscr{C}_1 \oplus \mathscr{C}_2)(E \backslash e) \mid e \in E\} \neq X_1 \times \mathscr{C}_2(\pi_2 E)$. Hence $\mathfrak{c} = \mathfrak{m} + \mathfrak{n}$.

The main result of this section, Theorem 2.1, is a generalization of J. R. Reay's Theorem 1 (first part) together with Example (1) on pg. 229 of [7]. In fact, Reay proves that the Carathéodory-number $\mathfrak{c}$ of the convex-product-structure, whose component spaces are Euclidian-spaces with dimensions $\mathfrak{m}_1$ and $\mathfrak{n}_2$ and with the usual convexity-structure, satisfies the condition $\mathfrak{c} = \mathfrak{m}_1 + \mathfrak{n}_2$. (See Ex. 2.6.)

3. The Helly-number.

Let $\mathscr{F} \subset 2^X$, $|F| \geqq \mathfrak{k}$, $\mathfrak{k} \in \boldsymbol{N}$, for some set X. Define $\bigcap_{(\mathfrak{k})} F = \{\bigcap A \mid A \in \mathscr{F}^{\mathfrak{k}}\}$. I.e. $\varnothing \notin \bigcap_{(\mathfrak{k})} \mathscr{F}$ implies that each intersection of $\mathfrak{k}$ elements of $\mathscr{F}$ is not empty.

DEFINITION 3.1. A convexity-structure $\mathscr{C}$ for X has the *Helly-number* $\mathfrak{h}$ if $\mathfrak{h}$ is the smallest integer such that $[\mathscr{F} \subset \mathscr{C} \wedge$

$|\mathscr{F}| < \infty \wedge \varnothing \notin \bigcap_{(\mathfrak{h})}F] \Rightarrow \bigcap \mathscr{F} \neq \varnothing.$

Note that if $X = \varnothing$ then $\mathfrak{h}$ does not exist, and if $\mathfrak{h}$ exists then $\mathfrak{h} \geqq 1$. The following characterization of the Helly-number gives rise to another definition of it; see also [1] and [4].

LEMMA 3.1. *Let $\mathscr{C}$ be a convexity-structure for X with Helly-number $\mathfrak{h}$. Then the following assertions are equivalent*:

(i) $\mathfrak{h} \leqq \mathfrak{k}$;

(ii) $[\mathscr{F} \subset \mathscr{C} \wedge |\mathscr{F}| = \mathfrak{k} + 1 \wedge \varnothing \notin \bigcap_{(\mathfrak{k})}\mathscr{F}] \Rightarrow \bigcap \mathscr{F} \neq \varnothing$;

(iii) $[\mathscr{A} \subset X \wedge |A| = \mathfrak{k} + 1] \Rightarrow \bigcap \{\mathscr{C}(A\backslash a) \mid a \in A\} \neq \varnothing$;

(iv) $[A \subset X \wedge \mathfrak{k} + 1 \leqq |A| < \infty] \Rightarrow \bigcap \{\mathscr{C}(A\backslash a) \mid a \in A\} \neq \varnothing$.

Proof. We shall go through the following implication-cycle: (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv): Take $A \subset X$, $|A| = \mathfrak{k} + \mathfrak{n}$ ($\mathfrak{n} \in N$) and define $\mathscr{F} = \{\mathscr{C}(A\backslash a) \mid a \in A\}$. Note that $|\mathscr{F}| \leqq \mathfrak{k} + \mathfrak{n}$. If there exist elements $a, b \in A$, such that $a \neq b$ and $\mathscr{C}(A\backslash a) = \mathscr{C}(A\backslash b)$ then of course $\bigcap \{\mathscr{C}(A\backslash a) \mid a \in A\} \neq \varnothing$, and we are done. So we may assume that $|\mathscr{F}| = \mathfrak{k} + \mathfrak{n}$. If $\mathscr{F}' \subset \mathscr{F}$, with $|\mathscr{F}'| = \mathfrak{k}$, then $\mathscr{F}' \neq \mathscr{F}$ and there exists an element $a_1 \in A$ such that $\mathscr{C}(A\backslash a_1) \notin \mathscr{F}'$. From $a_1 \in A\backslash\{a\} \subset \mathscr{C}(A\backslash a)$ for each $a \in A$, $a \neq a_1$, it follows that $a_1 \in \bigcap \mathscr{F}'$, and hence $\varnothing \notin \bigcap_{(\mathfrak{k})} \mathscr{F}$. Because $\mathfrak{h} \leqq \mathfrak{k}$ we have $[\varnothing \notin \bigcap_{(\mathfrak{k})} \mathscr{F} \Rightarrow \varnothing \notin \bigcap_{(\mathfrak{h})} \mathscr{F}]$. From Definition 3.1 it follows that $\bigcap \{\mathscr{C}(A\backslash a) \mid a \in A\} = \bigcap \mathscr{F} \neq \varnothing$.

(iv) $\Rightarrow$ (iii) is trivial and (ii) $\Rightarrow$ (i) follows by induction. It remains to be shown that (iii) $\Rightarrow$ (ii): Take $\mathscr{F} \subset \mathscr{C}$, $|F| = \mathfrak{k} + 1$ and $\varnothing \notin \bigcap_{(\mathfrak{k})}\mathscr{F}$. Let $\mathscr{F} = \{F_i \mid i = 1, 2, \cdots, \mathfrak{k} + 1\}$. Then $\varnothing \notin \bigcap_{(\mathfrak{k})}\mathscr{F} \Rightarrow \bigcap \{\mathscr{F}\backslash\{F_i\}\} \neq \varnothing$, $\forall i = 1, 2, \cdots, \mathfrak{k} + 1$. For each $i = 1, 2, \cdots, \mathfrak{k} + 1$ we choose an element $a_i \in \bigcap \{\mathscr{F}\backslash\{F_i\}\}$. Define $A = \{a_i \mid i = 1, 2, \cdots, \mathfrak{k} + 1\}$. Hence $|A| \leqq \mathfrak{k} + 1$. If $|A| < \mathfrak{k} + 1$, then there exists an index i such that $a_i \in F_i$. But then $a_i \in \bigcap \mathscr{F}$. Hence $\bigcap \mathscr{F} \neq \varnothing$, and we are done. So we may assume that $|A| = \mathfrak{k} + 1$. From (iii) it follows that $\bigcap (\mathscr{C}(A\backslash a_i) \mid i = 1, 2, \cdots, \mathfrak{k} + 1\} \neq \varnothing$. For each $i = 1, 2, \cdots, \mathfrak{k} + 1$ we have $A\backslash\{a_i\} \subset F_i$, so $\mathscr{C}(A\backslash a_i) \subset F_i$. Hence $\bigcap \{\mathscr{C}(A\backslash a_i) \mid i = 1, 2, \cdots, \mathfrak{k} + 1\} \subset \bigcap \mathscr{F}$, and we conclude than $\bigcap \mathscr{F} \neq \varnothing$.

DEFINITION 3.1′. A convexity-structure $\mathscr{C}$ for X has the *Helly-number* $\mathfrak{h}$ if $\mathfrak{h}$ is the smallest integer such that $[A \subset X \wedge |A| = \mathfrak{h} + 1] \Rightarrow \bigcap \{\mathscr{C}(A\backslash a) \mid a \in A\} \neq \varnothing$.

With the aid of Lemma 3.1. it is easy to verify that Definitions 3.1 and 3.1′ are equivalent. We now prove the classical theorem of Levi with the aid of Definition 3.1′. See [5], Theorem H.

THEOREM 3.1 (Levi). *Let $\mathscr{C}$ be a convexity-structure for X. Then the existence of a Radon-number $\mathfrak{r}$ implies the existence of a Helly-number $\mathfrak{h}$, such that $\mathfrak{h} \leqq \mathfrak{r} - 1$.*

Proof. Each $A \subset X$ with $|A| = \mathfrak{r}$ has a $\mathscr{C}$-Radon-partition; see [2]. So there exists a set $B \subset A$, with $\emptyset \neq B \neq A$ and $\mathscr{C}(B) \cap \mathscr{C}(A \backslash B) \neq \emptyset$. Because $\mathscr{C}(B) \subset \bigcap \{\mathscr{C}(A \backslash a) \mid a \in A \backslash B\}$ and $\mathscr{C}(A \backslash B) \subset \bigcap \{\mathscr{C}(A \backslash a) \mid a \in B\}$, we have $\bigcap \{\mathscr{C}(A \backslash a) \mid a \in A\} \supset \mathscr{C}(B) \cap \mathscr{C}(A \backslash B) \neq \emptyset$. Hence $\mathfrak{h}$ exists and $\mathfrak{h} \leqq \mathfrak{r} - 1$.

THEOREM 3.2. *Let $\mathscr{C}_i$ be a convexity-structure for X_i, $X_i \neq \emptyset$, with Helly-number $\mathfrak{h}_i$; $i = 1, 2$. Then the Helly-number $\mathfrak{h}$ of the convex-product-structure $\mathscr{C}_1 \oplus \mathscr{C}_2$ for $X_1 \times X_2$ exists and $h = \max(\mathfrak{h}_1, \mathfrak{h}_2)$.*

Proof. We may assume that $\mathfrak{h}_1 \geqq \mathfrak{h}_2$. Take $E \subset X_1 \times X_2$ with $|E| = \mathfrak{h}_1 + 1$. There are two posibilities for the projection $\pi_i E$ of E on X_i; $i = 1, 2$.

—If $|\pi_i E| \leqq \mathfrak{h}_i$ for some $i \in \{1, 2\}$, then there exist elements $e_1, e_2 \in E$ such that $e_1 \neq e_2$ and such that $\pi_i(e_1) = \pi_i(e_2)$. Clearly $\pi_i(e_1) \in \mathscr{C}_i(\pi_i(E \backslash e_1))$. Hence $\pi_i(e_1) \in \bigcap \{\mathscr{C}_i(\pi_i(E \backslash e)) \mid e \in E\} \neq \emptyset$.

—If $|\pi_i E| = \mathfrak{h}_i + 1$, for some $i \in \{1, 2\}$, then it follows from Definition 3.1′ that $\bigcap \{\mathscr{C}_i(\pi_i E \backslash x) \mid x \in \pi_i E\} \neq \emptyset$. Hence $\bigcap \{\mathscr{C}_i(\pi_i(E \backslash e)) \mid e \in E\} \supset \bigcap \{\mathscr{C}_i(\pi_i E \backslash \pi_i(e)) \mid e \in E\} \neq \emptyset$ and we may conclude that $\bigcap \{\mathscr{C}_i(\pi_i(E \backslash e)) \mid e \in E\} \neq \emptyset$; $i = 1, 2$.

Hence $\bigcap \{(\mathscr{C}_1 \oplus \mathscr{C}_2)(E \backslash e)\} = \bigcap \{\mathscr{C}_1(\pi_1(E \backslash e)) \times \mathscr{C}_2(\pi_2(E \backslash e))\} \neq \emptyset$, and so $\mathfrak{h} \leqq \mathfrak{h}_1 = \max(\mathfrak{h}_1, \mathfrak{h}_2)$.

Next we show that $\mathfrak{h} \geqq \mathfrak{h}_1$. Assuming $\mathfrak{h} = \mathfrak{h}_1 - 1$, $A \subset X_1$ with $|A| = \mathfrak{h}_1$ and $b \in X_2$ we have, because $|A \times \{b\}| = \mathfrak{h}_1$, $\bigcap \{(\mathscr{C}_1 \oplus \mathscr{C}_2)((A \times \{b\}) \backslash (a, b)) \mid a \in A\} \neq \emptyset$. Hence $\bigcap \{\mathscr{C}_1(A \backslash a) \mid a \in A\} = \bigcap \{\mathscr{C}_1(\pi_1((A \times \{b\}) \backslash (a, b))) \mid a \in A\} \neq \emptyset$. This contradicts the fact that $\mathfrak{h}_1$ is the Helly-number of $\mathscr{C}_1$, so that, indeed, $\mathfrak{h} \geqq \mathfrak{h}_1 = \max(\mathfrak{h}_1, \mathfrak{h}_2)$ and the final conclusion is that $\mathfrak{h} = \mathfrak{h}_1 = \max(\mathfrak{h}_1, \mathfrak{h}_2)$.

It is well known that the Carathéodory-number $\mathfrak{c}$, the Helly-number $\mathfrak{h}$ and the Radon-number $\mathfrak{r}$ of the usual convexity-structure conv for $\boldsymbol{R}^n$ satisfy the equalities $\mathfrak{c} = \mathfrak{h} = \mathfrak{r} - 1$ $(= n + 1)$, that is, in Levi's theorem equality holds. There are however convexity-structures $\mathscr{C}$ where the equality does not hold. This is even the case when $\mathscr{C}$ is a convex-product-structure:

We know that (see also [8]):

$$\mathfrak{c}_1 + \mathfrak{c}_2 - 2 \leqq \mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$$
$$\mathfrak{h} = \max(\mathfrak{h}_1, \mathfrak{h}_2)$$
$$\max(\mathfrak{r}_1, \mathfrak{r}_2) \leqq \mathfrak{r} \leqq \mathfrak{r}_1 + \mathfrak{r}_2 - 2\ ;$$

$\mathfrak{c}_i$, $\mathfrak{h}_i$ and $\mathfrak{r}_i$ are resp. the Carathéodory-, Helly- and Radon-number of $\mathscr{C}_i$; $i = 1, 2$.

If $\mathfrak{c}_i = \mathfrak{h}_i = \mathfrak{r}_i - 1$, $i = 1, 2$, then we have:

a. if $\mathfrak{c}_i \geqq 3$ (e.g. when $\mathscr{C}_i$ is T_1) then $\mathfrak{c} \neq \mathfrak{h}$.

b. if $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2$, then $\mathfrak{c} > \mathfrak{r} - 1$.

if the exchange-number, $\mathfrak{e}_1 \leqq \mathfrak{c}_1$ and if $\mathfrak{r} = \mathfrak{r}_1 + \mathfrak{r}_2 - 2$ (e.g. in the case $X_i = \boldsymbol{R}^{\mathfrak{c}_i - 1}$, $\mathscr{C}_i = \text{conv}$) then $\mathfrak{c} \leqq \mathfrak{r} - 1$.

c. if $\mathfrak{r} > \max(\mathfrak{r}_1, \mathfrak{r}_2)$ (e.g. in the same case as in b) then $\mathfrak{h} < \mathfrak{r} - 1$.

The results in this paper can be extended to convex-product-structures which are products of finitely many convexity-structures.

In a next paper we shall pay more attention to the properties of the exchange number. For example we shall show that under certain conditions, the exchange-number of a convex-product-structure exists and how it can be derived.

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