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LOCALITY OF THE NUMBER OF PARTICLES OPERATOR

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We view the number of particles operator N as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup in an abstract Wiener setting. It is shown that if two functions f, g in the domain of N agree a.e. on an open set $\mathscr{O}$, then $Nf = Ng$ on $\mathscr{O}$. The restriction of N to a large core acts as an infinite dimensional partial differential operator L, and it is shown that L may be defined locally in an L^2_{loc} setting.

One of the mathematical concepts which has been the subject of considerable recent interest in constructive quantum field theory is the identification of the Bose Fock space $\mathscr{F}$ with a space $\mathscr{L}$ of L^2 functions over some Gaussian measure space $(\mathscr{Q}, d\mu)$. When $\mathscr{F} = \sum_{n=0}^{\infty} \bigotimes_n^s \mathscr{H}$, the sum of the n-fold Hilbert space symmetric tensor product of the complexification $\mathscr{H}$ of a real separable Hilbert space H, possible choices of $(\mathscr{Q}, d\mu)$ include any measure space on which the isonormal distribution over H may be realized. This identification is nicely described by Nelson [3]. The isometric isomorphism of $\mathscr{F}$ with $\mathscr{L}$ preserves the canonical direct sum decomposition of $\mathscr{F}$; that is, we have a corresponding decomposition $\mathscr{L} = \sum_{n=0}^{\infty} \mathscr{L}_n$. $\mathscr{L}_n$ has a natural interpretation as the L^2 space spanned by the Hermite functions on $\mathscr{Q}$ of rank $\leqq n$ [3, 8].

One way of realizing the isonormal distribution on H is to complete H with respect to a weaker norm (a "measurable" norm in the sense of Gross [2]) obtaining a Banach space B in which H is densely embedded. The pair (H, B) is known as an abstract Wiener pair, and μ is taken to be the Wiener measure p_1 on the Borel sets of B, generated by the canonical Gauss cylinder set measure on H [2]. Under the identification of $\mathscr{F}$ and $L^2(p_1)$, we further identify the number of particles operator on $\mathscr{F}$ (i.e. the second quantization of the identity operator on H) with the infinitesimal generator N of the Ornstein-Uhlenbeck velocity semigroup $\{e^{-tN}\}$ for the Brownian motion on B [3, 4]. For H finite dimensional $Nf(x) = -\Delta f(x) + x \cdot \operatorname{grad} f(x)$ on smooth f. However, $\mathscr{F}$ is usually constructed over an infinite dimensional H, and the expression of N as a differential operator must be suitably reinterpreted.

As a differential operator, N only incorporates derivatives in the directions of vectors of H. We define the H-derivative $Dg(x)$ of a function g defined on a neighborhood of x in B and taking values in a Banach space W as follows. Let $\hat{g}(h) = g(x + h)$ for all h in H

such that $x + h$ is in the domain of g. Then $\hat{g}$ maps a neighborhood of the origin of H into W. $Dg(x) \equiv \hat{g}'(0)$, the Fréchet derivative of $\hat{g}$ at 0. When g is real valued, $Dg(x)$ is an element of H^* which is customarily identified via the Riesz representation with an element of H. $D^2g(x)$ is defined by iteration, and will be identified with an element of $\mathscr{L}(H, H)$. Since B^* is dense in $H^* \approx H$, we can always find orthornormal bases $\{e_i\}$ for H consisting of elements of B^*.

In [4] it is shown that the set

$$\mathscr{C} = \{\text{real valued } f \in L^2(p_1) \text{ such that } |Df(x)|_H \text{ exists a.e. and is in } L^2(p_1) \text{ and also } D^2f(x) \text{ exists a.e. and is a Hilbert-Schmidt operator on } H \text{ with } |D^2f(x)|_{\mathscr{H}-\mathscr{S}} \in L^2(p_1)\}$$

is a core for N. The action of N on an f in $\mathscr{C}$ is as follows. If $\{e_i\}$ is any orthornormal basis in H with $e_i \in B^*$, and P_i is the orthogonal projection of H onto $\{e_1, \cdots, e_i\}$, then

$$(1) \qquad \{\langle x, P_iDf(x)\rangle - \text{trace}\,(P_iD^2f(x))\}_{i=1,2,\cdots}$$

is a Cauchy sequence in $L^2(p_1)$. Nf is the limit of this sequence, and is independent of the choice of $\{e_i\}$.

Other smaller cores for N are well-known. They generally consist of smooth polynomial cylinder functions. For the purposes of this note, however, $\mathscr{C}$ possesses a property that polynomial cores fail to have. Namely, the elements of $\mathscr{C}$ suffice to generate partitions of unity on B [1, 6]. That is, given any two concentric balls $b_1 \subsetneqq b_2$ in B, we can find $\varphi \in \mathscr{C}$ such that $0 \leqq \varphi(x) \leqq 1$, $\varphi(x) = 1$ on b_1 and $\varphi(x) = 0$ on $B - b_2$. Moreover, $\varphi(x)$, $|D\varphi(x)|_H$ and $|D^2\varphi(x)|_{\mathscr{H}-\mathscr{S}}$ can be assumed continuous and bounded on B. We call such a φ a partition function for b_1, b_2. We point out that if H-differentiability were replaced with the usual Fréchet differentiability on B, it would not always be possible to find a nontrivial C^1 function φ vanishing off b_2.

Locality of N can be stated in several ways. If two functions f, g in the domain of N have disjoint supports, then Nf and Ng have disjoint supports. Or, a stronger statement, that if f and g coincide a.e. on an open set, then $Nf = Ng$ a.e. on that set. Or, equivalently,

PROPOSITION 1. *If f is in the domain of N and if f vanishes* a.e. *on an open subset $\mathscr{O}$ of B, then Nf vanishes* a.e. *on $\mathscr{O}$.*

Proof. Since $\mathscr{C}$ is a core for f, we can find $f_n \in \mathscr{C}$ with $f_n \to f(L^2)$ and $Nf_n \to Nf(L^2)$. Fix $y \in \mathscr{O}$, and choose two open balls

b_1, b_2 centered at y, with $b_1 \subset b_2$ and $\bar{b}_2 \subset \mathcal{O}$. Choose $\varphi_y \in \mathcal{C}$ with $0 \leqq \varphi_y(x) \leqq 1$, $\varphi_y(x) = 0$ on b_1, $\varphi_y(x) = 1$ on $B - b_2$ and with $\varphi_y(x)$, $|D\varphi_y(x)|_H$ and $|D^2\varphi_y(x)|_{\mathscr{H}-\mathscr{S}}$ all continuous and bounded on B. Since ∂b_2 has p_1 measure zero, we may without loss of generality assume each f_n vanishes on b_2. Now $\varphi_y f_n \to \varphi_y f = f$ in L^2. Also $\varphi_y f_n \in \mathcal{C}$, and

$$\begin{aligned} N\varphi_y f_n = {} & \lim_i \varphi_y(x)\{\langle x, P_i Df_n(x)\rangle - \operatorname{trace}(P_i D^2 f_n(x))\} \\ & + \lim_i f_n(x)\{\langle x, P_i D\varphi_y(x)\rangle - \operatorname{trace}(P_i D^2\varphi_y(x))\} \\ & - 2\lim_i \operatorname{trace} P_i(D\varphi_y(x) \otimes Df_n(x)) . \end{aligned}$$

Dominated convergence ensures that the first limit exists, and the choice of support for f_n ensures that the subsequent terms are zero a.e. Hence $N\varphi_y f_n = \varphi_y \cdot Nf_n$, and so $N\varphi_y f_n \to \varphi_y \cdot Nf$ in L^2. Since N is closed, $\varphi_y \cdot Nf = Nf$ follows. Thus Nf vanishes a.e. on b_1. Since B is separable, it follows that Nf vanishes a.e. on $\mathcal{O}$.

It is expected that N should serve as the model for the Laplace-Beltrami operator on manifolds modelled on B. We will now show that we can easily locally define an operator L which extends the restriction of N to $\mathcal{C}$. For any open subset $\mathcal{O}$ of B, we define

$$\begin{aligned} \mathcal{C}_{\mathcal{O}} = \{ & \text{real valued } f \text{ defined on } \mathcal{O}, \text{ with } |Df(x)|_H \\ & \text{and } |D^2 f(x)|_{\mathscr{H}-\mathscr{S}} \text{ existing a.e. on } \mathcal{O}, \text{ such that} \\ & f, |Df|_H \text{ and } |D^2 f|_{\mathscr{H}-\mathscr{S}} \text{ are locally in } L^2(p_1) \text{ on} \\ & \mathcal{O}\} . \end{aligned}$$

Then we may define L on $\mathcal{C}_{\mathcal{O}}$ by

PROPOSITION 2. *Given f in $\mathcal{C}_{\mathcal{O}}$, let $\{\mathcal{O}_n\}$ by any countable cover of $\mathcal{O}$ by open balls such that for each $\mathcal{O}_n$ there is a concentric $\mathcal{O}'_n$ with $\mathcal{O}_n \subsetneqq \mathcal{O}'_n \subset \mathcal{O}$ and such that f, $|Df|_H$ and $|D^2 f|_{\mathscr{H}-\mathscr{S}}$ are in L^2 on each $\mathcal{O}'_n$. Let φ_n be a partition funcion for $\{\mathcal{O}_n, \mathcal{O}'_n\}$. Extend $\varphi_n f$ to be zero outside $\mathcal{O}$. Then $\varphi_n f \in \mathcal{C}$, and we define $Lf = N\varphi_n f$ on $\mathcal{O}_n$. Then Lf is well defined, is locally in $L^2(p_1)$ on $\mathcal{O}$, and is independent of the choice of $\mathcal{O}_n$ and φ_n.*

Proof. If x belongs to two members of the covering, say to $\mathcal{O}_n$ and $\mathcal{O}_m$, then $\varphi_n f$ and $\varphi_m f$ agree on $\mathcal{O}_n \cap \mathcal{O}_m$ and Lf is well-defined by Proposition 1. Hence since B is separable, Lf is independent of the choice of $\mathcal{O}_n$ and φ_n.

In Reference [4] it is shown that for $f \in \mathcal{C}$,

$$(2) \qquad |Nf|^2_{L^2(p_1)} \leqq ||Df|_H|^2_{L^2(p_1)} + ||D^2 f|_{\mathscr{H}-\mathscr{S}}|^2_{L^2(p_1)} .$$

Thus for for f in $\mathcal{C}_{\mathcal{O}}$, it follows that Lf is square integrable on $\mathcal{O}_n$.

REMARK. A popular choice of $(\mathcal{Q}, d\mu)$ is the underlying probability space of the realization on $\mathcal{S}'(\boldsymbol{R}^d)$ of a Gaussian process over Schwartz space $\mathcal{S}(\boldsymbol{R}^d)$. That is, $\mathcal{Q} = \mathcal{S}'$ and $d\mu$ is a Gaussian Borel measure on $\mathcal{S}'$. Such measures $d\mu$ have as supporting sets Hilbert spaces $B \subset \mathcal{S}'$, such that there is an $H \subset B$ with (H, B) an abstract Wiener pair. $d\mu|_B = p_1$, the Wiener measure for (H, B) [7, 5]. Our Proposition 1 then may be applied in $L^2(B, p_1)$.

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