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NOTES ON STABLE CURRENTS

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With additional assumptions we answer a conjecture proposed by Lawson and Simons.

In a work [5], H. B. Lawson, Jr. and J. Simons proved that there exist no stable rectifiable currents on an n-dimensional unit sphere S^n in the $(n+1)$-dimensional Euclidean space R^{n+1}. And concerning to this fact, they conjectured as follows.

Conjecture. Let M be a compact, simply-connected Riemannian manifold with the sectional curvature satisfying $1/4 < K_\sigma \leqq 1$ for all tangent two planes σ. Then there exist no stable rectifiable currents on M.

We obtain the following results with respect to this conjecture.

Let M be a compact, connected n-dimensional Riemannian manifold isometrically immersed in $(n+1)$-dimensional Euclidean space R^{n+1}. Let δ be a constant with $0 < \delta \leqq 1$, and suppose that at each point x of M, with respect to a suitable unit normal, every principal curvature λ_j of M satisfies

$$\sqrt{\delta} \leqq \lambda_j \leqq 1$$

$j = 1, \cdots, n$.

THEOREM. *Let M be a compact, connected Riemannian manifold satisfying the conditions expressed above. Associate to each $\mathscr{S} \to \mathscr{R}_p(M)$ a quadratic form $Q_{\mathscr{S}}$ on $\mathscr{V}$ as follows. For $V \in \mathscr{V}$, let ϕ_t be the flow generated by V and set*

$$Q_{\mathscr{S}}(V) = \frac{d^2}{dt^2} \boldsymbol{M}(\phi_{t\#}\mathscr{S})_{|t=0} .$$

Then for all $\mathscr{S} \in \mathscr{R}_p(M)$

$$\operatorname{tr} Q_{\mathscr{S}} \leqq p(p+1-n\delta-\delta)\boldsymbol{M}(\mathscr{S}) .$$

(For definitions of $\mathscr{V}$ and $\mathscr{R}_p(M)$, see below.)

COROLLARY 1. *Under the assumptions of the Theorem, for all p with $1 \leqq p < n\delta + \delta - 1$, any rectifiable p-current $\mathscr{S} \leftarrow \mathscr{R}_p(M)$ is not stable. If δ satisfies $n/(n+1) < \delta \leqq 1$, then any rectifiable p-current $\mathscr{S} \in \mathscr{R}_p(M)$ is not stable for each p with $1 \leqq p \leqq n-1$.*

COROLLARY 2. *Under the assumptions of the Theorem, if* δ *satisfies* $n/(n+1) < \delta \leqq 1$, *then*

$$H_p(M; Z) = H_p(S^n; Z)$$

for each p *with* $0 \leqq p \leqq n$. *Therefore, in particular, if* $n = 2$ *or* $n \geqq 5$, *then* M *is homeomorphic to* S^n. (This conclusion follows from weaker conditions.)

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1. In the following, we use the same notation as in [5]. Also see [5] for detailed definitions. Let M be a compact n-dimensional Riemannian manifold with Riemannian metric g and canonical connection ∇. For a point $x \in M$, $T_x(M)$ denotes the tangent space of M at x. Let $\mathscr{R}_p(M)$ be the set of all rectifiable p-currents on M, where $0 \leqq p \leqq n$. For a current $\mathscr{S} \in \mathscr{R}_p(M)$, $\vec{\mathscr{S}_x}$ denotes an orientation of $\mathscr{S}$, that is, for $\mathscr{H}^p$-almost all $x \in \mathscr{S}$, $\vec{\mathscr{S}_x}$ is a simple p-vector of unit length which represents $T_x(\mathscr{S})$, where $\mathscr{H}^p$ is the Hausdorff p-measure on M. Let V be a smooth vector field on M. We define a linear mapping $\mathscr{A}^V: T_x(M) \to T_x(M)$ by $\mathscr{A}^V(X) := \nabla_X V$ for $X \in T_x(M)$. This mapping can be extended uniquely as a derivation to $\bigwedge^p T_x(M)$, that is, as a linear map $\mathscr{A}^V: \bigwedge^p T_x(M) \to \bigwedge^p T_x(M)$ which for simple vectors is given by

$$\mathscr{A}^V(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^{p} X_1 \wedge \cdots \wedge X_{i-1} \wedge \mathscr{A}^V X_i \wedge X_{i+1} \wedge \cdots \wedge X_p \,.$$

At $x \in M$, we define also the linear map $\nabla_V, \cdot V: T_x(M) \to T_x(M)$ by $\Delta_{V,X} V := \nabla_V \nabla_{\tilde{X}} V - \nabla_{\Delta_V \tilde{X}} V$ for $X \in T_x(M)$, where $\tilde{X}$ is any extension of X to a local vector field. The definition is independent of any extension $\tilde{X}$, and the map carries over uniquely as a derivation to $\bigwedge^p T_x(M)$. Consider a current $\mathscr{S} \in \mathscr{R}_p(M)$ and a vector field V on M. Let $\phi_t: M \to M$, $t \in R$ be the 1-parameter group of diffeomorphisms generated by V. Then for each $t \in R$ we have the rectifiable current $\phi_{t\#}(\mathscr{S})$ which, as a linear functional on $\bigwedge^p(M)$, is given by

$$(\phi_{t\#}\mathscr{S})(\omega) = \mathscr{S}(\phi_t^* \omega)$$

for $\omega \in \bigwedge^p(M)$, where $\bigwedge^p(M)$ is the space of all smooth p-forms on M. Let M denote the usual norm of a linear functional on $\bigwedge^p(M)$ which has the usual Fréchet topology. Then,

$$M(\phi_{t\#}\mathscr{S}) = \int_M \sqrt{(\phi_t^* g)(\vec{\mathscr{S}}, \vec{\mathscr{S}})}\, d\|\mathscr{S}\|)$$

where $||\mathscr{S}||$ is a measure on M defined, by using the p-dimensional Hausdorff measure $\mathscr{H}^p$ on M, as follows: for a Borel set $X \subset M$, $||\mathscr{S}||(X) = \mathscr{H}^p(X \cap \mathscr{S})$.

DEFINITION. A rectifiable p-current $\mathscr{S} \in \mathscr{R}_p(M)$ is said to be stable if, for each vector field V the following two conditions hold:

$$(\mathrm{s}_1) \qquad \frac{d}{dt}\boldsymbol{M}(\phi_{t\#}\mathscr{S})_{|t=0} = 0 ,$$

$$(\mathrm{s}_2) \qquad \frac{d^2}{dt^2}\boldsymbol{M}(\phi_{t\#}\mathscr{S})_{|t=0} \geqq 0 .$$

The following is obtained by Lawson and Simons in [5].

PROPOSITION 1. *Let M be a compact Riemannian manifold and V a vector field on M with associated flow ϕ_t. Then for any rectifiable p-current $\mathscr{S} \in \mathscr{R}_p(M)$,*

$$(1) \qquad \frac{d}{dt}\boldsymbol{M}(\phi_{t\#}\mathscr{S})_{|t=0} = \int_M \langle \mathscr{A}\vec{\mathscr{S}}, \vec{\mathscr{S}}\rangle d||\mathscr{S}|| ,$$

$$(2) \qquad \begin{aligned} \frac{d^2}{dt^2}\boldsymbol{M}(\phi_{t\#}\mathscr{S})_{|t=0} = \int_M \{ &-\langle \mathscr{A}^V\vec{\mathscr{S}}, \vec{\mathscr{S}}\rangle^2 + \langle \mathscr{A}^V\mathscr{A}^V(\vec{\mathscr{S}}), \vec{\mathscr{S}}\rangle \\ &+ |\mathscr{A}^V(\vec{\mathscr{S}})|^2 + \langle \nabla_{V,\vec{\mathscr{S}}} V, \vec{\mathscr{S}}\rangle\} d||\mathscr{S}|| . \end{aligned}$$

REMARK. In the special case that $V = \nabla f$ (= the gradient of f) for some $f \in C^3(M)$, the transformation $\mathscr{A}^V$ is symmetric and (2) simplifies to

$$(2)' \qquad \begin{aligned} \frac{d^2}{dt^2}\boldsymbol{M}(\phi_{t\#}\mathscr{S})_{|t=0} = \int_M \{ &-\langle \mathscr{A}^V\vec{\mathscr{S}}, \vec{\mathscr{S}}\rangle^2 + 2|\mathscr{A}^V(\mathscr{S})|^2 \\ &+ \langle \nabla_{,\vec{\mathscr{S}}} V, \mathscr{S}\rangle\} d||\mathscr{S}|| . \end{aligned}$$

For future reference we shall write the integrand of (2)′ at $x \in M$ in terms of tangent vectors at x. Let $\{\bar{e}_1, \cdots, \bar{e}_p, \bar{e}_{p+1}, \cdots, \bar{e}_n\}$ be an orthonormal basis of $T_x(M)$ and set $\xi = \bar{e}_1 \wedge \cdots \wedge \bar{e}_p$. Then

$$(3) \qquad \begin{aligned} &-\langle \mathscr{A}^V\xi, \xi\rangle^2 + 2|\mathscr{A}^V(\xi)|^2 + \langle \nabla_{V,\xi} V, \xi\rangle \\ &= \left\{\sum_{j=1}^p \langle \mathscr{A}^V(\bar{e}_j), \bar{e}_j\rangle\right\}^2 + 2\sum_{j=1}^p \sum_{\alpha=p+1}^n \langle \mathscr{A}^V(\bar{e}_j), \bar{e}_\alpha\rangle^2 \\ &\quad + \sum_{i=1}^p \langle \nabla_{V,\bar{e}_j} V, \bar{e}_j\rangle , \end{aligned}$$

where $|\mathscr{A}^V(\xi)|$ denotes the length of p-vector $\mathscr{A}^V(\xi)$.

2. Now we assume that M is isometrically immersed in $(n+1)$-

dimensional Euclidean space R^{n+1} with canonical Riemannian metric $\langle , \rangle$ and canonical Riemannian connection $\bar{\nabla}$. For all local formulas we may consider the isometric immersion f of M into R^{n+1} as an imbedding and thus identify $x \in M$ with $f(x) \in R^{n+1}$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(R^{n+1})$. The normal space $T_x^{\perp}$ is the subspace of $T_x(R^{n+1})$ consisting of all $\zeta \in T_x(R^{n+1})$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric $\langle , \rangle$. For each point x of M, choose a field ζ of unit normal vectors defined on a neighborhood U of x. Then we have the basic formulas

$$\bar{\nabla}_X Y = \nabla_X Y + \langle A_\zeta X, Y\rangle \zeta$$
$$\bar{\nabla}_X \zeta = -A_\zeta X$$

where X and Y are smooth vector fields tangent to M, and A_ζ is a tensor field of type (1, 1), called the second fundamental form associated with ζ. The Gauss equation expresses the curvature tensor R of M as follows.

$$R(X, Y)Z = \langle A_\zeta Y, Z\rangle A_\zeta X - \langle A_\zeta X, Z\rangle A_\zeta Y$$

where X, Y and Z are smooth vector fields tangent to M.

Let δ be a constant with $0 < \delta \leqq 1$, and suppose that at each point x of M, with respect to a suitable field ζ of unit normals, every principal curvature λ_j of M satisfies $\sqrt{\delta} \leqq \lambda_j \leqq 1$, $j = 1, \cdots, n$.

REMARK. The above assumption implies that M has the sectional curvature satisfying $\delta \leqq K_\sigma \leqq 1$ for all tangent two planes σ. And from the continuity of the eigen-values of the linear map A_ζ: $T_x(M) \to T_x(M)$, called the principal curvatures of M, the above assumption also implies that M is orientable. Therefore we can choose a global field ζ of unit normals on M which satisfies the above condition, and then we can write $A_\zeta = A$.

3. To estimate the left hand side of (s_2) we begin with the space of functions $\mathscr{F} = \{\psi | M; \psi: R^{n+1} \to R$ is linear$\}$, and define

$$\mathscr{V} = \{\nabla\psi; \psi \in \mathscr{F}\} .$$

Then there is a natural isomorphism

$$\mathscr{V} \cong R^{n+1} \tag{4}$$

which associates to $v \in R^{n+1}$ the gradient of the function $\psi_v(x) = \langle v, x\rangle$ on M. This identification introduces a natural inner product on $\mathscr{V}$.

To any simple unit p-vector $\xi \in \bigwedge^p T_x(M)$, at any $x \in M$, we can associate a quadratic form Q_ξ on $\mathscr{V}$ as follows. For $V \in \mathscr{V}$, let ϕ_t

be the flow generated by V, and define

$$Q_\xi(V) = \frac{d^2}{dt^2}|\phi_{t\#}\xi|_{|t=0} .$$

Then we have the following.

PROPOSITION 2. *Under the assumptions as expresses above, we have*

$$\operatorname{tr} Q_\xi \leqq p(p + 1 - n\delta - \delta) .$$

Proof. Suppose $V \in \mathscr{V}$ corresponds to $v \in R^{n+1}$ under the isomorphism (4). Then at any $y \in M$

$$V(y) = v - \langle v, \zeta_y\rangle\zeta_y ,$$

and then for $X \in T_x(M)$, $\nabla_X V = (\bar{\nabla}_X V)^T = \langle v, \zeta_x\rangle AX$, where $(\ )^T$ denotes orthogonal projection $T_x(R^{n+1}) \to T_x(M)$. Thus,

$$(5) \qquad \mathscr{A}^V(X) = \nabla_X V = \langle v, \zeta_x\rangle AX .$$

And it follows easily that

$$(6) \qquad \nabla_{V,X} V = -\langle V, AV\rangle AX + \langle v, \zeta_x\rangle \nabla_V(A\tilde{X}) - \langle v, \zeta_x\rangle A(\nabla_V \tilde{X})$$

where $\tilde{X}$ is any extension of X to a local vector field.

We now choose an orthonormal basis $\{x_0 = \zeta_x, x_1 = e_1, \cdots, x_n = e_n\}$ for R^{n+1}, where e_j is an eigenvector corresponding to the eigenvalue λ_j of A, $j = 1, \cdots, n$. Via (4) this fixes an orthonormal basis $\{V_0, V_1, \cdots, V_n\}$ of $\mathscr{V}$. It then follows from (5) and (6) that $\nabla_{V_0}, \cdot V_0 = \mathscr{A}^V 1 = \cdots = \mathscr{A}^V n = 0$ and $\mathscr{A}^V o = A$, $\nabla_{V_j}, \cdot V_j = -\lambda_j A$, $j = 1, \cdots n$, as transformations of $T_x(M)$. For given simple unit p-vector $\xi \in \Lambda^p T_x(M)$, we can choose an orthonormal basis $\{\bar{e}_1, \cdots, \bar{e}_p, \bar{e}_{p+1}, \cdots, \bar{e}_n\}$ of $T_x(M)$ with $\xi = \bar{e}_1 \wedge \cdots \wedge \bar{e}_p$. It then follows from (2), (2)′, (3) and above formulas that

$$\begin{aligned}
\operatorname{tr}(Q_\xi) &= \sum_{l=0}^{n} Q_\xi(V_l) \\
&= \sum_{l=0}^{n} \Big\{\Big(\sum_{j=1}^{p} \langle \mathscr{A}^V l\bar{e}_j, \bar{e}_j\rangle\Big)^2 + 2\sum_{j=1}^{p}\sum_{\alpha=p+1}^{n} \langle \mathscr{A}^V l\bar{e}_j, \bar{e}_\alpha\rangle^2 \\
&\quad + \sum_{j=1}^{p} \langle \nabla_{V_l,\bar{e}_j} V_l, \bar{e}_j\rangle\Big\} \\
&= \Big(\sum_{j=1}^{p} \langle A\bar{e}_j, \bar{e}_j\rangle\Big)^2 + 2\sum_{j=1}^{p}\sum_{\alpha=p+1}^{n} \langle A\bar{e}_j, \bar{e}_\alpha\rangle^2 - \sum_{l=1}^{n}\sum_{j=1}^{p} \langle \lambda_l A\bar{e}_j, \bar{e}_j\rangle \\
&= \Big(\sum_{j=1}^{p} \langle A\bar{e}_j, \bar{e}_j\rangle\Big)^2 + 2\sum_{j=1}^{p}\Big(|A\bar{e}_j|^2 - \sum_{i=1}^{p} \langle A\bar{e}_j, \bar{e}_i\rangle^2\Big) \\
&\quad - \sum_{l=1}^{n}\sum_{j=1}^{p} \lambda_l \langle A\bar{e}_j, \bar{e}_j\rangle
\end{aligned}$$

$$= 2\sum_{j=1}^{p} |A\bar{e}_j|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{p} (\langle A\bar{e}_i, \bar{e}_i\rangle\langle A\bar{e}_j, \bar{e}_j\rangle - 2\langle A\bar{e}_j, \bar{e}_i\rangle^2)$$

$$- \sum_{j=1}^{p} \langle A\bar{e}_j, \bar{e}_j\rangle^2 - \sum_{l=1}^{n}\sum_{j=1}^{p} \lambda_l \langle A\bar{e}_j, \bar{e}_j\rangle .$$

By the assumption, $\sqrt{\delta} \leqq \lambda_j \leqq 1$, $j = 1$, $j = 1, \cdots, n$, we get $|A\bar{e}_j|^2 \leqq 1$, and $\sqrt{\delta} \leqq \langle A\bar{e}_j, \bar{e}_j\rangle \leqq 1$ for $1, \cdots, n$. Thus we have

$$\begin{aligned} \operatorname{tr}(Q_\xi) &\leqq 2p + p(p-1) - p\delta - np\delta \\ &= p(p + 1 - n\delta - \delta) . \end{aligned}$$

Combining Proposition 1 and Proposition 2 we get the theorem and the Corollary 1. And by virtue of the basic theorems on integral currents, we have the Corollary 2, see [2] or [5].

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