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A RATIO LIMIT THEOREM FOR A STRONGLY SUBADDITIVE SET FUNCTION IN A LOCALLY COMPACT AMENABLE GROUP

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It is the purpose of this paper to prove that the following property holds: Given a locally compact, amenable, unimodular group G, if S is a strongly subadditive, nonpositive, right invariant set function defined on the class $\mathscr{K}$ of relatively compact Borel subsets of G, and if $\{A_\alpha\}$ is a net in $\mathscr{K}$ satisfying an appropriate growth condition, then

$$\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$$

exists independently of $\{A_\alpha\}$, where λ is Haar measure on G.

Let G be a locally compact group. Let λ be right Haar outer measure defined on the subsets of G. Let $\mathscr{K}$ be the class of relatively compact Borel subsets of G. If A is a subset of G and $K \in \mathscr{K}$, let $[A]_K = \{g \in A: Kg \subset A\} = \bigcap_{k \in K \cup \{1\}} k^{-1} A$, where 1 is the identity of G. In this paper, we call a locally compact, amenable, unimodular group a *lcau* group.

DEFINITION 1. Following [1], we define a net $\{A_\alpha\}$ in $\mathscr{K}$ to be a regular net in the locally compact group G if

(D. 1.1) $\lambda(A_\alpha) > 0$ for each α;

(D. 1.2) $\lim_\alpha \lambda(KA_\alpha)^{-1}\lambda([A_\alpha]_K) = 1$, $K \in \mathscr{K}$, $K \neq \phi$.

(Even though KA_α and $[A_\alpha]_K$ may not be Borel measurable, (D. 1.2) makes sense because we required λ to be right Haar outer measure, which is defined for *all* subsets of G.)

LEMMA 1. *A locally compact group G possesses a regular net if and only if G is a lcau group.*

Proof. A locally compact group G is amenable if and only if for any $\varepsilon > 0$, and for any nonempty compact subset K of G, there exists a compact subset U of G, of positive measure, such that $\lambda^*(U)^{-1}\lambda^*(KU) < 1 + \varepsilon$, where λ^* is left Haar measure. (See [2].) We call this necessary and sufficient condition for amenability of G condition (A).

Now suppose G possesses a regular net $\{A_\alpha\}$. Then (D. 1.2) implies that

$$\lim_{\alpha} \lambda(KA_\alpha)^{-1}\lambda(A_\alpha) = 1,\ K \in \mathscr{K},\ K \neq \phi\ . \tag{1}$$

Taking $K = \{g\}$, where g is any element of G, we see that $\Delta(g) = 1$. Thus G is unimodular. It then follows that (1) implies condition (A), and thus G is also amenable.

Conversely, suppose now G is *lcau*. Given $\varepsilon > 0$ and a nonempty compact subset K of G, we may find by condition (A) a compact set $U = U_{(K,\varepsilon)}$, of positive measure, such that $\lambda(U)^{-1}\lambda(K^2U) < 1 + \varepsilon$. We direct the set $W = \{(K, \varepsilon): K$ a nonempty compact set in $G, \varepsilon > 0\}$ as follows: $(K_1, \varepsilon_1) > (K_2, \varepsilon_2)$ if and only if $K_1 \supset K_2$ and $\varepsilon_1 < \varepsilon_2$. Then $\{V_{(K,\varepsilon)}: (K, \varepsilon) \in W\}$ is a regular net of compact subsets of G, where $V_{(K,\varepsilon)} = KU_{(K,\varepsilon)}$.

DEFINITION 2. Let G be a regular group. Throughout this paper, we consider a set function $S: \mathscr{K} \to R$, the set of real numbers, which satisfies the following properties:

(D. 2.1) $S(\phi) = 0$.

(D. 2.2) S is strongly subadditive; that is, $S(A \cap B) + S(A \cup B) \leqq S(A) + S(B)$, $A, B \in \mathscr{K}$.

(D. 2.3) $S(A) \leqq 0$, $A \in \mathscr{K}$.

(D. 2.4) $S(Ag) = S(A)$, $A \in \mathscr{K}$, $g \in G$.

The main result we will prove in this note is the following theorem.

THEOREM 1. *Let G be a lcau group. Let $S: \mathscr{K} \to R$ satisfy Definition 2. Then there is an extended real number r^* such that $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha) = r^*$ for every regular net $\{A_\alpha\}$ in $\mathscr{K}$.*

A special case of this theorem, for vector groups, was proved in [7] in order to define entropy in statistical mechanics for classical continuous systems. The theorem can be used to define the entropy of a measurable partition relative to a discrete amenable group of measure-preserving transformations on a probability space, thereby enabling one to generalize the concept of the Kolmogorov-Sinai invariant [5].

One may construct a set function S satisfying Definition 2 as follows: Let $(\Omega, \mathscr{M})$ be a measurable space. For each element g of the regular group G, let T^g be a measurable transformation from Ω to Ω. We suppose that $T^{g_1} \cdot T^{g_2} = T^{g_1 g_2}$, $g_1, g_2 \in G$. Let $\mathscr{F}$ be a fixed sub-sigmafield of $\mathscr{M}$. If E is a nonempty subset of G, let $\mathscr{F}_E$ be the smallest sub-sigmafield of $\mathscr{M}$ containing $\bigcup_{g \in E} (T^g)^{-1}\mathscr{F}$. Define $\mathscr{F}_\phi = \{\phi, \Omega\}$. Let P, Q be probability measures on $\mathscr{M}$, such that P is stationary with respect to $\{T^g: g \in G\}$ and the fields $\{(T^g)^{-1}\mathscr{F}: g \in G\}$

are independent with respect to Q. For each $E \in \mathscr{K}$, let $S(E)$ be the negative of the entropy of P with respect to Q over $\mathscr{F}_E$, which we assume finite. The function $S: \mathscr{K} \to R$ defined in this way can be shown to satisfy Definition 2 in a manner analogous to that employed in [7] for vector groups.

LEMMA 2. *If Theorem holds for all sigma-compact lcau groups it holds for all lcau groups.*

Proof. Let d be a complete metric on R^*, the set of extended real numbers, which induces the usual topology on R^*. Let $\{A_\alpha\}$ be a regular net for a non-sigmacompact *lcau* group G. Suppose $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$ does not exist. Then for some $\varepsilon > 0$, we may find a sequence $\{F_n\}_0^\infty$ of elements of $\{A_\alpha\}$ and a sequence $\{E_n\}_0^\infty$ in $\mathscr{K}$ such that

(a) F_0 is any A_α and E_0 is an open symmetric neighborhood of the identity.

(b) $d(\lambda(F_n)^{-1}S(F_n), \lambda(F_{n-1})^{-1}S(F_{n-1})) > \varepsilon$, $n \geqq 1$.

(c) $\lambda(E_{n-1}F_n)^{-1}\lambda([F_n]_{E_{n-1}}) > 1 - n^{-1}$, $n \geqq 1$.

(d) E_n is an open symmetric set containing the closure of $[E_{n-1} \cup F_n]^2$, $n \geqq 1$.

Let $G' = \bigcup_n E_n$. It is easily seen that G' is an open, sigma-compact subgroup of G.

If we restrict λ to G', we get right Haar measure on G'. Thus $\{F_n\}$ is a regular sequence for G', and G' is a *lcau* group. Assuming Theorem 1 holds for sigma-compact *lcau* groups, $\lim_n \lambda(F_n)^{-1}S(F_n)$ would have to exist, a contradiction of b). Thus $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$ exists. Let $\{B_\beta\}$ be another regular net in G. Let $s_1 = \lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$, $s_2 = \lim \lambda(B_\beta)^{-1}S(B_\beta)$. We show that $s_1 = s_2$. Define sequences $\{C_n\}_1^\infty$, $\{D_n\}_1^\infty$, $\{E_n\}_0^\infty$ in $\mathscr{K}$ such that

(a) E_0 is an open symmetric neighborhood of the identity, $\{C_n\} \subset \{A_\alpha\}$, $\{D_n\} \subset \{B_\beta\}$.

(b) $d(\lambda(C_n)^{-1}S(C_n), s_1) < n^{-1}$, $d(\lambda(D_n)^{-1}S(D_n), s_2) < n^{-1}$, $n \geqq 1$.

(c) $\lambda(E_{n-1}C_n)^{-1}\lambda([C_n]_{E_{n-1}}) \geqq 1 - n^{-1}$, $\lambda(E_{n-1}D_n)^{-1}\lambda([D_n]_{E_{n-1}}) \geqq 1 - n^{-1}$, $n \geqq 1$.

(d) E_n is open, symmetric and contains the closure of $[E_{n-1} \cup C_n \cup D_n]^2$, $n \geqq 1$.

It follows that $G' = \bigcup_n E_n$ is an open, sigma-compact, *lcau* subgroup of G and that $\{C_n\}$ and $\{D_n\}$ are regular sequences for G'. Therefore, $\lim_n \lambda(C_n)^{-1}S(C_n) = \lim_n \lambda(D_n)^{-1}S(D_n)$, and so $s_1 = s_2$ by b).

DEFINITION 3. If G is a locally compact group, if $S: \mathscr{K} \to R$ satisfies Definition 2, and if $A, B \in \mathscr{K}$ with $A \cap B = \phi$, define $S(A|B) = S(A \cup B) - S(B)$.

LEMMA 3. *Let G be a locally compact group, and let $S: \mathscr{K} \to R$ satisfy Definition 2. Then S obeys the following laws:*

(L. 3.1) $S(A) \leqq S(B)$ *if* $A \supset B$, $A, B \in \mathscr{K}$.

(L. 3.2) *If $A_1, A_2, \cdots, A_k$ are elements of $\mathscr{K}$ which partition A, then $S(A) = \sum_{i=1}^{k} S(A_i \mid \bigcup_{j=1}^{i-1} A_j)$, where an empty union is the null set.*

(L. 3.3) $S(E|D_1) \leqq S(E|D_2)$, $D_1 \supset D_2$, $E \cap D_1 = \phi$, $E, D_1, D_2 \in \mathscr{K}$.

(L. 3.4) $S(E|D) \leqq S(E) \leqq 0$, $E, D \in \mathscr{K}$, $E \cap D = \phi$.

Proof. (L. 3.2) follows easily from Definition 2. The strong subadditivity of S is equivalent to saying $S(A\backslash B|B) \leqq S(A\backslash B|A \cap B)$, $A, B \in \mathscr{K}$. Letting $A = E \cup D_2$ and $B = D_1$, where E, D_1, D_2 satisfy $D_1 \cap E = \phi$ and $D_1 \supset D_2$, we have $A \cap B = D_2$ and $A\backslash B = E$, whence (L. 3.3) follows. In (L. 3.3) if we take $D_2 = \phi$, (L. 3.4) follows because $S(E|\phi) = S(E)$. If $A \supset B$, where $A, B \in \mathscr{K}$, then $S(A) = S(B) + S(A\backslash B|B) \leqq S(B)$, and thus (L. 3.1) follows.

DEFINITION 4. We define a locally compact group G to be a P-group if there exists for some positive integer n a triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ such that:

(D. 4.1) K is a nonempty relatively compact Borel set in G.

(D. 4.2) $\{G_i\}_1^n$ and $\{H_i\}_1^n$ are sequences of closed subgroups of G satisfying $G_1 \subset H_1 \subset G_2 \subset H_2 \subset \cdots \subset G_n \subset H_n$.

(D. 4.3) The index of G_i in H_i is countable, $i = 1, 2, \cdots, n$.

(D. 4.4) If E_i is any set of coset representatives of the right cosets $\{G_ih: h \in H_i\}$ of G_i in H_i, $i = 1, 2, \cdots, n$, then each $g \in G$ has a unique factorization in the form $g = ke_1e_2 \cdots e_n$, $k \in K$, $e_i \in E_i$, $i = 1, 2, \cdots, n$. Also, $K(\prod_{j=1}^{i-1} E_j)G_i = K(\prod_{j=1}^{i-1} E_j)$, $i = 1, 2, \cdots, n$, where an empty product is the identity in G.

In order to prove Theorem 1 for sigma-compact *lcau* groups, we need to show that such groups are P-groups. This we now do, by means of several lemmas. To see how the following lemma may be proved, see [2], page 379.

LEMMA 4. *Let G' be a closed normal subgroup of a connected Lie group G. Let $\phi: G \to G/G'$ be the canonical homomorphism. Then there exists a map $\tau: G/G' \to G$ such that*

(L. 4.1) *τ is a cross-section; that is, $\phi \cdot \tau$ is the identity map on G/G'.*

(L. 4.2) *If U is a relatively compact subset of G/G', then $\tau(U)$ is a relatively compact subset of G.*

(L. 4.3) *If U is a Borel set in G/G' and V is a Borel set in G', then $\tau(U)V$ is a Borel set in G.*

LEMMA 5. *Let G be a connected Lie group and G' a closed normal subgroup of G such that G/G' is either a vector group or compact. Then if G' is a P-group, so is G.*

Proof. Let $\tau: G/G' \to G$ be the cross-section map provided by Lemma 4. Since G/G' is a vector group or compact, it is easy to see that there exists a closed countable subgroup G'' of G/G' and a relatively compact Borel set K' in G/G' such that $\{K'g: g \in G''\}$ partitions G/G'. If G' is a P-group with respect to the triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$, then G is a P-group with respect to the triple $(\tau(K')K, \{G_i\}_1^{n+1}, \{H_i\}_1^{n+1})$, where $G_{n+1} = G'$ and $H_{n+1} = \phi^{-1}(G'')$.

LEMMA 6. *If G is a sigma-compact locally compact group and G' is an open subgroup of G which is a P-group, then G is a P-group.*

Proof. Let G' be a P-group with respect to the triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$. Then G is a P-group with respect to the triple $(K, \{G_i\}_1^{n+1}, \{H_i\}_1^{n+1})$, where $G_{n+1} = G'$, $H_{n+1} = G$.

LEMMA 7. *If G is a locally compact group and G' is a compact normal subgroup of G such that G/G' is a P-group, then G is a P-group.*

Proof. Suppose G/G' is a P-group with respect to the triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$. Let $\phi: G \to G/G'$ be the canonical homomorphism. Then G is a P-group with respect to the triple $(\phi^{-1}(K), \{\phi^{-1}(G_i)\}_1^n, \{\phi^{-}(H_i)\}_1^n)$.

THEOREM 2. *Every sigma-compact locally compact amenable group is a P-group.*

Proof. Every connected amenable Lie group G possesses a series of closed subgroups $G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$, where G_0 is the identity, G_i is normal in G_{i+1}, and G_{i+1}/G_i is either a vector group or compact, $i = 0, 1, \cdots, n-1$. (See [3], Theorem 3.3.2, and [4], Lemma 3.3.) Now G_0 is clearly a P-group, so by using Lemma 5 repeatedly we conclude every connected amenable Lie group is a P-group. Applying Lemma 6, every sigma-compact amenable Lie group is a P-group. For every locally compact group G there exists an open subgroup G' of G and a compact normal subgroup K of G' such that G'/K is a Lie group. (See [6], page 153.) Assuming G in addition is sigma-compact and amenable, so is G'/K. Thus G'/K is a P-group and then so is G' by Lemma 7. Then G is a P-group by Lemma 6.

We fix G to be a sigma-compact *lcau* group for the rest of the paper. We need to show Theorem 1 holds for G. This we accomplish by means of some lemmas and Theorem 3.

Let $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ be a triple with respect to which G is a P-group. Let E_i be a set of coset representatives of the right cosets of G_i in H_i such that $1 \in E_i$, $i = 1, 2, \cdots, n$, where 1 is the identity of G. For each i, let $\bar{H}_i$ be the collection of right cosets of G_i in H_i. (Since G_i is not necessarily normal in H_i, $\bar{H}_i$ need not be a group.) For each i, let $\phi_i: H_i \rightarrow \bar{H}_i$ be the map such that $\phi_i(h) = G_i h$, $h \in H_i$; let $\tau_i: \bar{H}_i \rightarrow E_i$ be the unique map such that $\phi_i \cdot \tau_i$ is the identity map on $\bar{H}_i$. By a total order $<$ on a set W, we mean a transitive relation such that for $x, y \in W$ exactly one of the following hold: $x < y$, $x = y$, or $y < x$. For each i, let $\prec^i$ be a total order on E_i; if $h \in H_i$, let $\prec_h^i$ be the total order on E_i such that if $e, e' \in E_i$ then $e \prec_h^i e'$ if and only if $\tau_i \cdot \phi_i(eh) \prec^i \tau_i \cdot \phi_i(e'h)$. If $h \in H_i$, let $P_h^i(e) = \{e' \in E_i: e' \prec_h^i e\}$. Let $E = E_1 E_2 \cdots E_n$. Let H be the locally compact amenable group $H = H_1 \times H_2 \times \cdots \times H_n$. If $h = (h_1, h_2, \cdots, h_n) \in H$, let $\prec_h$ be the lexicographical order on E defined as follows: if $e = e_1 e_2 \cdots e_n$ and $e' = e_1' e_2' \cdots e_n'$ are elements of E, where $e_i, e_i' \in E_i$, then $e \prec_h e'$ if and only if there exists an integer k, $n \geqq k \geqq 1$, such that $e_k \prec_{h_k}^k e_k'$ and for $n \geqq j > k$, $e_j = e_j'$. If $h \in H$, $e \in E$, let $P_h(e) = \{e' \in E: e' \prec_h e\}$. If $A \in \mathscr{K}$, $e \in E$, let $\phi_A^e: H \rightarrow R$ be the function such that $\phi_A^e(h) = S(Ke \mid KP_h(e) \cap Ae) = S(K \mid KP_h(e)e^{-1} \cap A)$, $h \in H$.

LEMMA 8. *If $A \in \mathscr{K}$ and $e \in E$, then $\phi_A^e \in L^\infty(H)$, the space of bounded Borel-measurable real-valued functions with domain H.*

Proof. Fix $A \in \mathscr{K}$, $e \in E$. By (L. 3.4), $\phi_A^e \leqq 0$. To achieve a lower bound, let $E' = \{e' \in E: Ke' \cap Ae \neq \phi\}$. Since $KE' \subset KK^{-1}Ae$, E' is finite. Let $F = \{e\} \cup E'$. By (L. 3.2), $S(KF) = \sum_{f \in F} S(Kf \mid KP_h(f) \cap KF)$. By (L. 3.3) and (L. 3.4), $S(KF) \leqq S(Ke \mid KP_h(e) \cap KF) \leqq S(Ke \mid KP_h(e) \cap Ae) = \phi_A^e(h)$, where the fact that $KF \supset Ae$ was used. Thus ϕ_A^e is a bounded function. We now show that it is a Borel measurable function. It is easily seen that ϕ_A^e is a simple function with possible values $S(Ke \mid KF' \cap Ae)$, $F' \subset F$. If $F' \subset F$, then $\phi_A^e = S(Ke \mid KF' \cap Ae)$ on the set $\{h \in H: P_h(e) \cap F = F'\}$, which is equal to the intersection of the sets $\bigcap_{f \in F'} \{h: f \in P_h(e)\}$ and $\bigcap_{f \in F \setminus F'} \{h: f \notin P_h(e)\}$. Thus ϕ_A^e is Borel measurable if for each $f \in F$, $\{h \in H: f \in P_h(e)\}$ is a Borel set. If $f = e$, this set is empty. Thus, fix $f \in F$, $f \neq e$. Let $f = f_1 f_2 \cdots f_n$, and $e = e_1 e_2 \cdots e_n$, where $e_i, f_i \in E_i$ for each i. Let $j = \max\{i: f_i \neq e_i\}$. Then $\{h \in H: f \in P_h(e)\} = \{h \in H: f_j \in P_{h_j}^j(e_j)\}$, where $h_j \in H_j$ is the j^{th} component of $h \in H$. This is a Borel set in H if $\{h \in H_j: f_j \in P_h^j(e_j)\}$ is a Borel set in H_j. Now this latter set is the union of the sets $\{h \in H_j: G_j f_j h = G_j g_1, G_j e_j h = G_j g_2\}$ where (g_1, g_2) ranges over all ordered pairs such that $g_1, g_2 \in E_j$ and $g_1 <^j g_2$. Since the union is a countable union of closed subsets of H_j, Borel measurability follows.

LEMMA 9. *Let μ be a left invariant mean on $L^\infty(H)$. Then $\mu(\phi_A^e) = \mu(\phi_A^1)$, $A \in \mathscr{K}$, $e \in E$.*

Proof. Fix $A \in \mathscr{K}$, $e \in E$. We observe that

$$KP_h(e)e^{-1} = \left[\bigcup_{i=1}^{u} K\left(\prod_{j=1}^{i-1} E_j\right)P_{h_i}^i(e_i)e_{i+1}\cdots e_n\right]e^{-1}$$
$$= \bigcup_{i=1}^{n}\left[K\left(\prod_{j=1}^{i-1} E_j\right)G_iP_{h_i}^i(e_i)e_i^{-1}\cdots e_2^{-1}e_1^{-1}\right],$$

by (D. 4.4), where $h = (h_1, h_2, \cdots, h_n) \in H$ and $e = e_1e_2\cdots e_n$. It is routine to show that $G_iP_{h_i}^i(e_i) = G_iP_1^i(\tau_i\cdot\phi_i(e_ih_i))h_i^{-1}$. Also, since $e_j \in G_i$ for $j < i$, we have $\phi_i(e_ih_i) = \phi_i(e_1e_2\cdots e_ih_i)$. Thus, $KP_h(e)e^{-1} = \bigcup_{i=1}^n [K(\prod_{j=1}^{i-1} E_j)P_1^i(\tau_i\cdot\phi_i(e_1\cdots e_ih_i))(e_1\cdots e_ih_i)^{-1}] = KP_{mh}(1)$, where $m = (m_1, m_2, \cdots, m_n) \in H$ satisfies $m_i = \prod_{j=1}^i e_j$, $i = 1, 2, \cdots, n$. Thus $\phi_A^e(h) = \phi_A^1(mh)$, $h \in H$, from which the lemma follows.

THEOREM 3. *Let $\{A_\alpha\}$ be a regular net in the sigmacompact lcau group G. Then $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha) = \inf_{B\in\mathscr{K}} \lambda(K)^{-1}\mu(\phi_B^1)$.*

Proof. Fix the regular net $\{A_\alpha\}$. Now $KE'_\alpha \subset A_\alpha \subset KE_\alpha$, where $E_\alpha = \{e \in E\colon Ke \cap A_\alpha \neq \phi\}$, $E'_\alpha = \{e \in E\colon Ke \subset A_\alpha\}$. Thus by (L. 3.1), $S(KE_\alpha) \leqq S(A_\alpha) \leqq S(KE'_\alpha)$. We show that $\limsup_\alpha \lambda(A_\alpha)^{-1}S(KE'_\alpha) \leqq L$ and $\liminf_\alpha \lambda(A_\alpha)^{-1}S(KE_\alpha) \geqq L$, where $L = \inf_{B\in\mathscr{K}} \lambda(K)^{-1}\mu(\phi_B^1)$. Now $S(KE_\alpha) = \sum_{e\in E_\alpha} S(Ke\,|\,KP_h(e) \cap KE_\alpha) \geqq \sum_{e\in E_\alpha} \phi_{B_\alpha}^e$, where $B_\alpha = \bigcup_{e\in E_\alpha} KE_\alpha e^{-1}$. Applying μ to the inequality and using Lemma 9, $S(KE_\alpha) \geqq |E_\alpha|\mu(\phi_{B_\alpha}^1) \geqq |E_\alpha|\lambda(K)L = \lambda(KE_\alpha)L$, where $|E_\alpha|$ denotes the cardinality of E_α. Since $KE_\alpha \subset KK^{-1}A_\alpha$ we have $\lim_\alpha \lambda(A_\alpha)^{-1}\lambda(KE_\alpha) = 1$, by the regularity of $\{A_\alpha\}$. Thus $\liminf_\alpha \lambda(A_\alpha)^{-1}S(KE_\alpha) \geqq L$. Fix $B \in \mathscr{K}$. We suppose that $B \supset K$. Now $S(KE'_\alpha) = \sum_{e\in E'_\alpha} S(Ke\,|\,KP_h(e) \cap KE'_\alpha) \leqq \sum_{e\in F_\alpha} \phi_B^e$ where $F_\alpha = \{e \in E'_\alpha\colon KE'_\alpha e^{-1} \supset B\}$. Applying μ, $S(KE'_\alpha) \leqq \lambda(KF_\alpha)\lambda(K)^{-1}\mu(\phi_B^1)$. We could conclude that $\limsup_\alpha \lambda(A_\alpha)^{-1}S(KE'_\alpha) \leqq L$, provided $\lim_\alpha \lambda(A_\alpha)^{-1}\lambda(KF_\alpha) = 1$. This limit is one by the regularity of $\{A_\alpha\}$, since $[A_\alpha]_{KK^{-1}BK^{-1}} \subset KF_\alpha$. To see this, let $x \in [A_\alpha]_{KK^{-1}BK^{-1}}$. By definition, $KK^{-1}BK^{-1}x \subset A_\alpha$. Now $x \in Ke$ for some $e \in E$. We have $Ke \subset KK^{-1}x \subset KK^{-1}BK^{-1}x \subset A_\alpha$. Thus $e \in E'_\alpha$. It will follow that $x \in KF_\alpha$ if $Be \subset KE'_\alpha$. To see this, let $y \in Be$. Then $y \in Ke'$ for some $e' \in E$. Now $Ke' \subset KK^{-1}y \subset KK^{-1}Be \subset KK^{-1}BK^{-1}x \subset A_\alpha$. Thus $e' \in E'_\alpha$ and $y \in KE'_\alpha$.

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