ON POWER-INVARIANCE

Eloise Hamann

Let R be a commutative ring with identity, and consider
the power series ring $R[[X]]$ in one analytic indeterminate over
R. Is the coefficient ring R unique in the sence that if $R[[X]]$
is isomorphic to $S[[Y]]$ with Y an analytic indeterminate over
S, need S be isomorphic to R? Whenever this is the case, R
will be called power-invariant. It will be shown that if R
is a quasi-local or a complete semi-local ring then R is power-
invariant.

The answer to the *general* question was not known by the author
until the Commutative Algebra Conference in June of 1974 at the
University of Nebraska where a counterexample was produced by
Andy Magid. He has graciously requested that it be reproduced in
this paper. The fact that rings with nilpotent Jacobson radical are
power-invariant is known [6]. The paper will also show that if
$R[[X]] = S[[Y]]$, under the assumption that certain elements are not
zero-divisors that there exist one-to-one maps from R into S and S
into R. In particular, this is the case if R is a domain. Finally,
the paper generalizes the power-invariant results to an arbitrary
number of variables.

The following notational conventions will be observed and referred
to throughout the paper. $J(R)$ will denote the Jacobson radical of
a ring R. W, R, S, X, Y, j, u, v will be such that $W = R[[X]] = S[[Y]]$
where X and Y are analytic indeterminates over R and S respec-
tively; $Y = j + uX$, and $X = k + vY$ where $J \in J(R)$, $u \in W$, $k \in J(S)$,
and $v \in W$.

Note that since W is complete with respect to both (X) and (Y),
W is also complete with respect to (X, Y) and thus also complete
with respect to the ideals jW and kW contained in (X, Y). (If $\{w_n\}$
is a Cauchy sequence in the (X, Y)-adic topology, it can be written
as a sum of two Cauchy sequences $\{s_n\}$ and $\{t_n\}$ which are also Cauchy
with respect to the (X)-adic and (Y)-adic topologies respectively.
Let s be the limit of $\{s_n\}$ in the (X)-adic topology and t the limit of
$\{t_n\}$ in the (Y)-adic topology. Since s and t are then limit points
of $\{s_n\}$ and $\{t_n\}$ respectively in the (X, Y)-adic topology, $s + t$ is a
limit point of $\{w_n\}$. Conceivably the (X, Y)-adic topology may not
be Hausdorff, so that limits aren't necessarily unique.) Further, R
is certainly a closed subset of W in the jW-adic topology so R is
complete with respect to (j), the ideal of R generated by j, but
perhaps not Hausdorff. Similarly, S is complete but perhaps not

Hausdorff with respect to (k), the ideal of S generated by (k).

1. Andy Magid's counterexample. The example is the completion with respect to a certain ideal of Melvin Hochster's counterexample to the question whether $R[X]$ isomorphic to $S[Y]$ implies S is isomorphic to R, where X and Y are ordinary indeterminates over R and S respectively. Specifically, there exists a Noetherian ring R with zero Jacobson radical which has a finitely generated nonfree module P such that $P \oplus R \cong R^3$. Taking symmetric algebras of both sides yields $A[T]$ isomorphic to $R[X, Y, Z]$ where T, $\{X, Y, Z\}$ are indeterminates over A and R respectively and A is the symmetric algebra of P, and is not isomorphic to $R[X, Y]$. See either [3] or [4] for more details. If M is a B-module, let $\hat{S}_B(M)$ denote the complete symmetric algebra of M over B, i.e., the completion of the symmetric algebra $S_B(M)$ with respect to the ideal generated by M. To get the counterexample for the power series case, take the complete symmetric algebras of $P \oplus R$ and R^3 over R. Then $\hat{S}_R(P \oplus R) \cong \hat{S}_R(P)[[T]]$ and $\hat{S}_R(R^3) \cong R[[X, Y, Z]]$ with T an analytic indeterminate over $\hat{S}_R(P)$ and X, Y, Z independent analytic indeterminates over R. It remains to show that $\hat{S}_R(P)$ is not isomorphic to $R[[X, Y]]$. It suffices to show that $\hat{S}_R(P) \cong R[[X, Y]]$ implies $S_R(P) \cong R[X, Y]$. Let M be any finitely generated R-module. $J(\hat{S}_R(M))$ is the ideal generated by M since $M\hat{S}(M)$ is certainly contained in $J(\hat{S}_R(M))$ and $\hat{S}(M)/M\hat{S}(M) \cong R$ whose Jacobson radical is zero. Thus, the associated graded ring of $\hat{S}(M)$ with respect to $J(\hat{S}(M))$ is $\hat{S}(M)/M\hat{S}(M) \oplus M\hat{S}(M)/]M\hat{S}(M)]^2 \oplus \cdots$ which is isomorphic to $\hat{S}(M)$. Thus, $\hat{S}(M)$ determines $S(M)$ and the result follows.

2. Some power-invariant rings. The following theorem from [8] and [9] will be needed for the results of this section.

THEOREM 1. *Let* $B = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$, *and suppose that* ϕ *is an R-endomorphism of $R[[X]]$ such that* $\phi(X) = B$. *Then:*
   (a) ϕ *is onto if and only if* a_1 *is a unit of R.*
   (b) *If* ϕ *is onto, then* ϕ *is one to one.*
   (c) ϕ *is an automorphism if and only if* a_1 *is a unit of R.*

THEOREM 2. *Let* $B = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$. *If* $\bigcap_{n=1}^{\infty} (b_0^n) = 0$ *in R (or* $\bigcap_{n=1}^{\infty} (B^n) = 0$ *in $R[[X]]$) and R is complete with respect to* (b_0) *(or $R[[X]]$ is complete with respect to (B)) then ϕ_B which maps* $\sum_{i=0}^{\infty} a_i X^i$ *to* $\sum_{i=0}^{\infty} a_i B^i$ *is an R-endomorphism.*

THEOREM 3. *Let* $B = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$. *Let A be an ideal of R. If $b_0 A = A$, then* $A \subseteqq \bigcap_{k=1}^{\infty} (B^k)$.

Also recall that an element of $R[[X]]$ is invertible if its constant term is invertible [7].

We are ready for our first result.

THEOREM 4. *If R is quasi-local, R is power-invariant.*

Proof. Let the notation be as in the next-to-last paragraph of the introduction. If u is an invertible element of W, then Theorem 1 implies $R[[Y]] = R[[X]] = S[[Y]]$ and Y is an analytic indeterminate over R. Thus, $S \cong W/(Y) \cong R$. By symmetry we can reduce to the case where both u and v are in the maximal ideal of W. For $w \in W$, let $w_i \in S$ be such that $w = w_0 + w_1 Y + \cdots$. Taking the Y-coefficient of both sides of $Y = j + uX$ we get $1 = j_1 + u_0 X_1 + u_1 X_0 = j_1 + u_0 X_1 + u_1 k$. u_0 is in the maximal ideal since u is, so j_1 is invertible. Similarly k^1, the X-coefficient of k is invertible. Suppose for the moment that there is an R-endomorphism of $R[[X]]$ which takes X to k, and an S-endomorphism which takes Y to j. In this case Theorem 1 would imply $R[[k]] = R[[X]] = S[[Y]]$ with k analytically independent over R, and $R \cong S/(k)[[Y]]$ with Y an analytic indeterminate over $S/(k)$. Similarly, $S \cong R/(j)[[X]]$ with X analytically independent over $R/(j)$. However, $(X, j) = (X, Y) = (k, Y)$ so that $S/(k) \cong W/(X, Y) \cong R/(j)$. This yield $R \cong S$. It remains only to show the desired endomorphisms exist. W is certainly complete with respect to (j) (or (k)), so by Theorem 2 and symmetry, the result will follow from the following proposition.

PROPOSITION. *Let $R[[X]] = S[[Y]] = W$ where $Y = j + uX$ and $X = k + vY$ as above. If u is in $J(W)$, then $\bigcap_{n=1}^{\infty} (j^n) = 0$ as an ideal in R.*

Proof. Since $\bigcap_{n=1}^{\infty} (Y^n) = 0$, by Theorem 3 it suffices to show that $j[\bigcap_{n=1}^{\infty} (j^n)] = \bigcap_{n=1}^{\infty} (j^n)$. Let $A = \bigcap_{n=1}^{\infty} (j^n)$. First assume that $\text{Ann}_R j \subseteq A$. Let $f \in A$ so that $f = jt_1$ for some t_1, and $f = j^n t_n$ for some t_n given n. $j(t_1 - j^{n-1} t_n) = 0$ implies $t_1 - j^{n-1} t_n \in A$, which further implies $t_1 \in (j^{n-1})$. Since n was arbitrary $t_1 \in A$. Thus, $f \in jA$ and $jA = A$. Now let us show that $\text{Ann}_R j \subseteq A$. It suffices to show that $fj^i = 0$ implies $f \in (j)$. Let $fj^i = 0$, in which case $fY^i = \sum_{h=1}^{i} \binom{i}{h} fj^{i-h} u^h X^h$. If we take the Y^i coefficient of both sides, we get f_0 expressed as a sum of terms of the form $f_a j_{b_1} j_{b_{i-h}} u_{c_1} u_{c_h} X_{d_1} X_{d_h}$ where a, b_α, c_α, d_α, are all integers, and $1 \leq h \leq i$. (Recall w_i denotes the Y^i coefficient in S of w in $S[[Y]]$.) Further, $a + \sum_{\alpha=1}^{i-h} b_\alpha + \sum_{\alpha=1}^{h} c_\alpha + \sum_{\alpha=1}^{h} d_\alpha = i$. If such a term involves f_0, i.e., $a = 0$, then not all of the b_α, c_α, and d_α can be ≥ 1 since this would give $\sum_{\alpha=1}^{i-h} b_\alpha + \sum_{\alpha=1}^{h} c_\alpha +$

$\sum_{\alpha=1}^{h} d_\alpha \geqq (i - h) + h + h > i$. Thus, any term involving f_0 involves either j_0, u_0, or X_0 each of which is in $J(W)$. If a term does not involve f_0, then it must involve X_0 or j_0. (If not then $a + \sum_{\alpha=1}^{i=h} b_\alpha + \sum_{\alpha=1}^{h} c_\alpha + \sum_{\alpha=1}^{h} d_\alpha \geqq 1 + (i - h) + h > i$.) Since $X_0 = k$ and $j_0 \in (k)$, we have $f_0 = af_0 + bk$ where $a \in J(W)$. Since $1 - a$ is invertible, $f_0 \in (k) \subsetneqq (k, Y) = (j, X)$. However, $f \in R$ so $f \in (j)$ as required.

COROLLARY. *Let $R[[X]] = S[[Y]]$, $y = j + uX$, and $X = k + vY$. Let P be a maximal ideal of R, and Q the maximal ideal of S such that $(P, X) = (Q, Y)$, then $\hat{R}_P \cong \hat{S}_Q$ where the completion can be taken with respect to the maximal ideals in question or with respect to $(j)R_P$ and $(k)S_Q$.*

Proof. Let $M = (P, X) = (Q, Y)$, then W_M can be thought of as a subring of $R_P[[X]]$ or $S_Q[[Y]]$. In either case the completion of W_M with respect to M is all of $\hat{R}_P[[X]]$ or $\hat{S}_Q[[Y]]$ where $\hat{R}_P$ is completion with respect to PR_P and $\hat{S}_Q$ is completion with respect to QS_Q. Thus, $\hat{R}_P[[X]] \cong \hat{S}_Q[[Y]]$ and the result follows from Theorem 4. The proof of the other completion is similar, this time complete W_M with respect to $(X, Y) = (j, X) = (k, Y)$.

THEOREM 5. *If R is a complete semi-local ring, R is power-invariant.*

Proof. Let $R[[X]] = S[[Y]] = W$, then W and S are also complete semilocal. Since a complete semilocal ring is a direct sum of complete local rings, the result follows from the corollary.

3. **Existence of one to one maps.** The following lemmas are needed for our next result. With notation as in the introduction, we keep the convention that if $w \in W = R[[X]] = S[[Y]]$ that w_i is the Y^i coefficient in S of W, and we let w^i denote the X^i coefficient in R of w. Since $i = 0$ and $i = 1$ are the only cases of interest, there should be no confusion with exponents.

Define $\phi: R \to R$ by $\phi(r) = (r_0)^0$.

LEMMA 1. *With $\varnothing, R, S, W, X, Y$ as above and $Y = j + uX$, $X = k + vY$, we have $(Y) \cap R \subseteq \text{Ker } \phi \subseteq (j)$. ($(j)$ denotes the ideal of R generated by j).*

Proof. Since $Y_0 = 0$ the first containment is trivial. Now suppose $r \in R_0$ and $(r_0)^0 = 0$, then $r_0 = tX$ and $r = tX + fY$. Since $r = r^0$ $r = f^0 y^0 = f^0 j$.

LEMMA 2. *With everything as in Lemma* 1, *assume* $\phi(j)$ *is either zero or not a zero-divisor. Then* $\mathrm{Ker}\,\phi = 0$, *or* $\mathrm{Ker}\,\phi = (j)$.

Proof. If $\phi(j) = 0$, $\mathrm{Ker}\,\phi = (j)$ by Lemma 1. If $\phi(j) \neq 0$, let $f \in \mathrm{Ker}\,\phi$. $f = aj$ also by Lemma 1. $0 = \phi(f)\phi(a)\phi(j)$, which implies $\phi(a) = 0$. Thus, $a = bj$ and it is clear that $f \in \bigcap_{n=1}^{\infty} (j^n)$. Since $\phi(j) = (u_0)^0 v^0 j$ is not a zero-divisor, j is not a zero-divisor. Thus $j[\bigcap_{n=1}^{\infty} (j^n)] = \bigcap_{n=1}^{\infty} (j^n) = 0$ by Theorem 3. Thus, $f = 0$ and $\mathrm{Ker}\,\phi = 0$.

LEMMA 3. *With notation as above, j not a zero-divisor implies k is not a zero-divisor.*

Proof. j not a zero-divisor implies $\{j, X\}$ is a W-sequence. Suppose k is a zero-divisor, then some element of S kills k, say $sk = 0$. We get $sX = svY$. Now $sv \notin (x)$ since then s would be a multiple of Y. However, $sX = svj + svuX$ makes svj a multiple of X which is a contradiction. Thus, k is not a zero-divisor.

THEOREM 6. *Let $R[[X]] = S[[Y]]$, $Y = j + uX$, $X = k + vY$. If j and $\phi(j) = (j_0)^0$ are not zero-divisors (unless 0), then there exist 1 to 1 maps from R into S and from S into R.*

Proof. We need only consider the two cases of Lemma 2.

Case 1. $\mathrm{Ker}\,\phi = (j) \neq 0$.

In this case R is actually isomorphic to S. Let $A = Im\phi$. If $r \in R$, $r = r_0 + wY$ for some $w \in W$ and $r = r^0 = (r_0)^0 + w^0 Y^0 = \phi(r) + w^0 j$. Thus, $R = A + (j)$, $\phi = \phi^2$, and $A \cap (j) = 0$ all follow. $R = A + (j) = A + jR$ implies $R = A + jA + \cdots + j^n A + j^{n+1}R$ for any n. R is complete with respect to (j) and also Hausdorff as in the proof of Lemma 2. Thus, $R = A[[j]]$. j not a zero-divisor and $A \cap (j) = 0$ together imply that j is an analytic indeterminate over A so $R \cong A[[X]]$. We have $W = A[[k, Y]]$. We next show that $(Y) \cap A[[k]] = 0$. Let $t \in (Y) \cap A[[k]]$. $t = ly = \sum_{i=0}^{\infty} a_i k^i$. (Here $a_i \in A$ and i is an ordinary subscript.) $a_0 \in (j) \cap A = 0$ so $ly = \sum_{i=1}^{\infty} a_i k^i$. Taking constant terms in S we get $0 = \sum_{i=1}^{\infty} (a_i)_0 k^i$. Since k is not a zero-divisor $\sum_{i=1}^{\infty} (a_i)_0 k^{i-1} = 0$ and $(a_1)_0 \in (k)$. Thus, $a_1 \in (k, Y) \cap A = (j) \cap A = 0$. By induction $a_i = 0$ for all i and $(Y) \cap A[[k]] = 0$. But then $A[[k]] \cong W/(Y) \cong S$. k not a zero-divisor and $(k) \cap A \subseteq (j) \cap A = 0$ imply k is also an analytic indeterminate over A and $S \cong A[[X]] \cong R$.

Case 2. $\mathrm{Ker}\,\phi = 0$.

Clearly the map from R to S which takes r to r_0 is 1-1. Let

ϕ denote the map from S to S which takes s to $(s^0)_0$. Since k is not a zero-divisor, $\phi(k)$ is zero or not a zero-divisor. Thus, by Lemma 2 Ker $\phi = 0$ or Ker $\phi = (k)$. If Ker $\phi = (k)$ the above argument with the role of R and S reversed yields $R \cong S$. If Ker $\phi = 0$, the map from S to R which takes s to s^0 is 1-1.

Note. There is the following analogue of Theorem 6. The proof is due to Nagata and appears in [1].

THEOREM 7. *If R is an integral domain and $R[X] = S[Y]$, then there exist injective homomorphisms of R into S and S into R.*

4. *n*-Variable case. I believe the work on power-invariance to data has involved only one variable. It is natural to consider the following question. When can one conclude $S \cong R$, if there exists some n such that $R[[X_1, \cdots, X_n]] = S[[Y_1, \cdots, Y_n]]$ where the X_i and Y_i are independent analytic indeterminates over R and S respectively. To wit, we give the following definition:

DEFINITION. R will be called "forever power-invariant" provided $S \cong R$ whenever there is a positive integer n such that $S[[X_1, \cdots, X_n]] \cong R[[X_1, \cdots, X_n]]$ where the X_i are independent analytic indeterminates over R and S.
Induction readily yields the following:

THEOREM 8. *If R is a quasi-local or a complete semi-local ring, then R is forever power-invariant.*

The next theorem generalizes the result that a ring with nilpotent Jacobson radical is power-invariant under the 1-variable definition. It also relaxes the nilpotent condition to a nil Jacobson radical.

THEOREM 9. *If every element of $J(R)$ is nilpotent, then R is forever power-invariant.*

Proof. Let $R[[X_1, \cdots, X_n]] = S[[Y_1, \cdots, Y_n]] = W$. Let X, Y denote the vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$; J, K vectors in R^n and S^n respectively such that $Y = J + XU, X = K + YV$ where U and V are $n \times n$ matrices with entries in W. Since $J(R)$ is nil, each component of the vector J is nilpotent. The key point is that if j is a nilpotent element of R, every coefficient in its expression as an element of $S[[Y]]$ is also nilpotent. Now, careful examination of $Y = J + KU + YVU$ yields $(1, \cdots, 1) = J_1 + KU_1 + \text{diag } V_0U_0$ by taking the Y_i coefficient of the ith component of each side. Here J_1 denotes

the result of this application to J; U_1 is the nxn matrix whose ith column consists of the Y_i coefficients of the ith column of U; and $U_0(V_0)$ is the nxn matrix of constant terms of $U(V)$. Since J_1 and K have entries in $J(S)$, (the elements of J_1 being nilpotent) the elements of the diagonal of V_0U_0 are invertible. The same is, of course, then true for VU. By a similar scrutiny of the same, i.e., $Y = J + KU + YVU$, this time taking the Y_j coefficient of the ith component of both sides with $i \neq j$, we get the entries of VU which are off the diagonal to be in $J(W)$. Thus, VU has an invertible determinant, whence both V and U do, and both are invertible matrices. Thus, if the maps from $R[X]$ to $R[[X]]$ which take R to R and X to $J + XU$ or $(X - J)U^{-1}$ can be extended to $R[X]]$ in the natural way, they are clearly inverse maps, and thus R-automorphisms. In order for the extensions to be made $R[[X]]$ needs to be complete and Hausdorff with respect to the ideal generated by the images of the $\{X_i\}$. This is clear in the case that X maps to $J + XU = Y$. $R[[X]]$ is certainly complete with respect to $(j_1, \cdots, j_n, X_1, \cdots, X_n)$. Since the j_i are nilpotent and $\bigcap_{k=1}^{\infty} (X_1, \cdots, X_n)^k = 0$, $\bigcap_{k=1}^{\infty} (j_1, \cdots, j_n, X_1, \cdots, X_n)^k = 0$ which is equivalent to $R[[X]]$ being Hausdorff with respect to $(j_1, \cdots, j_n, X_1, \cdots, X_n)$. Thus, the map taking X to $(X - J)U^{-1}$ can also be extended. Thus, $R[[X]] = R[[Y]]$ and $R \cong U/(Y) \cong S$.

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NORTH CENTRAL COLLEGE