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A NOTE ON STARSHAPED SETS

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If S is a compact subset of R^d, it is shown that S is starshaped if and only if S is nonseparating and the intersection of the stars of the (d-2)-extreme points of S is nonempty.

Let $S \subset R^d$. The (d-2)-*extreme points* of S are by definition those points of S such that if $D \subset S$ is a (d-1)-dimensional simplex then $x \notin \operatorname{relint} D$ (the relative interior of D). The totality of (d-2)-extreme points of S is denoted by $E(S)$. For each $y \in S$ we define $S(y)$, the *star of* y by $S(y) = \{z: [y, z] \subset S\}$, where $[y, z]$ denotes the closed line segment from y to z. S is said to be *starshaped* if $\operatorname{Ker} S \neq \emptyset$ where $\operatorname{Ker} S = \{S(y): y \in S\}$. In [2] it is shown that if S is a compact starshaped set in R^d then $\operatorname{Ker} S = \bigcap \{S(y): y \in E(S)\}$. Thus the following question arises: if S is such that $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, under what hypothesis is S starshaped? It is clearly desirable that the hypothesis should be as weak as possible in order to indicate to what extent $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$ implies that S is starshaped. In [3] it is shown that one suitable hypothesis is that S should have the *half-ray property*, that is, for any point x in $R^d \backslash S$ there is a half-line l with vertex x such that $l \cap S = \emptyset$. Now we note that this hypothesis is a rather strong one especially as it is being used to deduce the fact that a certain set is starshaped. Thus one suspects that a much weaker hypothesis might suffice. This suspicion is further strengthened by the example given in [3] to show that, in fact, some hypothesis is necessary. More precisely, the example given is a *separating set* that is, its complement is not connected. The purpose of this note is to prove the following

THEOREM. *If $S \subset R^d$ is a nonseparating compact set and $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, then S is starshaped.*

Proof. Let $z \in \bigcap \{S(y): y \in E(S)\}$. We shall show that for any x in $R^d \backslash S$, $l(x, z) \cap S = \emptyset$ where $l(x, z)$ is the half-line with vertex x which does not contain z but is such that the line containing $l(x, z)$ does contain z. Clearly this suffices to show that S is starshaped.

Choose x_0 in the complement of the convex hull of S, then $l(x_0, z) \cap S = \emptyset$. Now since S is a nonseparating compact set, its complement is a path-connected unbounded open set (see [1, p. 356]). Thus any point in $R^d \backslash S$ can be "joined" to x_0 by a finite polygonal path in $R^d \backslash s$ such that if t is any segment of the path then the line containing t does not contain z.

Now we assume $l(x, z) \cap S \neq \emptyset$ for some point x in $R^d \backslash S$ and seek a contradiction. Let P be a polygonal path as described above with consecutive vertices $v_1 = x, v_2, v_3, \cdots, v_n = x_0$. Put $i = \max\{j: l(v_j, z) \cap S \neq \emptyset\}$ then $1 \leqq i < n$. Let the closed segment $[v_i, v_{i+1}]$ be the image under the continuous function f of the unit interval, with $f(0) = v_i$ and $f(1) = v_{i+1}$. Note that if $p \neq q$ then $l(f(p), z) \cap l(f(q), z) = \emptyset$. Now $l(f(1), z) \cap S = \emptyset$ and so, since S is compact we can put $p = \max\{q: l(f(q), z) \cap S \neq \emptyset\}$ and then $0 \leqq p < 1$. Let y be the point of S on $l(f(p), z)$ which is furthest from z. Now suppose D is a $(d-1)$-simplex with $D \subset S$ and $y \in \operatorname{relint} D$.

Then y must be the mid-point of a segment which is contained in $S \cap Q$ where Q is the plane through z, v_i, v_{i+1}. But this is impossible because of the definition of y and the fact that $l(f(q), z) \cap S = \emptyset$ for $p < q \leqq 1$. Hence $y \in E(S)$ and so $f(p) \in S$. This contradiction shows that $l(x, z) \cap S = \emptyset$ and thus completes the proof.

Finally, as a result of the above theorem and the comments made in [2] we are led to ask: if S has the half-ray property and has a point which "sees" just the extreme points of the convex hull of S and not all the (d-2)-extreme points, is S necessarily starshaped? The following example shows that the answer is negative:

$$S = \{(x, y) \in R^2: |x| \leqq 1, |y| \leqq 1\} \Big\backslash \left\{(x, y) \in R^2: |x| < \frac{1}{2}, |y| > \frac{1}{2}\right\} .$$

Similarly we observe that if we rotate S about the y-axis we obtain a three dimensional set with the required properties.

References

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2. J. W. Kenelly and W. R. Hare et al., *Convex components, extreme points, and the convex kernel*, Proc. Amer. Math. Soc., **21** (1969), 83-87.
3. N. Stavrakas, *A note on starshaped sets, (k)-extreme points and the half ray property*, Pacific J. Math., **53** (1974), 627-628.

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