PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

LOWER BOUNDS ON THE STABLE RANGE OF POLYNOMIAL RINGS

MICHAEL R. GABEL

Using the existence of certain nonfree projective modules, lower bounds on the stable range of commutative polynomial rings are established.

1. Introduction. The stable range of a commutative ring R is the largest integer n such that there exists a unimodular sequence $r_1, \cdots, r_n$ in R which is not stable. (See definitions 2.1 and 2.2.) Using the existence of certain nonfree projectives, we prove the following theorem concerning lower bounds for the stable range of polynomial rings:

THEOREM A. *Let K be any commutative ring. For every integer $n \geqq 1$ let $K(n) = K[Z_1, \cdots, Z_n]$ be the polynomial ring over K in indeterminates $Z_1, \cdots, Z_n$. Then*

(1) *The stable range of $K(1)$ is at least 2*

(2) *The stable range of $K(n)$ is at least $[n/2] + 1$ if $n \geqq 2$.*

(3) (Vasershtein, [6, Theorem 8]). *If there is a ring homomorphism from K to a subfield of real numbers, then the stable range of $K(n)$ is at least $n + 1$.*

The techniques of this paper are essentially the same as those in [2]. That paper contains a weaker form of Theorem A(2) ($[n/2]$ appears in place of $[n/2] + 1$, and there are certain restrictions on the number of indeterminates.)

2. Basic definitions and propositions. Throughout this paper, all rings are commutative with identity and ring homomorphisms preserve the identity element.

DEFINITION 2.1. A sequence $r_1, \cdots, r_n$ of elements in a ring R is called *unimodular* if the ideal $(r_1, \cdots, r_n)$ is all of R.

DEFINITION 2.2. A unimodular sequence $r_1, \cdots, r_n$ of elements in a ring R is called *stable* if $n \geqq 2$ and if there exist $b_1, \cdots, b_{n-1} \in R$ such that the sequence $r_1 + b_1 r_n, r_2 + b_2 r_n, \cdots, r_{n-1} + b_{n-1} r_n$ is again unimodular.

PROPOSITION 2.3. Let R be a ring and $I \subset R$ a proper ideal of R. Let $r_1, \cdots, r_t$ be a sequence of elements in R such that their residues $\bar{r}_1, \cdots, \bar{r}_t$ form a *unimodular* sequence in R/I which is not stable. Choose any $i \in I$ and $s_1, \cdots, s_t \in R$ such that $r_1 s_1 + \cdots + r_t s_t + i = 1$. Then the sequence $r_1, \cdots, r_{t-1}, r_t s_t + i$ is unimodular but not stable in R. Consequently, the stable range of R bounds that of R/I.

Proof. The sequence $r_1, \cdots, r_{t-1}, r_t s_t + i$ is clearly unimodular. If it were stable, then there would exist $b_1, \cdots, b_{t-1} \in R$ such that

$$r_1 + b_1(r_t s_t + i), \cdots, r_{t-1} + b_{t-1}(r_t s_t + i)$$

is unimodular. Going mod I, this contradicts our assumption that $\bar{r}_1, \cdots, \bar{r}_t$ is not stable in R/I.

DEFINITION 2.4. If $r_1, \cdots, r_t \in R$, $\ker [r_1 \cdots r_t]$ will denote the kernel of the map $R^t \to R$ whose matrix is $[r_1 \cdots r_t]$.

The following proposition appears in [2].

PROPOSITION 2.5. *Let R be a ring and let $r_1, \cdots, r_t$ be a unimodular sequence in R. Let $P = \ker[r_1 \cdots r_t]$. If $r_1, \cdots, r_t$ is stable, then P is a free R-module.*

Proof. Since $r_1, \cdots, r_t$ is stable, it's easy to see that the matrix $[r_1 \cdots r_t]$ can be transformed to $\beta = [1\, 0 \cdots 0]$ via elementary transformations. Consequently $\ker [r_1 \cdots r_t] \cong \ker \beta$. But $\ker \beta$ is clearly free.

PROPOSITION 2.6. (1) (Estes-Ohm, [1].) *If*

$$Z_1, \cdots, Z_n, \quad 1 - \sum_{i=1}^{n} Z_i^2$$

is stable in $K[Z_1, \cdots, Z_n]$, then $Z_1, \cdots, Z_{n+1}, 1 - \sum_{i=1}^{n+1} Z_i^2$ is stable in $K[Z_1, \cdots, Z_{n+1}]$.

(2) *If $X_1, \cdots, X_s, 1 - \sum_{i=1}^{s} X_i Y_i$ is stable in $K[X_1, \cdots, X_s, Y_1, \cdots, Y_s]$, then $X_1, \cdots, X_{s+1}, 1 - \sum_{i=1}^{s+1} X_i Y_i$ is stable in $K[X_1, \cdots, X_{s+1}, Y_1, \cdots, Y_{s+1}]$.*

Proof. The proof we give here of (2) is essentially the same as the proof of (1) given in [1]. For any integer $t \geqq 1$, let $Q_t = 1 - \sum_{i=1}^{t} X_i Y_i$. Now observe that if $a_1, \cdots, a_s \in K[X_1, \cdots, X_s, Y_1, \cdots, Y_s]$, we have the following inclusion of ideals in $K[X_1, \cdots, X_{s+1}, Y_1, \cdots Y_{s+1}]$:

$$(X_1 + a_1 Q_s, \cdots, X_s + a_s Q_s) \subseteqq (X_1 + a_1 Q_{s+1}, \cdots, X_s + a_s Q_{s+1}, X_{s+1}) .$$

3. Proof of Theorem A. The proof of part (1) is easy, as $Z_1, 1 - Z_1^2$ is never stable in $K(1)$. (Just mod out K by a prime ideal to reduce to the domain case, where it is trivial.)

We now prove part (2). By Proposition 2.3, we can assume n is even, say $n = 2s$. Now write $K(n) = K[X_1, \cdots, X_s, Y_1, \cdots, Y_s]$. We shall show that the unimodular sequence $X_1, \cdots, X_s, 1 - \sum_{i=1}^{s} X_i Y_i$ is not stable. Let $\underline{m}$ be a maximal ideal of K. If $X_1, \cdots X_s, 1 - \sum_{i=1}^{s} X_i Y_i$ were stable in $K(n)$, then the image of this sequence under the canonical map $K(n) \to K/\underline{m}(n)$ would remain stable. Thus we may assume K is a field. By Proposition 2.6(2), we may also assume $s \geqq 8$. Let $R = K(n)[U, V]$, where U and V are indeterminates and let $f = X_1 Y_1 + \cdots + X_s Y_s + UV - 1$. Let $T = R/(f)$. Note that the sequence of residues $\bar{X}_1, \cdots, \bar{X}_s, \bar{U}$ is unimodular in T. Since $s \geqq 8$, the T-module $\ker[\bar{X}_1, \cdots, \bar{X}_s, \bar{U}]$ is not free ([4], Cor. 6.3). So by Proposition 2.5, $\bar{X}_1, \cdots, \bar{X}_s, \bar{U}$ is not stable in T. Since $X_1 Y_1 + \cdots + X_s Y_s + UV + (-f) = 1$, Proposition 2.3 shows that $X_1, \cdots, X_s, UV - f$ is not stable in $K(n)[U, V]$. But $UV - f = 1 - \sum_{i=1}^{s} X_i Y_i$ is an element of $K(n)$. That is, we have the unimodular sequence $X_1, \cdots, X_s, 1 - \sum_{i=1}^{s} X_i Y_i$ in $K(n)$ which is not stable in $K(n)[U, V]$. So certainly it is not stable in $K(n)$. This proves Theorem A(2).

The proof of Theorem A(3) is similar. We will show that the stable range of $K[Z_1, \cdots, Z_n]$ is at least $n + 1$ by showing that the unimodular sequence $Z_1, \cdots, Z_n, 1 - \sum_{i=1}^{n} Z_i^2$ is not stable. Since K maps to the reals, it suffices to show this when K equals the reals. By Proposition 2.6(1) we may also assume $n \geqq 8$. Now, let $R = K(n)[U]$, where U is an indeterminate, let $f = Z_1^2 + \cdots + Z_n^2 + U^2 - 1$, and set $T = R/(f)$. Since $n \geqq 8$, The T-module $\ker[\bar{Z}_1, \cdots, \bar{Z}_n, \bar{U}]$ is not free [5]. Propositions 2.3 and 2.5 imply that $Z_1, \cdots, Z_n, U^2 - f$ is not stable in $K(n)[U]$. But the U^2 drops out so we get that $Z_1, \cdots, Z_n, 1 - \sum_{i=1}^{n} Z_i^2$ is not stable in $K(n)$. This proves Theorem A(3).

4. Remarks. (1) For any ring K, let $P_{1,n}^0(K)$ be the module over $K_{1,n}^0 \equiv K[Z_1, \cdots, Z_n]/(\sum_{i=1}^{n} Z_i^2 - 1)$ defined by

$$P_{1,n}^0(K) = \ker[\bar{Z}_1, \cdots, \bar{Z}_n] .$$

The proof of Theorem A(3) yields that if $Z_1, \cdots, Z_n, 1 - \sum_{i=1}^{n} Z_i^2$ is stable in $K(n)$, then $P_{1,n+1}^0(K)$ is a free $K_{1,n+1}^0$-module. Assume now that K is a field. Then it is well-known [1] that the stable range of $K(n)$ is less than or equal to $n + 1$. So when $P_{1,n+1}^0(K)$ is not free, the stable range of $K(n)$ equals $n + 1$. Note that there are cases when $P_{1,n+1}^0(K)$ is free ($K =$ reals, $n = 7$) and the stable range of $K(n)$ is still $n+1$ [6]. Moreover, if K is a finite field $P^0_{1,n+1}(K)$ is free [3] and the stable range of $K(n)$ is less than $n+1$ (Vasershtein).

(2) The nonstable unimodular sequence $Z_1, \cdots, Z_n, 1-\sum_{i=1}^n Z_i^2$ that we obtained in the proof of Theorem A(3) is the same one Vasershtein discovered in [6].

REFERENCES

1. Dennis Estes and Jack Ohm, *Stable range in commutative rings*, J. Algebra, **7**, No. 3 (1967), 343-362.

2. M. R. Gabel and A. V. Geramita, *Stable range for matrices*, J. Pure Appl. Algebra, **5** (1974), 97-112.

3. A. V. Germita, *Projective modules and homogeneous forms* (unpublished).

4. M. Raynaud, *Modules projectifs universels*, Invent. Math., **6** (1968), 1-26.

5. R. G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc., **105** (1962), 264-277.

6. L. N. Vasershtein, *Stable rank of rings and dimensionality of topological spaces*, Functional Analysis and its Applications, **5** (1971), 102-110.

Received December 13, 1974.

PURDUE UNIVERSITY