PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

NORMAL HYPERSURFACES

JOSEPH BECKER

The purpose of this note is to give a simple analytic proof of a theorem of Oka: If V is a complex analytic hypersurface whose singular locus has codimension at least two, then V is normal. In other words, every weakly holomorphic function is holomorphic.

This result has since been generalized by Abhankar and Thimm to the case when V is an algebraic complete intersection (which is to say that the ideal of functions holomorphic in the ambient space vanishing on V is generated by k functions, where k is the codimension of V in the ambient space).

Actually we prove a slightly stronger result than Oka's.

THEOREM. *Let V be a complex analytic hypersurface, A a complex analytic subset of V with codimension at least 2. Then there is a bounded linear operator $\phi: \mathcal{O}(V-A) \rightarrow \mathcal{O}(V)$ such that $\phi(f) \mid V-A = f$.*

Proof. Suppose $V \subset C^n$ and the projection $\pi: C^n \rightarrow C^{n-1}$ to the first $n-1$ co-ordinates gives an r-sheeted branched cover of V in some neighborhood of the origin with branch set B, $B' = \pi(B)$, $A' = \pi(A)$ and $z' = \pi(z)$. Now π induces a homomorphism ${}_{n-1}\mathcal{O} \rightarrow {}_{n}\mathcal{O}/I(V) = \mathcal{O}(V)$ making $\mathcal{O}(V)$ into a finitely generated $\mathcal{O}_{n-1}$ module with generators $1, z_n, \cdots, z_n^{r-1}$. Let $P(z', z_n)$ be the minimal degree polynomial for z_n over ${}_{n-1}\mathcal{O}$; for any $f \in \mathcal{O}(V)$ by the Weierstrass division theorem we have $f = QP + R$ where $R \in {}_{n-1}\mathcal{O}[z_n]$ is a holomorphic polynomial of 'degree $\leqq r-1$. Hence f can be written as $\sum_{i=0}^{r-1} b_i(z')z_n^{r-i-1} \bmod I(V)$. However the $b_i(z')$'s are unique.

For every $z' \notin B'$, let $\alpha_1(z'), \cdots, \alpha_r(z')$ be the values of z_n on the fiber $\pi^{-1}(z)$ and $f_j = f(z', \alpha_j(z'))$ for $j = 1, \cdots, r$. Then

$$f_j = \sum_{i=0}^{r-1} b_i(z')\alpha_j(z')^{r-i-1} .$$

These equations can be viewed as a system of r linear equations in the r unknowns $b_i(z')$ and solved by Cramer's rule:

$$b_i(z') = \frac{\det [1, \alpha_j, \alpha_j^2, \cdots, \alpha_j^{r-i-2}, f_j, \alpha_j^{r-i}, \cdots, \alpha_j^{r-1}]}{\det [1, \alpha_j, \cdots, a_j^{r-1}]}$$

where in both determinants the entries in the jth row are indicated. The denominator is the Vandermonde determinant $\Delta(\alpha_1, \cdots, \alpha_r)$ and equals $\prod_{l \neq k} (\alpha_l(z') - \alpha_k(z'))$. It is an easy exercise in linear algebra to verify that the numerator equals

$$\sum_{j=1}^{r} \sigma_i(\alpha_1, \cdots, \hat{\alpha}_j, \cdots, \alpha_r) \Delta(\alpha_1, \cdots, \hat{\alpha}_j, \cdots, \alpha_r) f_j$$

where σ_i is the elementary symmetric polynomial of degree i, and $\hat{\alpha}_j$ means that α_j is to be deleted. Therefore

$$b_i(z') = \sum_{j=1}^{r} \frac{\sigma_i(\alpha_1(z'), \cdots, \hat{\alpha}_j(z'), \cdots, \alpha_r(z'))}{\prod_{k \neq j} (\alpha_j(z') - \alpha_k(z'))} f(z', \alpha_j(z')) .$$

Since f is holomorphic so is

$$b_i(z') = \frac{1}{(r-i-1)!} \frac{\partial^{r-i-1} f}{\partial z_n} (z', \alpha_j(z'))$$

and hence the $b_i(z')$ extend holomorphically across B'. (Since f and $f - QP = R$ have the same values on V, it makes no difference which is used in these formulas.)

Now suppose $h \in \mathcal{O}(V - A)$, and for $z' \notin B'$ define

$$\phi(h) = \sum_{j=1}^{r} \left(\prod_{k \neq j} \frac{z_n - \alpha_k(z')}{\alpha_j(z') - \alpha_k(z')} \right) h(z', \alpha_j(z')) = \sum_{i=0}^{r-1} b_i(z') z_n^{r-i-1}$$

where b_i is defined above. Then the functions are holomorphic in a neighborhood of z' provided h is holomorphic near each point in $\pi^{-1}(z')$. Thus $b_i(z')$ extend to holomorphic functions near any point of $B' - A'$ and then by Hartogs' theorem, b_i extends across A', because codim $A \geqq 2$.

The operator ϕ whose existence has been demonstrated is obviously linear and clearly bounded with norm one because both $\mathcal{O}(V - A)$ and $\mathcal{O}(V)$ are Fréchet spaces [3] with seminorms being sup on compact subsets; $V - A$ is dense in V so $\sup_{x \in V} | \phi(f) = \sup_{V \in x - A} | f(x) |$.

Incidentally notice that we did not use the result [4] that a function holomorphic in the complement of an analytic set of codim two is bounded in a neighborhood of that set, e.g. $\mathcal{O}(V - A) = \mathcal{O}(V)$.

REMARK. The same proof works when $\mathcal{O}(V)$ is a Cohen Macauley ring, that is the direct image sheaf of $\mathcal{O}(V)$ by a local parametrization is a free sheaf.

REFERENCES

1. S. Abhyankar, *Concepts of order and rank on a complex space, and a condition for normality*, Math. Ann., **141**, (1960), 171–192.

2. R. C. Gunning, *Lectures on Complex Analytic Varieties*, Mathematical Notes, Princeton, N.J., Princeton University Press 1970.
3. R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Englewood Cliffs, N.J., Prentice-Hall 1965.
4. R. Narasimhan, *Introduction to the Theory of Analytic Spaces*, Lecture Notes in Mathematics 25, Berlin-Heidelberg-New York, Spring 1966.
5. K. Oka, *Sur les fonctions analytique de plusieurs variables VIII*, J. Math. Soc. Japan, **3** (1951).
6. W. Thimm, *Untersuchungen über das Spurproblem von holomorphen Funktionen auf analytischen Mengen*, **139** (1959), 95–114.

Received February 5, 1974.

PURDUE UNIVERSITY