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ON LINEAR REPRESENTATIONS OF AFFINE GROUPS I

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The category of linear representations of an affine group is isomorphic to the category of comodules over a k-Hopf-algebra where k denotes a commutative ring. The category of C-comodules Comod-C over an arbitrary k-coalgebra C is comonadic over the category k-Mod of k-modules. It is complete, cocomplete and has a cogenerator. The C-comodules whose cardinality $\leqq$ max (cardk, $\aleph_0$) generate the category Comod-C. Comod-C is in general not abelian but can nicely be embedded into an AB-4 category. Comod-C is a tensored and cotensored k-Mod-category (enriched over k-Mod) with a canonical (E, M)-factorization which is the factorization in k-mod if and only if C is flat. Comod-C has free C-comodules if and only if C is finitely generated and projective. Furthermore I give numerous examples and counterexamples as well as the explicit description of all constructions, in particular of the limits in Comod-C which was not known even for coalgebras over fields.

Let k be a commutative ring with a unit. k-Alg shall denote a small category of models of k-algebras (cf. [5] p. XXIV). Recall that an affine k-monoid (resp. k-group) is a monoid (resp. group) in the functor category [k-Alg, Sets] whose underlying functor is representable. Let M be a k-module. Then M induces an affine k-monoid $\mathscr{L}(M)$: k-Alg $\rightarrow$ Sets by $\mathscr{L}(M)(A) = \mathrm{End}_A(M \otimes_k A)$, $A \in$ k-Alg (cf. [5] p. 149). Let $\mathscr{G}$ be an affine k-monoid and M a k-module. Then a monoid morphism $\varphi: \mathscr{G} \rightarrow \mathscr{L}(M)$ is called a linear representation of $\mathscr{G}$ in M and the pair (M, φ) a k-$\mathscr{G}$-module. The definition of morphisms between k-$\mathscr{G}$-modules is evident. Thus one obtains the category k-$\mathscr{G}$-Mod of linear representations of $\mathscr{G}$, resp. of k-$\mathscr{G}$-modules. Since $\mathscr{G}$ is representable we obtain the canonical isomorphisms [k-Alg, Sets] $(\mathscr{G}, \mathscr{L}(M)) \cong \mathscr{L}(M)(C) \cong k$-Mod $(M, M \otimes_k C)$, where C is the representing object of $\mathscr{G}$. The monoid structure of $\mathscr{G}$ induces a k-coalgebra structure on C, i.e., the representing object has two k-linear mappings $\varDelta: C \rightarrow C \otimes C$ and $\varepsilon: C \rightarrow k$, called comultiplication and counit, such that $\langle C, \varDelta, \varepsilon \rangle$ is coassociative and counitary (cf. [19]). By the above canonical isomorphisms every monoid morphism $\varphi: \mathscr{G} \rightarrow \mathscr{L}(M)$ induces a k-linear map $\chi_M: M \rightarrow M \otimes C$ such that $M \otimes \varDelta \cdot \chi_M = \chi_M \otimes C \cdot \chi_M$ and $M \otimes \varepsilon \cdot \chi_M = \mathrm{id}_M$, and conversely. A pair $\langle M, \chi_M \rangle$ fulfilling the above properties is called a C-comodule. Let $\langle M, \chi_M \rangle$ and $\langle N, \chi_N \rangle$ be C-comodules. A k-linear mapping $f: M \rightarrow N$ is a C-comodule homomorphism if $\chi_N \cdot f = f \otimes C\chi_M$. Let (M, φ_M) and (N, φ_N) be k-$\mathscr{G}$-modules and $\langle M, \chi_M \rangle$, resp. $\langle N, \chi_N \rangle$ the corresponding C-comodules. Then a k-linear mapping $f: M \to N$ is a k-$\mathscr{G}$-module homomorphism $f: (M, \varphi_M) \to (N, \varphi_N)$ if and only if $f: \langle M, \chi_M \rangle \to \langle N, \chi_N \rangle$ is a C-comodule homomorphism.

Hence the category of linear representations of an affine monoid (group) is isomorphic to a category of C-comodules where C is a k-bialgebra (resp. k-Hopf algebra).

In this paper I study the elementary properties of a category of comodules over an arbitrary k-coalgebra. Categories of comodules were already studied by several authors where k is a field or the coalgebra is finite or flat (cf. [5], [7], [10], [14], [15], [17], [18], [19]). In all these cases Comod-C is a Grothendieck category with a generator. But if C is not flat then Comod-C need not to be abelian. This was already shown in [17]. The homomorphism theorem is no longer valid, the comodule structure on a subcomodule is in general no longer unique and so on.

But even in the case of a flat coalgebra C one didn't know as yet such elementary things as the explicit descriptions of limits.

Let C be an arbitrary coalgebra over a commutative ring k with a unit. Then the most important results of this paper are: The underlying functor U: Comod-$C \to k$-Mod is comonadic. The category Comod-C is complete, cocomplete, wellpowered and cowellpowered, has a generator and cogenerator. Comod-C can be embedded (full and faithful) into an $AB4$-category with sufficiently many injectives and projectives which in general fails to be a Grothendieck-category. This embedding is coreflective if and only if all objects in Comod-C are projective and is an isomorphism if and only if Comod-C is a spectral category. The functor λ: Comod-$C \to C^*$-Mod (cf. [14] § 1 or [19] Chap. II) is comonadic. Comod-C has free comodules if and only if C is finitely generated and projective. Comod-C has a proper (E, M)-factorization which is preserved by the underlying functor Comod-$C \to k$-Mod if and only if C is flat. Comod-C is well-powered and cowellpowered with respect to this factorization. By applying the techniques of V-categories I show that the k-Mod-category Comod-C is tensored and cotensored. If $f: C \to C'$ is coalgebra morphism then the induced k-linear functor f^*: Comod-$C \to$ Comod-C' preserves tensors and is k-Mod-comonadic. The k-linear functor $- \otimes C$: k-Mod $\to$ Comod-C has a k-linear-right adjoint. Furthermore I give numerous examples and counterexamples as well as explicit descriptions of all constructions.

I. Comodules over arbitrary coalgebras. In the language of monoidal categories a k-coalgebra $\langle C, \Delta, \varepsilon\rangle$ is just a comonoid in the monoidal category (k-Mod, $\otimes$) (cf. [11] Chap. VII 3). A C-comodule $\langle M, \chi_M\rangle$ is a coaction of C on M and a C-comodule homomorphism is a morphism between coactions of C in (k-mod, $\otimes$) (cf. [11] Chap. VII 4). This formal description gives us at once some elementary results such as the existence of a right adjoint of the underlying functor U: Comod-$C \to k$-Mod or the creation of colimits by U.

In the sequel I will give another description of Comod-C which allows us to apply the highly developed theory of monads.

Let $\langle C, \Delta, \varepsilon\rangle$ be a coalgebra. The coalgebra structure of $\langle C, \Delta, \varepsilon\rangle$ induces a functor

$$\mathscr{C} := - \otimes C;\ k\text{-Mod} \longrightarrow k\text{-Mod}$$

and functorial morphisms

$$\Delta = - \otimes \Delta \colon \mathscr{C} \longrightarrow \mathscr{C}^2 = - \otimes C \otimes C$$
$$\varepsilon = - \otimes \varepsilon \colon \mathscr{C} \longrightarrow Id_{k-\mathrm{Mod}}\ .$$

Since $\langle \mathrm{C}, \Delta, \varepsilon\rangle$ is a coalgebra $\langle - \otimes C, - \otimes \Delta, - \otimes \varepsilon\rangle$ clearly defines a comonad over k-Mod. A coalgebra $\langle M, \chi_M\rangle$ over this comonad is a pair where M is k-module and $\chi_M \colon M \to \mathscr{C}(M)$ is a k-morphism such that the following diagrams commutes

$$\begin{tikzcd}
\mathscr{C}^2(M) & \mathscr{C}(M) \arrow[l, "\mathscr{C}(\chi_M)"'] \\
\mathscr{C}(M) \arrow[u, "\Delta(M)"] \arrow[ur, phantom, "="] & M \arrow[l, "\chi_M"] \arrow[u, "\chi_M"']
\end{tikzcd}$$

$$\begin{tikzcd}
M & \mathscr{C}(M) \arrow[l, "\varepsilon(M)"'] \\
 & M \arrow[u, "\chi_M"'] \arrow[ul, equal, "="']
\end{tikzcd}$$

A morphism f between $\mathscr{C}$-coalgebras $\langle M, \chi_M\rangle$ and $\langle N, \chi_N\rangle$ is a k-morphism $f \colon M \to N$ such that $\chi_N \cdot f = \mathscr{C}(f) \cdot \chi_M$. Hence we obtain the following

THEOREM 1 (Notation as above). *Let* $\langle C, \Delta, \varepsilon\rangle$ *be a coalgebra. Then the category* Comod-C *of* C*-comodules is comonadic over* k-Mod.

From the elementary theory of monads we obtain at once some important corollaries.

COROLLARY 2 (cf. [11], [13], [16]). *The underlying functor*

$$U\colon \text{Comod-}C \longrightarrow k\text{-Mod}$$

has a right adjoint $\mathscr{G}\colon k\text{-Mod} \to \text{Comod-}C$ *defined by*

$$\begin{aligned} \mathscr{G}\colon k\text{-Mod} &\longrightarrow \text{Comod-}C \\ M &\longmapsto \langle M \otimes C, M \otimes \Delta \rangle \\ f &\longmapsto f \otimes C \end{aligned}$$

The comonad defined in k-Mod *by this adjunction is the given comonad* $\langle - \otimes C, - \otimes \Delta, - \otimes \varepsilon \rangle$.

COROLLARY 3. *The underlying functor* $U\colon \text{Comod-}C \to k\text{-Mod}$ *creates colimits and isomorphisms. In particular* Comod-C *is cocomplete and the colimits are formed in* k-Mod.

COROLLARY 4. *U creates those limits which are preserved by* $- \otimes C$. *If* C *is flat and* $T\colon D \to \text{Comod-}C$ *is a finite diagram, then* $p\colon \text{Diag}\, M \to T$ *is a limit in* Comod-C *if and only if* Up: $\text{Diag}\, UM \to UT$ *is a limit in* k-Mod.

Applying 21.3.6 in [16] we obtain

COROLLARY 5. Comod-C *is cowellpowered.*

Since right adjoints preserve cogenerators we get

COROLLARY 6. Comod-C *has a cogenerator.*

Let $\mathscr{C}$ be a category with finite limits and finite colimits. A functor $F\colon C \to C'$ is called left-exact (right-exact) if F preserves finite limits (finite colimits). F is called exact if F is left-exact and right-exact.

Since k-Mod is an additive category and $- \otimes C$ is additive and right-exact we obtain from Remark 21.1.11 in [16] Chap. 21 the well known

COROLLARY (cf. [7], [10]).
(1) Comod-C *is an additive category.*
(2) U *and* $\mathscr{G}$ *are additive functors.*
Furthermore $\mathscr{G}$ *is exact and* U *is right exact.*

PROPOSITION 8 (Notation as above). *The following statements are equivalent:*
(i) U *is exact.*

(ii) *C is flat.*

(iii) *$\mathscr{G}$ preserves injectives.*

Proof. (ii) → (i): Since U creates finite limits and is right exact it is exact.

(i) → (ii): Let $f\colon M \to N$ be an injective k-module homomorphism. Since $\mathscr{G}$ is exact, $\mathscr{G}(f) = f \otimes C\colon M \otimes C \to N \otimes C$ is an equalizer in Comod-C. Since U is exact $f \otimes C$ is injective, i.e., C is flat.

(i) → (iii): Well known.

(iii) → (i): Let $m\colon \langle M, \chi_M\rangle \to \langle N, \chi_N\rangle$ be a monomorphism in Comod-C and $f\colon M \to Q$ an injective extension of M in k-Mod. Then we obtain the following commutative diagram

$$\begin{tikzcd}
\langle Q \otimes C, Q \otimes \Delta\rangle & \langle M \otimes C, M \otimes \Delta\rangle \arrow[l, "f\otimes C"'] \arrow[r, "m\otimes C"] \arrow[dr, phantom, "="] & \langle N \otimes C, N \otimes \Delta\rangle \\
 & \langle M, \chi_M\rangle \arrow[u, "\chi_M"'] \arrow[r, "m"] & \langle N, \chi_N\rangle \arrow[u, "\chi_N"']
\end{tikzcd}$$

Since $\mathscr{G}$ preserves injectives, $\langle Q \otimes C, Q \otimes \Delta\rangle = \mathscr{G}(Q)$ is injective in Comod-C. Since $\mathscr{G}(Q)$ is injective and m is a monomorphism we obtain a comodule-homomorphism $g\colon \langle N, \chi_N\rangle \to \langle Q \otimes C, Q \otimes \Delta\rangle$ such that

$$f \otimes C \cdot \chi_M = g \cdot m\ .$$

$$\begin{tikzcd}
\langle M, \chi_M\rangle \arrow[r, "m"] \arrow[d, "f\otimes C\cdot\chi_M"'] & \langle N, \chi_N\rangle \arrow[dl, dashed, "g"] \\
\langle Q \otimes C, Q \otimes \Delta\rangle &
\end{tikzcd}$$

Since $\langle M, \chi_M\rangle$ is a C-comodule and $\varepsilon\colon - \otimes C \to Id_{k\text{-Mod}}$ is a functorial morphism we obtain the following equations:

$$\varepsilon_M \cdot \chi_M = id_M \quad \text{and} \quad f \cdot \varepsilon_M = \varepsilon_Q f \otimes C\ .$$

Thus $f = f \cdot \mathrm{id}_M = f \cdot \varepsilon_M \cdot \chi_M = \varepsilon_Q \cdot f \otimes C\chi_M = \varepsilon_Q \cdot g \cdot m$. Hence m is injective since f is injective, i.e., U is exact.

If C is flat U creates finite limits and colimits. Since Comod-C is additive and k-Mod is abelian we conclude that Comod-C is abelian. Since furthermore k-Mod is a Grothendieck category and U preserves and reflects colimits and monomorphisms Comod-C fulfills $AB5'$ (cf. [16] 4, 6.3), i.e., we obtain the following well known result.

COROLLARY 9. *If C is flat then* Comod-C *is a Grothendieck category. Furthermore U preserves and reflects finite limits and* *colimits. In particular a comodule homomorphism is an equalizer* (coequalizer) *in* Comod-C *if and only if* f *is injective* (surjective).

EXAMPLES 10. (1) Let k be a regular ring (regular in the sense of von Neumann) (cf. [2] p. 175, EX. 13). Then Comod-C is a Grothendieck category for every k-coalgebra C.

Let k be a commutative, associative ring with unit. Let T be a k-module. Then $C = k \oplus T$ together $\varDelta(r, t) = r \otimes 1 + 1 \otimes t + t \otimes 1 + \rho(t)$ and $\varepsilon(r, t) = r$ is a coalgebra with unit (cf. [18], where $\rho: T \rightarrow T \otimes T$ is an arbitrary coassociative k-morphism (take for example $\rho = 0$). Hence $C = k \oplus T$ is flat (projective, finitely generated, $\cdots$) if and only if T is flat (projective, finitely generated, $\cdots$).

(2) Let A be a torsion free abelian group A and $C = Z \oplus A$ with the above defined structure˙ Then Comod-C is a Grothendieck category$^{1)}$.

(3) Let A be an abelian group which is not torsion free. (e.g., $\boldsymbol{Z}/n\boldsymbol{Z}$, $\boldsymbol{Q}/\boldsymbol{Z}$). Then the coalgebra $C = \boldsymbol{Z} \oplus A$ with one of the above defined coalgebra structures is not flat[1].

DEFINITION 11. Let $\langle M, \chi_M \rangle$ be a C-comodule. A *subcomodule* $\langle N, \chi_N \rangle$ is a submodule N of M such that the inclusion $i: N \rightarrow M$ is a comodule homomorphism.

PROPOSITION 12. *Let* Comod-C *be an abelian category. Then the comodule structure on a subcomodule is unique.*

Proof. Let $\langle N, \chi_1 \rangle$ and $\langle N, \chi_2 \rangle$ be subcomodules of $\langle M, \chi_M \rangle$. Since the inclusion $i: \langle N, \chi_1 \rangle \rightarrow \langle M, \chi_M \rangle$ is injective it is a monomorphism and hence an equalizer in Comod-C since Comod-C is abelian by assumption. Hence the identity $\langle N, \chi_2 \rangle \rightarrow \langle N, \chi_1 \rangle$ must be a comodule homomorphism. Since U: Comod-$C \rightarrow k$-Mod creates isomorphisms we obtain $\chi_1 = \chi_2$.

EXAMPLE 13. (cf. [18]) Let $C = \boldsymbol{Z} \oplus Z/_{nz}$ be the Z-coalgebra with the following structure:

$$\varDelta(z, \bar{q}) = z \otimes 1 + 1 \otimes \bar{q} + \bar{q} \otimes 1 + \bar{q} \otimes \bar{1}$$
$$\varepsilon(z, \bar{q}) = z. \text{ (cp. (11) Ex. 1)}$$

Then the category Comod-C of $\boldsymbol{Z} \oplus Z/n\boldsymbol{Z}$-comodules is not abelian. By applying Proposition 12 we have only to show that there exist a C-comodule $\langle M, \chi_M \rangle$ and subcomodules $\langle N, \chi_N \rangle$ and $\langle N, \chi'_N \rangle$ of

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[1] Let k be a principal ideal domain. Then a k-module M is flat if and only if M is torsion free (cf. [4] § 24 Prop. 3 (ii)).

$\langle M, \chi_M \rangle$ with $\chi_N \neq \chi'_N$. The following example was given in [18]. Take

$$M = \boldsymbol{Q}/\boldsymbol{Z};\ \chi_M(\bar{q}) = \bar{q} \otimes 1$$
$$N = \boldsymbol{Z}/n\boldsymbol{Z};\ \chi_N(\bar{z}) = \bar{z} \otimes 1$$

and

$$N = \boldsymbol{Z}/n\boldsymbol{Z};\ \chi'_N(\bar{z}) = \bar{z} \otimes 1 + \bar{1} \otimes \bar{z}\ .$$

Then the inclusion $i\colon \boldsymbol{Z}/n\boldsymbol{Z} \to \boldsymbol{Q}/\boldsymbol{Z}\colon \bar{z} \to (\bar{z}/n)$ is a comodule homomorphism for χ_N and χ'_N. Since $\chi_N \neq \chi'_N$ we obtian that Comod-C is not abelian.

Conjecture 14. Comod-C is abelian if and only if C is flat.

In order to prove this conjecture one has to show that if Comod-C is abelian then the comodule monomorphisms are injective (cf. Proposition 8).

In [9], P. Freyd proves the existence of free abelian categories. He does it by taking a category C and embedding it into a large ambient abelian category. He then constructs the smallest exact subcategory containing C. The external version of this construction was made by M. Alderman in [1]. He gives an explicit description of free abelian categories. I'll take up Alderman's construction and will show that the category Comod-C (for every coalgebra C) can be fully and faithfully embedded into an AB-4 category with enough projectives and injectives, the free abelian category over Comod-C which in general fails to be a Grothendieck category.

Let us now recall Alderman's construction. Let $\boldsymbol{A}$ be an additive category. In the functor category $\boldsymbol{A}^{\to\to}$ define the following equivalence relation:

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "\varphi'"'] & A \arrow[r, "f"] \arrow[d, "\varphi"] & A'' \arrow[d, "\varphi''"] & \equiv & A' \arrow[r, "f'"] \arrow[d, "\psi'"'] & A \arrow[r, "f"] \arrow[d, "\psi"] & A'' \arrow[d, "\psi''"] \\ B' \arrow[r, "g'"] & B \arrow[r, "g"] & B'' & & B' \arrow[r, "g'"] & B \arrow[r, "g"] & B'' \end{tikzcd}$$

iff there are maps $h_1\colon A \to B'$ and $h_2\colon A'' \to B$ such that $\varphi - \psi = g'h_1 + h_2 f$, i.e., the two short complexes are homotopic. Then the resulting category $\boldsymbol{A}^{\to\to}/\equiv$ is denoted by $Ab(\boldsymbol{A})$. $Ab(\boldsymbol{A})$ is abelian ([1]). The functor $I_A\colon \boldsymbol{A} \to Ab(\boldsymbol{A})\colon A \to (0 \to A \to 0)$ is obviously full and faithful. Let now F be an additive functor from $\boldsymbol{A}$ to $\boldsymbol{B}$ with $\boldsymbol{B}$ abelian. Then there is a unique exact functor $F^*\colon Ab(\boldsymbol{A}) \to \boldsymbol{B}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "I_A"] \arrow[dr, "F"'] & Ab(A) \arrow[d, "F^*"] \\ & B \end{tikzcd}$$

commutes up to natural equivalence (cf. [1] Theorem 1.14).

Let now A be the additive category Comod-C.

THEOREM 15. *Let C be a coalgebra. Then*

(1) *There exists an abelian category* Ab (Comod-C) *and a full and faithful embedding*

$$I\colon \text{Comod-}C \longrightarrow Ab\ (\text{Comod-}C)$$

such that every additive functor $F\colon \text{Comod-}C \to B$ *into an abelian category B can be factored through an exact functor* $F^*\colon Ab\ (\text{Comod-}C) \to B$ (up to natural equivalence).

(2) Ab (Comod-C), *the free abelian category over* Comod-C, *is an $AB4$-category.*

(3) *The inclusion functor I preserves products and coproducts.*

(4) *The inclusion functor I preserves equalizers* (coequalizers) *if and only if the equalizers* (coequalizers) *in* Comod-C *are coretractions* (retractions).

(5) Ab (Comod-C) *has sufficiently many projectives and injectives.*

As immediate consequences of this theorem we obtain the following two theorems by applying the special adjoint functor theorem:

THEOREM 16 (Notation as above). *The following statements are equivalent.*

(i) Comod-C *is a coreflective subcategory of* Ab (Comod-C).

(ii) *The inclusion functor* $I\colon \text{Comod-}C \to Ab\ (\text{Comod-}C)$ *preserves epimorphisms.*

(iii) *Every epimorphism in* Comod-C *is a retraction.*

(iv) *Every object in* Comod-C *is projective.*

THEOREM 17 (Notation as above). *The following statements are equivalent:*

(i) *The inclusion* $I\colon \text{Comod-}C \to Ab\ (\text{Comod-}C)$ *is an isomorphism.*

(ii) *Every object in* Comod-C *is injective.*

(iii) *Every monomorphism in* Comod-C *is a coretraction. If* (i)-(iii) *are fulfilled then* Comod-C *is a spectral category.*

REMARK 18. If Comod-C is an abelian category then the statements of the above two theorems are equivalent. But if Comod-C is not abelian then these conditions need not to be equivalent.

Proof of Theorem 15. We have to prove (2), (3), (4) since the other statements were proved in [1].

(2) Let $M'_i \xrightarrow{f'i} M_i \xrightarrow{f'i} M''_i$, $i \in I$, be a family of Ab (Comod-C)-objects. Then

$$\begin{tikzcd}
\amalg M'_i \arrow[r, "\amalg fi'"] & \amalg M_i \arrow[r, "\amalg fi"] & \amalg M''_i \\
M'_i \arrow[u, "m'_i"] \arrow[r, "fi''"] & M_i \arrow[u, "m_i"] \arrow[r, "fi"] & M''_i \arrow[u, "m''_i"']
\end{tikzcd}$$

is the coproduct of these family in Ab (Comod-C) as one easily shows, where m'_i, m_i and m''_i, $i \in I$ are the corresponding coproducts of the objects M'_i, M_i and M''_i in Comod-C. Hence Ab (Comod-C) is cocomplete, i.e., an AB-3 category. In order to show that Ab (Comod-C) is an $AB4$-category we have to show that for any family $\{f_i\colon (M_i) \to (N_i)\}$ of monomorphisms in Ab (Comod-C), the morphism $\amalg f_i$ is also a monomorphism.

LEMMA 19 ([1] Theorem 1.1 or [8] Lemma 6.1).

(1) *The equalizer of*

$$\begin{tikzcd}
M' \arrow[r, "f'"] \arrow[d, "\varphi'"'] & M \arrow[r, "f"] \arrow[d, "\varphi"] & M'' \arrow[d, "\varphi''"] \\
N' \arrow[r, "g'"] & N \arrow[r, "g"] & N''
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
M' \oplus N \arrow[r, "\begin{pmatrix} f' & 0 \\ \varphi' & -1 \end{pmatrix}"] \arrow[d, "{(1, 0)}"] & M \oplus N' \arrow[r, "\begin{pmatrix} \varphi & -g' \\ f & 0 \end{pmatrix}"] \arrow[d, "{(1, 0)}"] & N \oplus M'' \arrow[d, "{(0, 1)}"] \\
M' \arrow[r] & M \arrow[r] & M''
\end{tikzcd}$$

and the coequalizer by

$$\begin{tikzcd}
N' \arrow[r, "g'"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & N \arrow[r, "g"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & N'' \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\
N' \oplus M \arrow[r, "\begin{pmatrix} g' & \varphi \\ 0 & -f \end{pmatrix}"'] & N \oplus M'' \arrow[r, "\begin{pmatrix} g & \varphi'' \\ 0 & -1 \end{pmatrix}"'] & N'' \oplus M'' .
\end{tikzcd}$$

Since Ab (Comod-C) *is an abelian category we obtain at once the following criterium.*

LEMMA 20. *Let*

$$\begin{tikzcd}[ampersand replacement=\&] \& M' \arrow[r, "f'"] \arrow[d, "\varphi'"] \& M \arrow[r, "f"] \arrow[d, "\varphi"] \& M'' \arrow[d, "\varphi''"] \\ (\varphi) = \& N' \arrow[r, "g'"] \& N \arrow[r, "g"] \& N'' \end{tikzcd}$$

be a morphism in Ab (Comod-C). *Then*

(1) (φ) *is a monomorphism if and only if there are morphisms*

$$\psi'\colon N' \longrightarrow M', \ q\colon M \longrightarrow M'$$
$$q''\colon M'' \longrightarrow M \text{ and } \psi\colon N \longrightarrow M \text{ such that}$$
$$f'q + \psi\cdot\varphi + q''\cdot f = \mathrm{id}_M$$

and

$$f'\cdot\psi' + \psi\cdot g' = 0 .$$

(2) (φ) *is an epimorphism if and only if there are morphisms*

$$p\colon N \longrightarrow N', \ p''\colon N'' \longrightarrow N ,$$
$$\delta\colon N \longrightarrow M \text{ and } \delta\colon N'' \longrightarrow M'' \text{ such that}$$
$$g'\cdot p + p''g + \varphi\cdot\delta = \mathrm{id}_N$$
$$\delta''g + f\cdot\delta = 0 .$$

The construction of coproducts in Ab (Comod-C) and Lemma (20) 1 show immediately that Ab (Comod-C) is an $AB4$-category.

(3) Trivial.

(4) Let $f\colon M \to N$ an equalizer in Comod-C and assume that I preserves this equalizer

Consider the following diagram

$$\begin{tikzcd}[ampersand replacement=\&] \& 0 \arrow[r] \arrow[d] \& M \arrow[r] \arrow[d, "f"] \& 0 \arrow[d] \\ If = (f) \& 0 \arrow[r] \& N \arrow[r] \& 0 . \end{tikzcd}$$

Then (f) is a monomorphism in Ab (Comod-C) if and only if there exists a morphism $g\colon N \to M$ such that $g\cdot f = \mathrm{id}_M$, i.e., if f is a coretraction (Lemma 20.1). In the same vein one shows by applying Lemma 20.2 that f is an epimorphism if and only if f is a retraction Comod-C. This completes our proof.

REMARK 21. (1) *Ab (Comod-C) is an $AB4^*$-Category.* Let C be a coalgebra. Then Comod-C is complete by Corollary 26. Now in the same vein as above one shows that Ab (Comod-C) has products which are the pointwise ones. Hence Ab (Comod-C) is an $AB3^*$-category. From the construction of products and the characterization of epimorphisms by Lemma 20.2 we obtain that Ab (Comod-C) is an $AB4^*$-category.

(2) *Ab (Comod-C) is, in general, not a Grothendieck category.* Take $\boldsymbol{Z}$ with the trivial coalgebra structure. Then Comod-$\boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}$-Mod, the category of abelian groups. Assume Ab (Comod-$\boldsymbol{Z}$) $= Ab$ ($\boldsymbol{Z}$-Mod) is a Grothendieck category. Since Ab ($\boldsymbol{Z}$-Mod) is an $AB3^*$-category by 21 1, Ab ($\boldsymbol{Z}$-Mod) is a C_2-category (Mitchell [12]), i.e., for any set (M_i) of objects in Ab ($\boldsymbol{Z}$-Mod) the canonical morphism

$$m\colon \coprod M_i \longrightarrow \prod M_i$$

is a monomorphism. Take now $M_n = \boldsymbol{Z}$ for $n \in N$. Then the canonical morphism

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & \coprod_N \boldsymbol{Z} = \boldsymbol{Z}^{(N)} \arrow[r] \arrow[d, "m"] & 0 \arrow[d] \\ I(m) = & 0 \arrow[r] & \prod_N \boldsymbol{Z} - \boldsymbol{Z}^N \arrow[r] & 0 \end{tikzcd}$$

is the image of the canonical morphism $m\colon \boldsymbol{Z}^{(N)} \longrightarrow \boldsymbol{Z}^N$. Then $I(m)$ is a monomorphism in Ab ($\boldsymbol{Z}$-Mod) if and only if the canonical morphism $m\colon \boldsymbol{Z}^{(N)} \longrightarrow \boldsymbol{Z}^N$ is a coretraction. Consider now the canonical projection $p\colon \boldsymbol{Z}^N \longrightarrow \boldsymbol{Z}^N/\boldsymbol{Z}^{(N)}$ and the element $\bar{x} = (2^n; n \in N) \in \boldsymbol{Z}^N$. Then the image $p(\bar{x})$ is obviously divisible by every power of 2. Since an element $(x_i; i \in I)$ in $\boldsymbol{Z}^I$ is divisible if and only if all components x are divisible in $\boldsymbol{Z}$ we obtain that $\boldsymbol{Z}_N|\boldsymbol{Z}^{(N)}$ cannot be embedded in a product $\boldsymbol{Z}^I$. Hence the monomorphism $m\colon 0 \longrightarrow \boldsymbol{Z}^{(N)} \longrightarrow \boldsymbol{Z}^N$ is not split, i.e., no coretraction and therefore $I(f)$ is no monomorphism in Ab ($\boldsymbol{Z}$-Mod). Hence Ab (Comod-$\boldsymbol{Z}$) is not a Grothendieck category.

Next I will prove that Comod-C has a generator where C is an arbitrary coalgebra. The existence of a generator in Comod-C where C is flat was proved by Saavedra [15] 2.07. But his proof cannot be generalized. The following proof uses Barr's results in [3[and is in fact an imitation of his proof of the existence of a set of generators in the category of coalgebras over a commutative ring.

A submodule $U \subset M$ of a module M is called a *pure submodule* of M provided that for any module N $U \otimes N \rightarrow M \otimes N$ is a monomorphism.

PROPOSITION 22 (Barr [3] 1.3). *Given $U \subset M$ there is an $U^* \subset M$*

such that $U \subset U^*$ *such that* U^* *is a pure submodule of* M, *and such that*

$$\text{card}\,(U^*) \leqq \max\,(\text{card}\,(U),\ \text{card}\,(k),\ \chi_0)\ .^2$$

THEOREM 23. *Let* $\langle M, \chi\rangle$ *be a* C*-comodule,* U *a submodule of* M. *Then there is a subcomodule* $M' \subset M$ *such that* $U \subset M'$ *and*

$$\text{card}\,(M') \leqq \max\,(\text{card}\,U,\ \text{card}\,k,\ \aleph_0)\ .$$

Proof. Let $\langle M, \chi\rangle$ be a C-comodule. A k-submodule U of M is called χ-invariant if $\chi(U) \subset i \otimes C\ (U \otimes C)$ where $i\colon U \rightarrow M$ is the inclusions. Let U be a submodule of M. For each $u \in U$ choose a representation

$$\chi(u) = \sum_{i=1}^{n} m_i \otimes C_i\ .$$

Let U' be the submodule generated by all m_i and the elements of U. Then $U \subset U' \subset M$, $\chi(U) = \sum_{i=1}^{n} m_i \otimes C_i \in i \otimes C(U' \otimes C)$ and $\text{card}\,(U') \leqq \max\,(\text{card}\,U,\ \text{card}\,k,\ \chi_0)$.

Now iterate the above process in order to get a sequence

$$U \subset U' \subset U'' \subset \cdots \subset U^{(n)} \subset \cdots$$

such that $\chi(U^{(n)}) \subset i \otimes C(U^{(n+1)} \otimes C)$. Define $\hat{U} = \bigcup_{n \in N} U^{(n)}$. Then $\hat{U}$ is a submodule of M such that $U \subset \hat{U}$ such that $\hat{U}$ is χ-invariant and such that $\text{card}\,(\hat{U}) \leqq \max\,(\text{card}\,U,\ \text{card}\,k,\ \chi_0)$. Next we define the following sequence of submodule of M

$$U_n = U_{n-1}^* \quad \text{when } n \text{ is odd}$$

and

$$U_n = \hat{U}_{n-1} \quad \text{when } n \text{ is even}\ ,$$

where U_{n-1}^* is "the" pure submodule of M containing U_{n-1} (→ Proposition 22). Then let $M' = \bigcup U_n$. Then $M' \subset M$ is a pure submodule of M which is χ-invariant. Hence $\chi(M') \subset M' \otimes C$ and $\langle M', \chi\rangle$ is a subcomodule of $\langle M, \chi\rangle$. The cardinality conclusion is obvious.

THEOREM 24. *The* C*-comodule whose cardinality* $\leqq \max(\text{card}\,k, \aleph_0)$ *generate the category* Comod-C. *In particular* Comod-C *has a generator.*

Proof. Let $f, g\colon \langle M, \chi_M\rangle \rightrightarrows \langle N, \chi_N\rangle$ be two different comodule homomorphisms. Then there exists an element $m \in M$ such that

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[2] card (X) means the cardinality of the set X.

$f(m) \neq g(m)$. Then by Theorem 22 there exists a subcomodule M' containing the submodule generated by m;
$\langle m\rangle \subset M' \subset M$. Furthermore $\operatorname{card}\langle m\rangle \leqq \operatorname{card} k$. Hence $\operatorname{card} M' \leqq \max(\operatorname{card} k, \chi_0)$ and $f_i \neq g_i$: $\langle M', \chi_{M'}\rangle \xrightarrow{i} \langle M, \chi_M\rangle \overset{f}{\underset{g}{\rightrightarrows}} \langle N, \chi_N\rangle$.

EXAMPLE 25. Let $C = \boldsymbol{Z} \oplus \boldsymbol{Q}/\boldsymbol{Z}$. Then the "set" of denumerable $\boldsymbol{Z} \oplus \boldsymbol{Q}/\boldsymbol{Z}$-comodules generates the category Comod-$\boldsymbol{ZQ}/\boldsymbol{Z}$.

Since Comod-C is cocomplete, cowellpowered and has a generator we obtain by applying the special functor theorem [cf. [13] p. 114 Corollary].

COROLLARY 26. *The category* Comod-C *is complete. Moreover* Comod-C *is locally presentable in the sense of Gabriel-Ulmer.*[3]

This Corollary shows only the existence of arbitrary limits in Comod-C but gives us no explicit description. Our next step will be therefore to describe explicitly the limits. This was not known even in the case where k is a field. We apply Linton's techniques of constructing colimits in an Eilenberg-Moore category over Sets (cf. [14] Chap. 21)

Construction of limits in Comod-C 27. Let $\boldsymbol{I}$ be a small category and $D\colon \boldsymbol{I} \longrightarrow$ Comod-C be a diagram. Let $(\lim UD, \varphi)$ be the limit of UD in k-Mod and $(\lim(-\otimes C\cdot U\cdot D, \psi)$ the limit of $-\otimes CU\cdot D$ in k-Mod. If $\boldsymbol{I}$ is void then $\lim D$ is the zero comodule. Now let $\boldsymbol{I}$ be nonvoid. Let $\eta\colon Id_{\text{Comod-}C} \longrightarrow -\otimes C\cdot U$ be the functorial morphism defined by

$$\chi = \eta(\langle M, \chi\rangle)\colon \langle M, \chi\rangle \longrightarrow \langle M\otimes C, M\otimes \Delta\rangle$$

$$\begin{tikzcd}
M \arrow[r, "\chi"] \arrow[d, "\chi"'] & M\otimes C \arrow[d, "M\otimes\Delta"] \\
M\otimes C \arrow[r, "M\otimes\chi"] & M\otimes C\otimes C
\end{tikzcd}$$

Then there is exactly one k-morphism

$$\eta^*\colon \lim(UD) \longrightarrow \lim(-\otimes C\cdot UD)$$

such that the following diagram commutes:

$$\begin{tikzcd}
UD \arrow[r, "\varphi"] \arrow[d, "U*\eta*D"'] \arrow[dr, phantom, "="] & \operatorname{Diag}(\lim UD) \arrow[d, "\operatorname{Diag}(\eta^*)"] \\
-\otimes C\cdot U\cdot D \arrow[r, "\psi"] & \operatorname{Diag}(\lim -\otimes CUD)
\end{tikzcd}$$

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[3] The set of generators in Comod-C is $\aleph_1$-presentable-(Ulmer).

where Diag is the diagonal functor.

Let $\lim UD = M$ and $\lim - \otimes C\cdot U\cdot D = N$. Then there exists exactly one k-morphism $\varphi^*\colon M \otimes C \to N$ such that $- \otimes C * \varphi = \psi\cdot\operatorname{Diag}(\varphi^*)$. We claim that η^* is a monomorphism. Consider

$$\begin{tikzcd}
\operatorname{Diag}(X) \arrow[r, shift left, "\operatorname{Diag}(f)"] \arrow[r, shift right, "\operatorname{Diag}(g)"'] & \operatorname{Diag}(M) \arrow[r, "\operatorname{Diag}(\eta^*)"] \arrow[d, "\varphi"] \arrow[dr, phantom, "="] & \operatorname{Diag}(N) \arrow[d, "\psi"] \\
 & UD \arrow[r, "U*\eta*D"] & - \oplus C\cdot U\cdot D
\end{tikzcd}$$

where $f, g\colon X \to M$ are k-morphisms with $\eta^* \cdot f = \eta\cdot g$. Since $(U, - \otimes C)$ is an adjoint functor pair $U*\eta$ is a coretraction and hence also $U*\eta*D$. Thus we obtain $\varphi \operatorname{Diag}(f) = \varphi \operatorname{Diag}(g)$ and hence $f = g$ since φ is a universal morphism.

Consider now the cofree comodules $\langle M \otimes C, M \otimes \varDelta\rangle$ and $\langle N \otimes C, N \otimes \varDelta\rangle$ and the comodule homomorphisms

$$\varphi^* \otimes C\cdot M \otimes \varDelta,\ \eta^* \otimes C\colon M \otimes C \longrightarrow N \otimes C\,.$$

Let $\langle K, \chi_K\rangle \xrightarrow{m} \langle M \otimes C, M \otimes \varDelta\rangle \overset{\eta^*\otimes C}{\underset{\varphi^*\otimes C\cdot M\otimes\varDelta}{\rightrightarrows}} (N \otimes C, N \otimes \varDelta\rangle$ be an equalizer of $(\eta^* \otimes C, \varphi^* \otimes M \otimes \varDelta)$. Then $\langle K, \chi_K\rangle$ is the limit of D in Comod-C.

This is now shown in several steps (cf. [16] 21. 2. 10).

EXAMPLE 28. Let C be a flat coalgebra. Then the finite limits and in particular the equalizers in Comod-C are formed in k-Mod. We want now to compute the products in Comod-C. Let $\langle M_i, \chi_i\rangle$; $i \in I$, be a family of C-comodules. Denote by ΠM_i the product of the underlying k-modules and by $\Pi M_i \otimes C$ the product of the k-modules $M_i \otimes C$. Then we obtain two canonical morphisms η^* and φ^* defined by the universal property of $\Pi M_i \otimes C$:

$$\begin{tikzcd}
M_i \otimes C \arrow[d, "\chi_i"] \arrow[dr, phantom, "="] & \Pi M_i \otimes C \arrow[l, "\mathrm{can}"'] \arrow[d, "\Pi\chi_i=\eta^*"] \\
M_i & \Pi M_i' \arrow[l, "\mathrm{can}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
M_i \otimes C \arrow[dr, phantom, "="] & \Pi M_i \otimes C \arrow[l, "\mathrm{can}"'] \\
M_i \otimes C \arrow[u] & (\Pi M_i) \otimes C \arrow[l, "\mathrm{can} \otimes C"'] \arrow[u, "\varphi^*"']
\end{tikzcd}$$

with $\varphi^*((m_i) \otimes c) = (m_i \otimes c)$ and $\eta^*(m_i) = (\chi_i(m_i))$. Then the equalizer of

$$\begin{tikzcd}(\Pi M_i) \otimes C \arrow[r, shift left, "\eta^*\otimes C"] \arrow[r, shift right, "\varphi^*\otimes C\cdot(\Pi M_i)\otimes\Delta"'] & (\Pi M_i \otimes C) \otimes C\end{tikzcd}$$

is the product of the family $\langle M_i, \chi_i\rangle$ in Comod-C, i.e.,

$$\prod_{\text{Comod-}C} \langle M_i, \chi_i\rangle = \Big\{\sum_{\text{finite}} \bar{m}_k \otimes C_k \in (\Pi M_i) \otimes C;\ \sum_{\text{finite}} (\chi_i(m_i^k)) \otimes C_k$$

$$= \sum_{\text{finite}} \sum_{(C_k)} (m_i^k \otimes C_{k(1)}) \otimes C_{k(2)}\Big\}$$

where $\bar{m}_k = (m_i^k)i \in I$ and $\Delta C_k = \sum_{(C_k)} C_{k(1)} \otimes C_{k(2)}$ with the comodule structure induced by the comodule structure $(\Pi M_i) \otimes \Delta$ and $(\Pi M_i \otimes \varepsilon(\Pi M_i) \otimes C$. The projections p_i are given by the following assignments.

$$p_i\colon \prod_{\text{Cmod-}C} \langle M_i, \chi_i\rangle \longrightarrow \langle M_i, \chi_i\rangle \sum_{\text{finite}} (m_i^k) \otimes C_k \longmapsto \varepsilon(C_k)\cdot m_i^k\ .$$

Let us now consider the functorial morphism (functorial in C)

$$\lambda\colon k\text{-Mod}\,(M, N \otimes C) \longrightarrow k\text{-Mod}\,(C^* \otimes M, N)$$

defined by $\lambda(f)(c^* \otimes m) = (1 \otimes c^*)f(m)$ where $C^* = k\text{-Mod}\,(\mathrm{C}, k)$. If C is a coalgebra then C^* is a k-algebra with the multiplication

$$f{*}f'(c) = \sum_{(c)} f(c_{(1)})\cdot f'(c_{(2)})$$

and unit $e(c) = \varepsilon(c)$. (cf. [14]) Let C be a coalgebra and $\langle M, \chi\colon M \to M \otimes C\rangle$ a comodule. Then M is a C^*-left module with multiplication: $\lambda(\chi)\colon C^* \otimes M \to M$. The assignments

$$\begin{aligned}\lambda\colon \text{Comod-}C &\longrightarrow C^*\text{-Mod}\\ \langle M, \chi\rangle &\longmapsto \langle M, \lambda(\chi)\rangle\\ f &\longmapsto f\end{aligned}$$

define a functor (cf. [14]).

THEOREM 29. $\lambda\colon \text{Comod-}C \to C^*\text{-Mod}$ *is comonadic. In particular* λ *has a right adjoint.*

Proof. Since Comod-C is cocomplete, cowellpowered and has a generator, λ has a right-adjoint if and only if λ preserves colimits (special adjoint functor theorem). Let

$$\langle M_i, \chi_i\rangle \xrightarrow{m_i} \langle \operatorname{colim} M_i, \chi\rangle$$

be a colimit diagram in Comod-C. Then $\lambda(\chi)\colon C^* \otimes \operatorname{colim} M_i \to \operatorname{colim} M_i$ is a colimit of $\langle M_i, \lambda(\chi_i)\rangle$, $i \in I$, as one easily computes.

Hence λ preserves colimits and thus has a right adjoint. Next I'll show that λ creates equalizer of λ-contractible pairs. Let $f, g\colon \langle A, \chi_A\rangle \rightrightarrows \langle B, \chi_B\rangle$ be a pair of λ-contractible Comod-C morphisms and $m\colon K \to A$ be an equalizer of $f, g\colon \langle A, \lambda(\chi_A)\rangle \rightrightarrows \langle B, \lambda(\chi_B)\rangle$ in C^*-Mod. Then there exist C^*-module homomorphisms $h\colon \langle B, \lambda(\chi_B)\rangle \to \langle A, \lambda(\chi_A)\rangle$ and $k\colon \langle A, \lambda(\chi_A)\rangle \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "1_A"] \arrow[dr, "h"'] \arrow[ddd, leftrightarrow, "m"'] & & A \arrow[dl, "f"] \arrow[ddd, leftrightarrow, "m"] \\
& B & \\
& A \arrow[u, "g"'] & \\
K \arrow[ur, "k"] \arrow[rr, "1_K"'] & & K \arrow[ul, "m"']
\end{tikzcd}$$

Since functors preserve equalizers of contractible pairs, $K \xrightarrow{m} A \overset{f}{\underset{g}{\rightrightarrows}} B$ is an equalizer of the contractible pair (f, g) in k-Mod. Since $U\colon$ Comod-$C \to k$-Mod is comonadic, K carries a comodule structure χ_K such that $\langle K, \chi_K\rangle \xrightarrow{m} \langle A, \chi_A\rangle \overset{f}{\underset{g}{\rightrightarrows}} \langle B, \chi_B\rangle$ is an equalizer diagram in Comod-C. Hence λ creates equalizers of λ-contractible pairs and hence is comonadic.

REMARKS 30. (1) The fact that λ creates equalizers of λ-contractible pairs follows also from the following:

LEMMA. *Let $f, g\langle A, \chi_A\rangle \rightrightarrows \langle B, \chi_B\rangle$ be a pair of comodule homomorphisms and $K \xrightarrow{m} A \overset{f}{\underset{g}{\rightrightarrows}} B$ the equalizer of f, g in* k-Mod. *If m is a coretraction in* k-Mod *then K carries a comodule structure χ_K such that*

$$\langle K, \chi_K\rangle \xrightarrow{m} \langle A, \chi_A\rangle \overset{f}{\underset{g}{\rightrightarrows}} \langle B, \chi_B\rangle$$

is an equalizer diagram in Comod-C.

Let m be an equalizer of a λ-contractible pair f, g. Then m is a coretraction in k-Mod and hence an equalizer in Comod-C, i.e., λ creates equalizers of λ-contractible pairs.

(2) The fact that λ is comonadic follows immediately from the following Dubuc-triangle

$$\begin{tikzcd}
\text{Comod-}C \arrow[rr, "\lambda"] \arrow[dr, "U"'] & \arrow[d, phantom, "="] & C^*\text{-Mod} \arrow[dl, "V"] \\
& k\text{-Mod} &
\end{tikzcd}$$

where U and V are the underlying functors. Since U and V are comonadic and Comod-C has equalizer, λ is also comonadic (cf. [20] Proposition 6.11).

(3) If C is finite ($\equiv$ finitely generated and projective) then λ: Comod-$C \longrightarrow C^*$-Mod is an isomorphism of categories (cf. [14]).

The next proposition solves the problem of the existence of free comodules i.e. answers the following question: For which coalgebras C does the forgetful functor V: Comod-$C \longrightarrow$ Sets have a left-adjoint?

PROPOSITION 31. *The following statements are equivalent*:

(i) *The forgetful functor* V: Comod-$C \longrightarrow$ Sets *has a left-adjoint.*

(ii) *C is finite* i.e. *finitely generated and projective.*

(iii) $- \otimes C$: k-Mod $\longrightarrow$ k-Mod *preserves limits.*

(iv) λ: Comod-$C \longrightarrow C^*$-Mod *has a left-adjoint.*

(v) U: Comod-$C \longrightarrow k$-Mod *preserves limits.*

If one of these conditions is fulfilled then λ: Comod-$C \longrightarrow C^*$-Mod *is an isomorphism.*

Proof. The equivalences (i) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv) $\leftrightarrow$ (v) are categorical routine. The equivalence (iii) $\leftrightarrow$ (ii) follows from the well-known fact that $- \otimes C$ preserves limits if and only if C is finitely presented and flat or equivalently if C is finitely generated and prejective. If one of these conditions is fulfilled then λ is an isomorphism by (30.3).

Description of the free C-comodules 32. Let C be a finitely generated and projective coalgebra. The above proposition gives us the following explicit description of the free C-comodules: Let X be an arbitrary set. Then the free C-comodule FX generated by X is given by $FX \cong \bigoplus_X C^*$ where C^* has the "canonical" C-comodule structure.

COROLLARY 33. *Notation as above. The functor* λ: Comod-$C \longrightarrow$ C^*-Mod *is an isomorphism if and only if C is finitely generated and projective.*

Next we consider factorizations in Comod-C. Let us first recall some of the basic notions and propositions (cf. [20]). Let A be a category. For two $\boldsymbol{A}$-morphisms $e: A \longrightarrow B$ and $m: C \longrightarrow D$ we write $e \downarrow m$ if every commutative diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d] & B \arrow[d] \arrow[dl, "w"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

can be made commutative by a unique morphism $w: B \longrightarrow C$. Let P be any class of $\boldsymbol{A}$-morphisms. Then $p^\uparrow$ resp. $p^\downarrow$ shall denote the following classes of $\boldsymbol{A}$-morphisms.

$$p^\uparrow = \{e; e \downarrow m \quad \text{for all } m \in P\}$$
$$p^\downarrow = \{m; e \downarrow m \text{ for all } e \in P\} .$$

A pair (E, M) of classes E and M of $\boldsymbol{A}$-morphisms is a *prefactrization* in $\boldsymbol{A}$ if $E = M^\uparrow$ and $M = E^\downarrow$. A prefactorization (E, M) is called a *factorization in* $\boldsymbol{A}$ if every morphism f in $\boldsymbol{A}$ is of the form $f = m \cdot e$ with $m \in M$ and $e \in E$. A factorization (E, M) is proper if every $e \in E$ is an epimorphism and every $m \in M$ is a monomorphism. Hence a proper factorization on $\boldsymbol{A}$ is the same thing as a bicategorical structure in the sense of Isbell. We say that a category $\boldsymbol{A}$ has a M-factorization if $\boldsymbol{A}$ has a $(M^\uparrow, M)$-factorization. Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be categories with factrizations M_K resp. M_L. A functor $F: \boldsymbol{K} \longrightarrow \boldsymbol{L}$ is said top reserve M_K-factorizations if $F(M_K) \subset M_L$ and $F(M_K^\uparrow) \subset M_L^\uparrow$. F is said to reflect M_L-factorizations if $F^{-1}(M_L)$ M_K and $F^{-1}(M_L^\uparrow) \subset M_K^\uparrow$. Let $H_K \subset \operatorname{Mor} \boldsymbol{K}$ with $\operatorname{Iso}(\boldsymbol{K}) \subset H_X$ and H_K $\operatorname{Iso}(\boldsymbol{K}) \subset H_K$. A functor $F: \boldsymbol{K} \longrightarrow \boldsymbol{L}$ is said to *create* H_K*-factorizations from* M_L*-factorizations* if for all $f \in \operatorname{Mor} \boldsymbol{K}$ with

$$Ff = m_L e_L,\ m_L \in M_L,\ e_L \in M_L^\uparrow$$

there is a unique factorization $f = m_K \cdot e_K$ in $\boldsymbol{K}$ with $F_{mK} = m_L$, $Fe_K = e_L$, $m_K \in H_K$, $e_K \in {}_K^\uparrow$.

PROPOSITION 34. *Let* $\boldsymbol{K}$ *be a cocomplete, cowellpowered category. Then* $\boldsymbol{K}$ *has an* (epi, extremal mono)*-factorization* i.e., *a factorization* (E, M) *where* E *is the class of all epimorphisms and* M *is the class of all extremal monomorphisms* (Isbell-Kennison).

Hence the category Comod-C has at least one proper factorization.

PROPOSITION 35. *Let* (E, M) *be a proper factorization in* Comod-C. *Then the following statement are equivalent.*

(i) *The underlying functor* U: Comod-$C \longrightarrow k$-Mod *preserves the* *factorization.*

(ii) *U is exact.*

(iii) *C is flat.*

Proof. Since (ii) and (iii) are equivalent by Proposition 8 and since the implication (iii) → (i) is trivial we have only to prove (i) → (iii). Let E_k resp. M_k be the class of all epimorphisms resp. monomorphisms in k-Mod. Since U preserves the factorization and U reflects isomorphisms we obtain that $E = U^{-1}(E_k)$ and $M = U^{-1}(M_k)$. Since $U(E) \subset E_k$ and $- \otimes C$ is right adjoint to U we get $(M_k) \otimes C \subset M$. Hence we get for the functor $- \otimes C$: k-Mod → k-Mod

$$(M_k) \otimes C = U(- \otimes C)(M_k) \subset (M) \subset M_k$$

i.e., $- \otimes C$ preserves monomorphisms.

COROLLARY 36. *The underlying functor* U: Comod-C → k-Mod *creates factorizations from E_k-factorizations in* k-Mod *if and only if C is flat.*

Proposition 35 shows that, if C is not flat, then an arbitrary C-comodule homomorphism can not be factorized through a surjective comodule homomorphism and an injective comodule homomorphism. In particular the canonical (epi-mono)-factorization of a comodule homomorphism in k-Mod cannot be lifted to a factorization in Comod-C. In the sequel (E, M) shall always denote the proper factorization (epi, extremal mono) on Comod-C. Words as epimorphism, monomorphism, generator, wellpowered ⋯ are used in a sense relative to (E, M).

PROPOSITION 37. Comod-C *is wellpowered relative to the factorization* (epi, extremal mono).[4]

Proof. In the same vein as the proof for Proposition 10.6.3 in [16].

For the rest of this paper we will use the property that the category k-Mod is a symmetrical monoidal closed category with respect to the tensor product, and that Comod-C is an enriched category over k-Mod. In the following we will study the left adjoints of the k-Mod-representable functors called tensors and cotensors. They provide a characterisation of certain constructions which is not available in an ordinary set based approach. Cotensors will play an important role in duality theory (i.e. Gelfand theory)

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[4] Comod-C is even wellpowered with respect to all monos.

as it will be shown in part II of the present work. We use the language in [6].

Comod-C is a k-Mod-category. The internal Hom-functor $[\ ,\ ]$: Comod-$C^{\mathrm{op}} \times$ Comod-$C \rightarrow k$-Mod is gived by $[M, N] =$ Comod-$C(M, N)$. The pair of adjoint functors Comod-C $\rightleftarrows k$-Mod is a pair of k-Mod-functors. In the sequel we call k-Mod-functors k-linear functors.

PROPOSITION 38. *The category* Comod-C *is tensored i.e. for every k-module M and every C-comodule X the functor* Comod-$C \rightarrow$ k-Mod: $Y \mapsto k$-Mod $(M,$ Comod-$C(X, Y))$ *is representable over* k-Mod.

Proof. Let $M \in k$-Mod and $X \in$ Comod-C. The $M \otimes X$ is a C-comodule. The rest follows from the canonical k-linear isomorphism

$$\text{Comod-}C(M \otimes X, Y) \cong k\text{-Mod}\,(M, \text{Comod-}C(X, Y)).$$

COROLLARY 39. *The cofree k-linear functor* $- \otimes C$: k-Mod $\rightarrow$ Comod-C *has a k-linear right adjoint functor represented by the k-linear functor* Comod-$C(C, -)$.

PROPOSITION 40. *The category* Comod-C *is cotensored* i.e. *for every $M \in k$-Mod and $X \in$* Comod-C *the functor* Comod-$C^{\mathrm{op}} \rightarrow k$-Mod: $Y \mapsto k$-Mod $(M,$ Comod $(Y, X))$ *is representable.*

Proof. Since Comod-C is a tensored category Comod-C is cotensored if and only if for every k-module M the k-linear functor F_M: $M \otimes -$: Comod-$C \rightarrow$ Comod-C has a k-linear right adjoint. Let $N \otimes X$ be a tensor with $N \in k$-Mod and $X \in$ Comod-C as above. Then $F_M(N \otimes X) = M \otimes (N \otimes X) \cong N \otimes (M \otimes X) \cong N \otimes F_M(X)$. Hence F_M is a tensor preserving functor in the sense of [6]. Since F_M preserves colimits, F_M has a right adjoint by the Special Adjoint Functor Theorem. Since F_M preserves tensors the right adjoint $\overline{\text{Comod-}C}\,(M, -)$ is a k-linear functor and the representation Comod-C $(X, \overline{\text{Comod-}C}\,(M, X)) \cong$ Comod-$C(M \otimes X, Y) \cong k$-Mod $(M,$ Comod $(X, Y))$ is k-linear.

COROLLARY 41. Comod-C *is* k-Mod-*complete and* k-Mod-*cocomplete.*

Let f: $C \rightarrow C'$ be a coalgebra morphism. Then f induceds a functor f^*: Comod-$C \rightarrow$ Comod-C' by the assignment $(M, \chi_M) \mapsto$ $(M, 1 \otimes f\chi_M)$. Then f^* is obviously a k-linear functor. By [15] 21.2.1 the mapping $f \mapsto f^*$ induces a bijection between Coalg (C, C') and the "set" of all functors φ: Comod-C $\rightarrow$ Comod-C' with $U_C = U_{C'}\varphi$.

PROPOSITION 42. *Let f: $C \rightarrow C'$ be a coalgebra morphism. Then*

(1) *f^* preserves tensors.*
(2) *f^* has a k-linear right adjoint f_*.*

Proof. The assertion 1 is trivial. Since f^* preserves colimits it has a right adjoint by the Special Adjoint Functor Theorem. Since f^* preserves tensors the right adjoint is k-linear.

Description of the functor f_* 43. Let M be a C-right comodule and N a C-left comodule. The tensor coproduct of M and N under C denoted by $M \otimes^C N$ is given by the following equalizer digram in k-Mod.

$$\begin{tikzcd} M \otimes^C N \arrow[r] & M \otimes N \arrow[r, shift left, "\chi_M \otimes M"] \arrow[r, shift right, "M \otimes \chi_N"'] & M \otimes C \otimes N \end{tikzcd}$$

Then if $f\colon C \to C'$ is a coalgebra morphism between flat coalgebras C and C' the functor $f_*\colon \text{Comod-}C' \to \text{Comod-}C$ is given by the following assignment $f_*(M, \chi_M) = (M \otimes^C C, 1_M \otimes^C \Delta)$.

Final Observation 44. In the same vein as I studied the category of comodules for a fixed coalgebra one can study the category Comod of all comodules i.e. pairs $((M, \chi_M), C)$ where (M, χ_M) is a comodule over C. One obtains similar results. The starting point for the study of this category is the following theorem

THEOREM 45. *The underlying functor*

$$U\colon \text{Comod} \longrightarrow k\text{-Mod} \times k\text{-Coalg}\colon ((M, \chi_M), C) \longmapsto (M, C)$$

is comonadic.

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