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ABELIAN AND NILPOTENT SUBGROUPS OF MAXIMAL ORDER OF GROUPS OF ODD ORDER

ZVI ARAD

Denote the *maximum* of the orders of all *nilpotent* subgroups A of class at most c, of a finite group G, by $d_c(G)$. Let $A_c(G)$ be the set of all nilpotent subgroups of class at most c and having order $d_c(G)$ in G. Let $A_\infty(G)$ denote the set of all nilpotent subgroups of maximal order of a group G.

The aim of this paper is to investigate the set $A_\infty(G)$ of groups G of odd order and the structure of the groups G with the property $A_2(G) \subseteq A_\infty(G)$. Theorem 1 gives an expression for the number of elements in $A_\infty(G)$. Theorem 2 gives criteria for the nilpotency of groups of odd order.

In this paper G is a finite group, and π is a set of primes. If G is of odd order, then G is solvable **[6]**.

1. Introduction. Denote the *maximum* of the *orders* of all *nilpotent* subgroups A of class at most c, of a finite group G, by $d_c(G)$. Let $A_c(G)$ be the set of all nilpotent subgroups of class at most c and having order $d_c(G)$ in G. Then $J_c(G)$ is the subgroup of G generated by $A_c(G)$. In particular, $J_1(G) = J(G)$ is the *Thompson subgroup* of G. Moreover, $A_\infty(G)$ is the set of all nilpotnet subgroups of maximal order of a group G. Here $J_\infty(G)$ is the subgroup of G generated by the elements of $A_\infty(G)$.

In this paper G is a finite group, and π is a set of primes. If G is of odd order, then G is solvable **[6]**.

The aim of this paper is to investigate the set $A_\infty(G)$ for groups G of odd order and the structure of the groups G with the property $A_2(G) \subseteq A_\infty(G)$.

We shall give, in Theorem 1, an expression for the number of elements in $A_\infty(G)$. In Theorem 2 we shall state criteria for the nilpotency of groups of odd order.

For groups G with the property $A_2(G) \subseteq A_\infty(G)$, we have the following:

THEOREM 3. *Let G be a π-solvable group with an S_π-subgroup K of G. Assume that $O_{\pi'}(G) = 1$ and that $A \in A_2(K) \cap A_\infty(K) \neq \varnothing$, then*

(i) *If either* $2, 3 \notin \pi$ *or* $O_2(A)$ *is Abelian, then* $O_p(A) = O_p(G) = O_p(K)$, *for every* $p \in \pi$.

(ii) *If* $F(G)$ *is odd and if* G *has an Abelian* S_2*-subgroup, then* $J_\infty(K) = J_\infty(G) = F(G) = F(K) = A$.

Three corollaries of Theorem 3 and further information can be found in Chapter 2.

Our notation is standard and is taken mainly from **[8]**. In particular, $\pi(G)$ will designate the set of primes dividing $|G|$ and G_π denotes an S_π-subgroup of G. For the definitions of *Sylow system, system normalizers* and *Carter subgroups* of a group G see **[11]**, Definition 11.1 p. 726 and Definition 12.1 p. 736.

2. Statements and proofs of the main theorems. The next result is needed for the proofs of the main Theorems.

PROPOSITION 1. *Suppose* G *is a group. Assume that* A *normalizes a nilpotent subgroup* B *of* G, *and assume that at least one of the following conditions holds*:

(i) $A \in A_1(G)$, *and* B *is Abelian* (**[3]**, Proposition 1).

(ii) $A \in A_1(G)$, $|A|$ *is odd, and an* S_2*-subgroup of* B *is Abelian* (**[3]**, Proposition 1).

(iii) $A \in A_2(G)$ (**[7]**).

(iv) $A \in A_c(G)$, $c \geqq 2$, $|B|$ *is odd, and* **an** S_2*-subgroup of* A *is Abelian* (**[4]**).

(v) $A \in A_\infty(G)$, $|B|$ *is odd and an* S_2*-subgroup of* A *is Abelian* (**[4]**).

Then AB *is nilpotent.*

Define $A^p_\infty(G)$ to be the set $\{A_p/A \in A_\infty(G)\}$ of distinct p-subgroups of a group G, where p is a prime.

THEOREM 1. *Let* G *be a group of odd order. Then*

(i) $|(A_\infty(G)| = [G: N_G(A)] = \Pi_{p \in \pi(A)} |A^p_\infty(G)|$, *where* $A \in A_\infty(G)$.

(ii) $|A^p_\infty(G)| \equiv 1 \pmod p$

(iii) $|A^p_\infty(G)| / h_p$, *where* $h_p = [G: N_G(G_p)]$

(iv) $G = \langle N_G(A)/A \in A_\infty(G)\rangle$.

(v) *If* $A \in A_\infty(G)$ *and* $A_p \subseteq G_p$ *then there exists* $x \in G$ *such that* $G_p \cap G^x_p = A_p$.

For a discussion of the number h_p see **[9]**, Theorem 9.3.1.

Proof. Proposition 1(v) implies that every element of $A_\infty(G)$ contains $F(G)$. Therefore **[13]**, Theorem 1 implies that $A_\infty(G)$ is a conjugate class. If A, $B \in A_\infty(G)$ then $[A_p, B_q] = 1$ for every two distinct primes p and q by **[11]**, Theorem 7.18, p. 705, proving (i). Let $A_p \subseteq G_p$. We shall prove that $N_G(G_p) \subseteq N_G(A_p)$. If $x \in N_G(G_p)$ then $\langle A_p^x, A_p \rangle \subseteq G_p$ is a p-group. Hence, $[\langle A_p^x, A_p \rangle, A_{p'}] = 1$. Therefore $\langle A_p^x, A_p \rangle A_{p'}$ is nilpotent. Consequently, $A_p^x = A_p$, and $N_G(G_p) \subseteq N_G(A_p)$. Part (i) implies that $A_\infty^p(G)$ is a conjugate class. Hence, $|A_\infty^p(G)| = [G: N_G(A_p)]$. By **[14]**, Theorem 6.2.3, $|A_\infty^p(G)| \equiv 1 \pmod p$. Thus (ii)–(iii) also hold. Let $\mathcal{F} = \{G_p / p \in \pi(G)\}$ be a sylow system of G. Let us denote the intersection of the normalizers of the subgroups of the given Sylow system by $N(\mathcal{F})$. By definition $N(\mathcal{F})$ is a system normalizer of G. Since $A_\infty(G)$ is a conjugate class of G there exists $A(p) \in A_\infty(G)$ such that $[A(p)]_p \subseteq G_p$, for every $p \in \pi[A(p)]$.

As above $A = x[A(p)]_p$ is an element of $A_\infty(G)$, moreover $N_G(A) \supseteq N(\mathcal{F})$. Since G is generated by the set of system normalizers of G **[11]**, we obtain that $G = \langle N_G(A) / A \in A_\infty(G) \rangle$, proving (iv). Proposition 1 implies that $A_p = O_p(N_G(A_p))$. As mentioned before $G_p \subseteq N_G(A_p)$. Therefore by Ito's theorem **[12]** there exists $x \in G$ such that $G_p \cap G_p^x = A_p$.

The author knows of no counterexample to the conjecture that if G is an arbitrary group then $A_\infty(G)$ is a conjugate class.

Let ϕ be the class of finite solvable groups in which the system normalizers are Carter subgroups. ϕ-groups are discussed in **[5]** and **[11]**, pp. 743–751.

THEOREM 2. *Let G be a group of odd order. Then*

(i) *If $G = BC$, where $B \in A_b(G)$, $C \in A_c(G)$, $b \geqq 1$, $c \geqq 1$, then G is nilpotent.*

(ii) *G is nilpotent if and only if $G \in \phi$ and the Carter subgroups of G are elements of $A_c(G)$, for some integer c.*

Proof. (i) Proposition 1 implies that $BF(G)$ and $CF(G)$ are nilpotent. Therefore, Theorem 1(i) and **[13]**, Theorem 1 imply that there exist $x \in G$ and $A \in A_\infty(G)$ such that $B \subseteq A$ and $C \subseteq A^x$. Hence $G = AA^x$. By **[11]**, Theorem 7.18, p. 705 G is nilpotent.

(ii) Let C be a Carter subgroup of G. Assume that $C \in A_c(G)$, for some integer c. Since $CF(G)$ is nilpotent by Proposition 1, there exists $A \in A_\infty(G)$ such that $C \subseteq A$. Since $N_G(C) = C$, we obtain that $C = A$. By assumption $G \in \phi$. Therefore A is a system normalizer. By the definition of system normalizer, Proposition 1 implies that G is nilpotent.

REMARK. Theorem 2(i) is true without assuming **[6]**.

If $G = BC$, where $B \in A_b(G)$, $C \in A_c(G)$, $b \geqq 1$, $c \geqq 1$, then G is solvable by **[14]**, Theorem 13.2.9.

We shall say that G is a *D_π-group* if all the maximal π-subgroups of G are conjugate S_π-subgroups of G. If G and every normal subgroup of G is a D_π-group we will call G a *D_π^N-group.*

Let $[A, B, C]$ denotes the *triple commutator* $[[A, B], C]$ of three subgroups A, B, C of G. We say that G is a *π-stable* group if it satisfies the following condition:

Let K be an arbitrary π-subgroup of G. Let A be an arbitrary π-subgroup of $N_G(K)$. Then if $[K, A, A] = 1$, we have $AC_G(K)/C_G(K) \subseteq O_\pi(N_G(K)/C_G(K))$.

The next result is needed for the proof of Theorem 3.

PROPOSITION 2. *Let G be a π-stable D_π^N-group. Let K be an S_π-subgroup of G. Assume that $A_2(K) \subseteq A_\infty(K)$, $C_G(F(G) \subseteq F(G)$ and $O_{\pi'}(G) = 1$. Then we have*

(i) $J_2(K)$ char G

(ii) *If $|F(G)|$ is odd and G has an Abelian S_2-subgroup, then $A_2(K) = A_\infty(K)$ and $J_\infty(K) = J_\infty(G)$.*

Proof. Assume (i) is false for G. Let α be an automorphism of G, and choose $g \in G$ such that $K^\alpha = K^g$. If $J_2(K) \lhd G$, then

$$(J_2(K))^\alpha = J_2(K^\alpha) = J_2(K^g) = (J_2(K))^g = J_2(K).$$

Therefore $J_2(K)$ char G. Hence $J_2(K) \ntriangleleft G$.

Let L be the largest normal subgroup of G which normalizes $J_2(K)$. Then $K \cap L$ is an S_π-subgroup of L by **[11]**, lemma 7.2 p. 444. Since $J_2(K \cap L)$ char $K \cap L$, it follows therefore, by a generalization of the Frattini argument that $G = LN$, where $N = N_G(J_2(K \cap L))$. If $J_2(K) \subseteq K \cap L$, then $J_2(K) = J_2(K \cap L)$. In this case $N = N_G(J_2(K))$. But then $G = LN \subseteq N_G(J_2(K))$ and $J_2(K) \lhd G$, a contradiction. Thus, we may assume that $J_2(K) \nsubseteq L \cap K$. By Proposition 1, $F(G) \subseteq A$ for every $A \in A_2(K) \subseteq A_\infty(K)$. In particular, $[F(G), A, A] = 1$. Since G is π-stable, it follows from the definition that $AC_G(F(G))/C_G(F(G)) \subseteq O_\pi(G/C_GF(G))$. By definition of L and by hypothesis $C_G(F(G) \subseteq F(G) \subseteq L$. Hence $AL/L \subseteq O_\pi(G/L)$ for every $A \in A_2(K) \subseteq A_\infty(K)$.

However, we claim that $O_\pi(G/L) = 1$. Indeed, set $O_\pi(G/L) =$

T/L. $K \cap T$ is an S_π-subgroup of T, therefore $(K \cap T)L/L$ is an S_π-subgroup of T/L by **[11]**, Lemma 7.2 p. 444. Therefore $(K \cap T)L = T$. But $K \cap T \subseteq N_G(J_2(K))$, thus $K \cap T \subseteq L$, whence $T = L$, and $O_\pi(G/L) = 1$. Therefore $A \subseteq L$ for every $A \in A_2(K) \subseteq A_\infty(K)$. Therefore $J_2(K) \subseteq L \cap K$, a contradiction. Therefore $J_2(K)$ char G, proving (i).

Clearly $F(G) \subseteq F(K)$ and $|F(K)|$ is odd, since $C_G(F(G) \subseteq F(G)$. Hence by hypothesis $C_G(F(K)) \subseteq F(K)$. Proposition 1 yields that $F(K) \subseteq A$ for every $A \in A_\infty(K)$. Therefore $A_\infty(K) = A_2(K)$ as a consequence of **[13]**, Theorem 1. Proposition 1 implies that $F(G) \subseteq A$ for every $A \in A_\infty(G)$. Therefore $Z(A) \subseteq C_G(F(G)) = Z(F(G))$. By hypothesis $F(G)$ is a π-group. Hence A is a π-group for every $A \in A_\infty(G)$. Since G is a D_π^N-group and $J_\infty(K)$ char G by part (i), we obtain:

$$J_\infty(G) = \langle J_\infty(K^X) / X \in G \rangle = J_\infty(K),$$

proving (ii).

We now obtain:

Proof of Theorem 3. (i) We use induction on the order of G. Let $T = O_p(G)$, $H = O_{pp'}(G)$, $G^* = AH$, and $K^* = A(K \cap H)$. Then $K \cap H$ is an S_π-subgroup of H and K^* is an S_π-subgroup of G^*.

Suppose that $G^* \subset G$. Since $A \subseteq K^*$, $A \in A_\infty(K^*)$. By induction, $O_p(A) \subseteq O_p(G^*)$. Hence

$$[H, O_p(A)] \subseteq H \cap O_p(G^*) \subseteq O_p(H) = T.$$

Therefore, $O_p(A)T/T \subseteq C_{G/T}(H/T) \subseteq H/T$, by **[8]**, p. 228. Consequently, $O_p(A) \subseteq H$. So,

$$O_p(A) \subseteq H \cap O_p(G^*) = O_p(H) = T.$$

On the other hand, $T = O_p(G) \subseteq O_p(A)$ by Proposition 1. Therefore, $O_p(A) = O_p(G)$, as desired.

Suppose that $G^* = G$. Then A_pT is an S_p-subgroup of G. By Proposition 1, $T = O_p(G) \subseteq O_p(A)$. Therefore $O_p(A)$ is an S_p-subgroup of G. It is well known that G is p-strongly solvable for every $p \in \pi - \{2\}$. By definition G is p-stable for $p = 2$. Hence G is p-stable for every $p \in \pi$. Therefore $G = O_{p',p,p'}(G)$ by Proposition 2. Hence $O_p(A) \subseteq O_{p'p}(G)$. By Proposition 1, $AF(O_{p'}(G))$ is nilpotent. Hence $O_p(A) \subseteq C_G(F(O_{p'}(G)))$. By **[3]**, Lemma 4, $O_p(A) \subseteq O_p(G)$, for every $p \in \pi$. Therefore $O_p(A) = O_p(G)$, as desired. In particular $O_p(A) = O_p(K)$, for every $p \in \pi$, proving (i).

(ii) Define $\sigma = \pi(F(G))$. It is well known that G is a D_σ^N-group. By **[1]**, Corollary 4.8, $C_G(F(G)) \subseteq F(G)$. Proposition 1 yields that $F(G) \subseteq A$. Therefore $Z(A) \subseteq C_G(F(G)) \subseteq F(G)$. Hence A is a σ-group. By definition $O_{\sigma'}(G) = 1$. Since by hypothesis G is p-strongly solvable, for every $p \in \sigma$, G is σ-stable by **[2]**, Lemma 3.4. Let R be an S_σ-subgroup of G. Then $J_\infty(R) = J_\infty(K) = J_\infty(G)$ by Proposition 2. So $J_\infty(K) = J_\infty(G) = F(G) = F(K) = A$, by part (i).

The author knows of no counterexample to the conjecture that if G is a group of odd order then $O_p(A_2) \subseteq O_p(G)$, where $p \geqq 5$ and $A_2 \in A_2(G)$.

Theorem 3 has three corollaries.

COROLLARY 1. *If the Sylow subgroups of a solvable group G are all Abelian or if G is of odd order and the Sylow subgroups of G are of class at most* 2, *then $F(G) \in A_\infty(G)$.*

COROLLARY 2. *If P is an S_p-subgroup of a group G, p odd,* $\mathrm{cl}(P) \leqq 2$, *and if $N_G(P)$ has a normal p-complement, then so does G.*

Proof. Following the method of **[8]**, Theorem 8.3.1 and using Theorem 3 we obtain Corollary 3. One must take $N_G(P)$ instead of $N_G(ZJ(P))$ in the above mentioned theorem and its proof.

Corollary 2 is a known result. It can be obtained by **[11]**, Theorem 8.1 p. 447.

We shall say that G is a ψ-group if $A_\infty(G)$ and Carter subgroups of G coincide.

COROLLARY 3. *Let G be a group of odd order. Assume that G and every subgroup of G is a ψ-group, then G is nilpotent.*

Proof. Let G be a minimal counterexample. By induction every proper subgroup of G is nilpotent. Therefore the Sylow subgroups of G are of class at most 2 by **[11]**, Theorem 5.2, p. 281. Hence, Theorem 3 implies that $\langle A_\infty(G) \rangle = F(G)$ is a Carter subgroup of G. Therefore G is nilpotent as desired.

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BAR-ILAN UNIVERSITY