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COMPLEX VECTOR FIELDS AND DIVISIBLE CHERN CLASSES

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This paper contains two theorems which relate the maximal number of independent sections of a complex bundle over a manifold to the Chern classes of the bundle and certain functional cohomology operations. The main theoretical result of the paper is a formula which relates the obstruction to a lifting in a fibration and a functional cohomology operation.

1. Introduction. Let M be a connected, closed, orientable, smooth manifold of dimension $2n$. If ω is a complex n-plane bundle over M, the complex span of ω is the maximal number of cross-sections of ω which are linearly independent over the complex numbers. In this paper, we consider the following question: when is complex span $\omega \geqq q$? Hopf's theorem says that complex span $\omega > 0$ if and only if ω has vanishing Euler class and the theorems of Thomas (**[10]** and **[11]**) give an effective answer in the case $q = 2$. We study this problem in the cases $q = 3, 4$ and establish theorems which give necessary and sufficient conditions for complex span $\omega \geqq q$ in terms of the Chern classes of ω and certain functional cohomology operations. The Chern class of ω in $H^{2i}(M; \mathbf{Z})$ is denoted by $c_i(\omega)$. If $\delta_p P^1$ denotes the Steenrod p-power P^1 followed by the Bockstein associated with reduction mod p, $\delta_p P^1_\omega(c(i - p + 1))$ denotes a subset of a functional operation defined on the universal Chern class $c(i - p + 1)$ and contained in $H^{2i}(M; \mathbf{Z})$. This subset will be described in the second section of this paper. If p is a prime, M is j-connected mod p if $H_i(M; \mathbf{Z}_p) = 0$, $1 \leqq i \leqq j$. In both theorems below, M is 1-connected mod 2 and 3.

THEOREM 1. *If n is even, $n \geqq 6$, then complex span $\omega \geqq 3$ if and only if $c_i(\omega) = 0$, $n - 2 \leqq i \leqq n$, and $0 \in \delta Sq^2_\omega(c(n-2))$.*

THEOREM 2. *If n is odd, $n \not\equiv 1 \pmod 3$, $n \geqq 9$, and M is 3-connected mod 2, then complex span $\omega \geqq 4$ if and only if $c_i(\omega) = 0$, $n - 3 \leqq i \leqq n$, $0 \in \delta Sq^2_\omega(c(n-3))$, and $0 \in \delta_3 P^1_\omega(c(n-3))$.*

In Theorem 1, if $n \not\equiv 0 \pmod 3$, and $n \equiv 2 \pmod 4$, the connectedness hypothesis can be dropped. Thomas and Gilmore **[11]** show that if M is 3-connected, then for every n, complex span $\omega \geqq 3$ if and only if $c_{n-2}(\omega) = 0$ and $c_n(\omega) = 0$. Gilmore **[2]** proves theorems similar to Theorems 1 and 2 in which he assumes that $H_2(M; \mathbf{Z})$ and $H_4(M; \mathbf{Z})$, respectively, have no elements of order 2. For the specified values of n, our theorems contain the results of Thomas and Gilmore, because the operation $\delta_p P^1_\omega(c(i-p+1))$ is a set of elements of order p and hence must vanish if $H^{2i}(M; \mathbf{Z})$ has no p-torsion. If M is an almost-complex manifold with almost-complex structure ω, our theorems relate the complex span of M to the Chern classes of M and the functional operations.

2. Obstruction formulas. Let the map $f\colon M \to BU(n)$ classify ω. It is clear that complex span $\omega \geqq q$ if and only if f lifts to the total space of the fibration $W_{n,q} \to BU(n-q) \to BU(n)$ where $W_{n,q}$ is the Stiefel variety of complex q-frames in complex n-space. We will apply two obstruction formulas to this lifting problem. The first formula is due to Olum **[8]**. If $\pi\colon E \to B$ is a fibration with fiber F, c in $H^{n-1}(F; G)$ a class transgressing to d in $H^n(B; G)$, X a CW complex and $f\colon X \to B$ a map such that the lifting obstruction $O^n(f)$ is nonvoid, then

$$(2.1) \qquad -c_* O^n(f) = f^* d,$$

where $c_*\colon H^n(X; \pi_{n-1}(F)) \to H^n(X; G)$ is induced by $c_\#\colon \pi_{n-1}(F) \to G$, the composite of the Hurewicz homomorphism and evaluation.

The second formula is the main theoretical result of this paper. We assume that there is a class a in $H^{n-2p+1}(F; \mathbf{Z})$ such that $\delta P^1 a = 0$ and a is the only class transgressing to b, where $f^* b = 0$ and $P^1 b \equiv 0$ (mod integral classes in kernel $\pi^* \cap \operatorname{kernel} f^*$). Under these hypotheses, there are liftings $f_a\colon F \to K(\mathbf{Z}, n-2p+1; \mathbf{Z}, n-1, \delta P^1)$ and $f_b\colon B \to K(\mathbf{Z}, n-2p+2; \mathbf{Z}, n, \delta P^1)$ of a and b, where the range spaces are two-stage Postnikov systems induced by δP^1. In **[3]**, we show that the set $\{f_{a\#}\colon \pi_{n-1}(F) \to \mathbf{Z}\}$ is a congruence class modulo the image of the Hurewicz homomorphism and for every $[g]$ in $\pi_{n-1}(F)$, $f_{a\#}[g] \in \delta P^1_g(a)$, where δP^1_g is the standard functional operation. (See **[6]**, p. 157.) Therefore the induced homomorphism $f_{a*}\colon H^n(X; \pi_{n-1}(F)) \to H^n(X; \mathbf{Z})$ can be effectively computed in some cases. In the proposition below, ι denotes the fundamental class of $K(\mathbf{Z}, n-2p+2; \mathbf{Z}, n, \delta P^1)$ and we assume that $H^n(E; \mathbf{Z})$ and $H^n(B; \mathbf{Z})$ are torsion free.

PROPOSITION 2.2. *If $\pi\colon E \to B$ is a fibration satisfying the above hypotheses and p is a prime such that $n \geqq 2(2p-1)$, then we have containments*

$$(2.3) \qquad f_{a*} O^n(f) \equiv \delta P^1_{f_b f}(\iota) \equiv \delta P^1_f(b).$$

Note that (2.3) shows that the obstruction is contained in the operation $\delta P^1_f(b)$. A general result of this kind was obtained by Meyer. (See **[5]**, §13.) It will be clear in the proof of (2.2) that the indeterminacy of $\delta P^1_{f_b f}$ is image δP^1, and so (2.3) expesses the obstruction as an operation with smaller indeterminacy than the indeterminacy of δP^1_f, image $(\delta P^1 + f^*)$. Proposition 2.2 will follow from the next lemma. In the proof of the lemma, the reduction mod p of an integral class θ will be denoted by $\bar{\theta}$.

LEMMA 2.4. *The composite $f_b\pi$ is homotopically trivial.*

Proof. Let $K = K(\mathbf{Z}, n-2p+2; \mathbf{Z}, n, \delta P^1)$. If $n \geqq 2(2p-1)$, $H^n(K; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ modulo finite groups with a free generator θ such that $\theta_\#$: $\pi_n(K) \to \mathbf{Z}$ is multiplication by p and $\bar{\theta} = P^1\iota$. These facts follow immediately from the long exact homotopy and Serre cohomology sequences for the fibration $K \to K(\mathbf{Z}, n-2p+2)$ and the Hurewicz theorem modulo finite groups. Since $(f_b\pi)^*\iota = \pi^* b = 0$, because b is in the image of the transgression, $O^n(f_b\pi, *)$ is nonvoid. It follows from the hypotheses that $f_b^*\bar{\theta} \equiv 0$ (mod integral classes in kernel $\pi^* \cap$ kernel f^*) and from the properties of θ and (4.4) in **[8]** that $f_b^*\theta - f_b'^*\theta = pO^n(f_b, f_b')$ for any two liftings of b. Therefore, after alteration by an n-cocycle, we may assume $f_b^*\theta \in$ kernel $\pi^* \cap$ kernel f^* and so $\pi^* f_b^* \theta = 0 = pO^n(f_b\pi, *)$, and this implies $f_b\pi$ is homotopically trivial since $H^n(E; \mathbf{Z})$ has no torsion.

Lemma 2.4 implies the existence of a map of fibrations from the fibration π into the path space fibration over K with fiber $\Omega K = K(\mathbf{Z}, n-2p+1; \mathbf{Z}, n-1, \delta P^1)$. The map of fibers is a lifting of a, f_a: $F \to K(\mathbf{Z}, n-2p+1; \mathbf{Z}, n-1, \delta P^1)$, because we are assuming that a is the only class transgressing to b. The map $f_b f$ lifts to the path space if and only if it is homotopic to the constant map. We have taken care that $f_b^*\theta \in \ker f^*$ and so $(f_b f)^*$ has image zero in dimension n. Therefore the obstruction to a homotopy of $f_b f$ to point is precisely $\delta P^1_{f_b, f}(\iota)$ (10.8 [7]) and the indeterminacy of $\delta P^1_{f_b f}$ is image δP^1. Formula (2.3) now follows immediately from standard naturality properties of obstructions and functional operations (**[6]**, p. 157).

3. The proofs of Theorems 1 and 2. We begin with some general remarks. The group $H^*(W_{n,q}; \mathbf{Z})$ is an exterior algebra with generators a_k in $H^{2k-1}(W_{n,q}; \mathbf{Z})$, $n-q+1 \leqq k \leqq n$ (**[1]**, p. 444). The space $W_{n,q}$ is $2(n-q)$-connected and $\pi_i(W_{n,q})$ is $\mathbf{Z}$ if i is odd and a finite group if i is even as long as n is large, $n-q$ is odd, $2 \leqq q \leqq 4$, and $2(n-q)+1 \leqq i \leqq 2n-1$. The necessary size of n is indicated in the theorems. The group $\pi_{2(n-q)+2}(W_{n,q})$ is zero and the other finite groups are 2 or 3 torsion groups in this range of dimensions, **[2]**. Since the

Hurewicz homomorphism $h\colon \pi_i(W_{n,q}) \to H_i(W_{n,q};\mathbf{Z})$ is a isomorphism modulo finite abelian groups if $i \leqq 4(n-q)$, it sends a generator in $\pi_{2k-1}(W_{n,q})$ into an integer $m_k \neq 0$, $n-q+1 \leqq k \leqq n$. These integers can be computed using an inductive argument based on the fibration $W_{n-1,q-1} \to W_{n,q} \to S^{2n-1}$. In particular, $m_{n-q+1}=1$, $m_{n-q+2}=2$ and the prime divisors of the others are either 2 or 3, **[4]**. Since a_k transgresses to $c(k)$, it follows from (2.1) that $-m_k O^{2k}(f) = c_k(\omega)$. Thomas' theorem **[10]** follows from this formula and the first two values of m_k: if M is arbitrary and n is odd, complex span $\omega \geqq 2$ if and only if $c_i(\omega)=0$, $n-1 \leqq i \leqq n$.

Consider the lifting obstruction $O^{2k}(f)$, where $n-q+p \leqq k \leqq n$. In this range of dimensions, $c(k-p+1)$ is the image of a unique class a_{k-p+1} under transgression and if $c_k(\omega)=0$, $P^1c(k-p+1) \equiv 0$ (mod integral classes in kernel $\pi^* \cap \operatorname{kernel} f^*$), (**[1]**, p. 429). Let $f'\colon BU(n) \to K(\mathbf{Z}, 2(k-p+1); \mathbf{Z}, 2k, \delta P^1)$ be a lifting of $c(k-p+1)$, $f_{k-p+1}\colon W_{n,q} \to K(\mathbf{Z}, 2k-2p+1; \mathbf{Z}, 2k-1, \delta P^1)$ a lifting of a_{k-p+1}, and set $\delta P^1_\omega(c(k-p+1)) = \delta P^1_{f'f}(\iota)$. If $[g]$ in $\pi_{2k-1}(W_{n,q})$ is a generator, then $f_{k-p+1\#}[g] \in \delta P^1_g(a_{k-p+1})$, **[3]**. Since $P^1 a_{k-p+1} = \mu_p(k) a_k$, where $\mu_p(k) \equiv k \pmod p$, (**[1]**, p. 429), a direct computation of the operation $\delta P^1_g(a_{k-p+1})$ yields the equation $f_{k-p+1\#}[g] \equiv \mu_p(k)p^{-1}m_k \pmod{m_k}$. If $c_k(\omega)=0$, then $m_k O^{2k}(f)=0$ and the action of the homomorphisms f_{k-p+1*} on the obstruction is independent of the lifting. The next proposition follows immediately from the above remarks and Proposition 2.2. The assumption $p \leqq q$ forces the inequality of the proposition because we have $2q \leqq n$ in our theorems.

PROPOSITION 3.1. *Let p be a prime such that $p \leqq q$. If $c_k(\omega)=0$ and $m_k \equiv 0 \pmod p$, then*

$$(3.2)\quad \mu_p(k)p^{-1}m_k O^{2k}(f) \equiv \delta P^1_\omega(c(k-p+1)) \equiv \delta P^1_f(c(k-p+1)).$$

We now turn to the proof of Theorem 2. The proof of Theorem 1 will be discussed later. In addition to the properties of $W_{n,4}$ mentioned above, we will need the fact that the 3-component of $\pi_{2n-4}(W_{n,4})$ is zero if $n \not\equiv 1 \pmod 3$ **[2]** and some more precise information on the image of the Hurewicz homomorphism $h\colon \pi_{2n-3}(W_{n,4}) \to H_{2n-3}(W_{n,4};\mathbf{Z})$, n odd and $n \geqq 9$: $m_{n-1} \equiv 0 \pmod 3$ and $m_{n-1} \not\equiv 0 \pmod 9$ if $n \not\equiv 1 \pmod 3$, **[4]**.

The conditions in Theorem 2 are clearly necessary. The hypothesis $c_{n-3}(\omega)=0$ implies that $O^{2n-4}(f)$ is nonvoid and because n is odd and $m_{n-2}=2$, Proposition 3.1 implies that $O^{2n-4}(f) = \delta Sq^2_\omega(c(n-3))$. This equation is an actual equality because the indeterminacy of $O^{2n-4}(f)$ is image δSq^2 (**[10]**, p. 191) which is the indeterminacy of $\delta Sq^2_\omega(c(n-3))$. Therefore the hypothesis $0 \in \delta Sq^2_\omega(c(n-3))$ is enough to imply that $O^{2n-2}(f)$ is nonvoid since $n \not\equiv 1 \pmod 3$ means that $\pi_{2n-4}(W_{n,4})$ has no

3-component and M is 3-connected mod 2. Since $c_{n-1}(\omega)=0$ and $m_{n-1}\equiv 0 \pmod 3$, we have $\mu_3(n-1)3^{-1}m_{n-1}O^{2n-2}(f)\equiv \delta P^1_\omega(c(n-3))$, where $\mu_3(n-1)\neq 0$ because $n\not\equiv 1 \pmod 3$. The proof of Theorem 2 will be complete when we have shown that this equation is an actual equality, because the other obstructions vanish by connectivity, Poincaré duality, and $c_n(\omega)=0$.

To establish equality, we work with the equation in the form $f_{n-3*}O^{2n-2}(f)\equiv O^{2n-2}(f'f)$, where $f_{n-3}\colon W_{n,4}\to K(\mathbf{Z}, 2n-7; \mathbf{Z}, 2n-3, \delta P^1)$ and $f'\colon BU(n)\to K(\mathbf{Z}, 2n-6; \mathbf{Z}, 2n-2, \delta P^1)$ are liftings of a_{n-3} and $c(n-3)$, respectively. Let h and $\bar{h}$ be $2n-3$-liftings of $f'f$ and f, respectively, and let $\{c^{2n-2}(h)\}$ and $\{c^{2n-2}(\bar{h})\}$ be the obstruction cohomology classes determined by these liftings. We assert that h and $\bar{h}$ can be altered in such a way that the obstruction class of h is unchanged and $f_{n-3*}\{c^{2n-2}(\bar{h})\}=\{c^{2n-2}(h)\}$. The argument begins by altering h by a $2n-7$-cocycle ν such that $\{\nu\}=-3O^{2n-7}(f'\bar{h}, h)$. The altered map h_ν extends to a $2n-3$ lifting of $f'f$ because of the homotopy of $K(\mathbf{Z}, 2n-7; \mathbf{Z}, 2n-3, \delta P^1)$ and $\{c^{2n-2}(h)\}-\{c^{2n-2}(h_\nu)\}=\delta P^1 O^{2n-7}(h, h_\nu)=\delta P^1\{\nu\}$, **[9]**, so $\{c^{2n-2}(h)\}=\{c^{2n-2}(h_\nu)\}$ by our choice of ν. Note that

$$O^{2n-7}(f'\bar{h}, h_\nu)=O^{2n-7}(f'\bar{h}, h)+O^{2n-7}(h, h_\nu)=-2O^{2n-7}(f'\bar{h}, h).$$

The homomorphism $f_{n-3\#}\colon \pi_{2n-7}(W_{n,4})\to \mathbf{Z}$ is the identity and so we may alter $\bar{h}$ by a $2n-7$-cocycle α such that $f_{n-3*}\{\alpha\}=\{\alpha\}=O^{2n-7}(f'\bar{h}, h_\nu)$. If $\bar{h}_\alpha$ is the altered map, $O^{2n-7}(\bar{h}, \bar{h}_\alpha)=-2O^{2n-7}(f'\bar{h}, h)$, and so $\{c^{2n-4}(\bar{h})\}-\{c^{2n-4}(\bar{h}_\alpha)\}=\delta Sq^2 O^{2n-7}(\bar{h}, \bar{h}_\alpha)=0$. Therefore, $\bar{h}_\alpha$ extends to a $2n-3$ lifting of f because $\{c^{2n-4}(\bar{h})\}=0$ and M is 3-connected mod 2. We have

$$O^{2n-7}(f'\bar{h}_\alpha, h_\nu)=O^{2n-7}(f'\bar{h}_\alpha, f'\bar{h})+O^{2n-7}(f'\bar{h}, h_\nu)=0$$

because $O^{2n-7}(\bar{h}_\alpha, \bar{h})=-\{\alpha\}$, and so $f'\bar{h}_\alpha$ is homotopic to h_ν in dimensions less than $2n-3$ and the difference formula for cocycles yields $f_{n-3*}\{c^{2n-2}(\bar{h}_\alpha)\}=\{c^{2n-2}(h_\nu)\}$. This completes the proof of equality and of Theorem 2. The proof of Theorem 1 is exactly the same except that there is no obstruction of order 3. The remark about the special case $n\not\equiv 0 \pmod 3$, $n\equiv 2 \pmod 4$ follows from **[2]**.

The problem of computing the operations δP^1_ω seems difficult. The following example shows that they are nontrivial invariants of the sectioning problem on the level of complexes[1]. Let n be even, $n\geqq 6$, and let $p\colon E\to BU(n)$ be the first stage in the Postnikov system for the fibration $BU(n-3)\to BU(n)$ and let $p_1\colon E_1\to BU(n-2)$ be the first

[1] I am grateful to the referee for this example.

stage in the system of the fibration $BU(n-3) \to BU(n-2)$. Let X be the $2n-1$-skeleton of E_1 and ω the bundle classified by the inclusion $BU(n-2) \to BU(n)$ composed with p_1. If k in $H^{2n-2}(E; \mathbf{Z})$ is the k-invariant, then $2k = p^*c(n-1)$ **[10]** and if $s: E_1 \to E$ is the natural map, then s^*k is in the image of the Bockstein mod 2. With these observations, it is easy to see that ω has the following properties: $c_i(\omega) = 0$, $n-2 \leqq i \leqq n$, complex span $\omega \geqq 2$, but $0 \not\in \delta Sq^2_\omega(c(n-2))$ because X does not lift to $BU(n-3)$.

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