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FUNCTIONAL RELATIONSHIPS BETWEEN A SUBNORMAL OPERATOR AND ITS MINIMAL NORMAL EXTENSION

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Let K be a compact subset of the plane. $C(K)$ denotes the continuous functions on K and $R(K)$ denotes those continuous functions of K which are uniform limits of rational functions whose poles lie off K. We say that f is minimal on K if $f \in R(K)$ and for every complex number c

$$R(L_c) = C(L_c)$$

where $L_c = \{z \in K \mid (fz) = c\}$.

Let S be a subnormal operator on a Hilbert space $\mathscr{H}$ with its minimal normal extension N on the Hilbert space $\mathscr{K}$. The spectrum of S is denoted by $\sigma(S)$. In this paper it is shown that if f is minimal on $\sigma(S)$ then $f(N)$ on $\mathscr{K}$ is the minimal normal extension of $f(N)$ restricted to $\mathscr{H}$. Some new results about subnormal operators follow as corollaries of this theorem.

An operator S acting on a Hilbert space $\mathscr{H}$ is called subnormal if there exists a normal operator N acting on a Hilbert space $\mathscr{K}$, which contains $\mathscr{H}$, such that $Nx = Sx$ for all x in $\mathscr{H}$. N is called the minimal normal extension (abbreviated mne.) of S when $\mathscr{K}$ is the only closed subspace containing $\mathscr{H}$ that reduces N. This is equivalent to saying that the closure of the linear manifold

$$\left\{\sum_{j=0}^{n} N^{*j} x_j \,\middle|\, x_j \in \mathscr{H},\ n \text{ a nonnegative integer}\right\}$$

is all of $\mathscr{K}$. (For the elementary properties of subnormal operators consult [2, 5].)

If K is a compact set in the plane then $C(K)$ denotes the continuous functions on K and $\mathscr{A}(K)$ is the collection of functions f analytic on some open set $G(f) \supset K$. $P(K)$ and $R(K)$ are the uniform closures of the polynomials and rational functions with poles off K, respectively. Further, ∂K and int K denote the boundary and interior of K, respectively. $\hat{K}$ designates the polynomial convex hull of K. [3, p. 66].

The set of bounded operators on a (complex) Hilbert space $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$ and $\sigma(T)$ represents the spectrum of any operator T belonging to $\mathscr{B}(\mathscr{H})$. Finally, $\boldsymbol{C}$ denotes the complex numbers and $\boldsymbol{N}$ denotes the nonnegative integers.

2. **The problem.** Throughout the rest of this paper it will be

assumed that S is a subnormal operator on $\mathscr{H}$ with its mne. N on $\mathscr{K}$. It is well-known that $\sigma(S) \supset \sigma(N)$ and, in fact, $\sigma(S)$ is obtained by "filling in" some of the holes of $\sigma(N)$. (See [2].) Moreover, $\sigma(S)$ and $\sigma(N)$ are spectral sets for S and N, respectively. (Consult [9] for the basic properties of spectral sets.)

Consequently, for each f in $R(\sigma(S))$ an operator $f(S)$ belonging to $\mathscr{B}(\mathscr{H})$ may be defined. Furthermore, a normal operator $f(N)$ (in $\mathscr{B}(\mathscr{K})$) is defined because $R(\sigma(S)) \subset R(\sigma(N))$. For these functions f, it is easy to show that $f(N)\mathscr{H} \subset \mathscr{H}$ and that

$$f(N)|_{\mathscr{H}} = f(S) . \tag{2.1}$$

($f(N)|_{\mathscr{H}}$ means the restriction of $f(N)$ to $\mathscr{H}$.) It now follows that $f(S)$ is subnormal.

Moreover, for functions f in $\mathscr{A}(\sigma(S))$ or $P(\sigma(S))$ we can define a normal (subnormal) operator $f(N)(f(S))$ such that (2.1) holds. This follows because $P(\sigma(S))$ is a uniformly closed subalgebra of $R(\sigma(S))$ and each function f belonging to $\mathscr{A}(\sigma(S))$ may be approximated uniformly on $\sigma(S)$ by rational functions with poles off $\sigma(S)$. (See Runges Theorem [7, p. 256].)

With this background a natural question arises.

Question 2.2. For which function f belonging to $R(\sigma(S))$ (or $\mathscr{A}(\sigma(S))$ or $P(\sigma(S))$) is $f(N)$ the minimal normal extension of $f(S)$?

In the next section we present the approximation theorems that will be the foundation for the answer to (2.2). Before this is done, it is shown that the spectrum of the mne. of $f(S)$ is $\sigma(f(N))$.

THEOREM 2.3. *Let f be a bounded Borel function on $\sigma(N)$ such that $f(N)\mathscr{H} \subset \mathscr{H}$ and set $f(S) = f(N)_{\mathscr{H}}|$. Then the spectrum of the* mne. *of $f(S)$ is equal to the spectrum of $f(N)$.*

Proof. Let $\mathscr{M}$ denote the closure of the linear manifold

$$\left\{\sum_{j=0}^{n} f(N)^{*j}x_j \,|\, x_j \in \mathscr{H}, \ n \in N\right\}$$

and set $T = f(N)|_{\mathscr{M}}$. Its clear that T (on $\mathscr{M}$) is the mne. of $f(S)$ and $N\mathscr{M} \subset \mathscr{M}$ because $Nf(N)^* = f(N)^*N$. Set $S_1 = N|_{\mathscr{M}}$ and observe that N is the mne. of S_1 because $\mathscr{H} \subset \mathscr{M} \subset \mathscr{K}$.

Since $f(N) = T \oplus f(N)|_{\mathscr{M}^\perp}$ we have the obvious inclusion $\sigma(T) \subset \sigma(f(N))$. Suppose $\sigma(f(N))\backslash\sigma(T)$ is nonempty and let λ be an element of this set. Choose a relatively open set G in $\sigma(f(N))$ such that $\lambda \in G$ and $\bar{G}$ (the closure of G) has an empty intersection with $\sigma(T)$.

Let E be the spectral measure for $f(N)$. In other words $f(N) =$

$\int zdE(z)$. Set P equal to the projection from $\mathscr{K}$ onto $\mathscr{M}$ and observe $PE(\Delta) = E(\Delta)P$ for all Borel sets Δ. By the uniqueness of spectral measures it is easy to verify that the spectral measure, $\tilde{E}$, of T is given by $\tilde{E}(\Delta) = PE(\Delta)P = PE(\Delta)$.

Because $\bar{G} \cap \sigma(T) = \square$ we have $\tilde{E}(\bar{G}) = 0$. Therefore $0 = PE(\bar{G}) = E(\bar{G})\, P$ so that the range of $E(\bar{G})$ is orthogonal to $\mathscr{M}$. Now G a nonempty relatively open set in $\sigma(f(N))$ implies $E(\bar{G}) \neq 0$ which is a contradiction of the fact that N is the mne. of S_1. Therefore $\sigma(f(N))\backslash\sigma(T) = \square$. The proof is now complete.

3. Approximation theorems.

THEOREM 3.1. *Let K be a compact set in the plane and f an analytic function on an open set $G \supset K$. If f is not constant on any component of G then the linear span of the set*

$$\mathscr{F} = \{\bar{f}^n f^m z^p \mid n, m, p \in N\}$$

is uniformly dense in $C(K)$.

The proof of this theorem as well as the proofs of Theorem 3.2 and Theorem 3.4 rely on Bishop's Generalized Stone-Weierstrass Theorem. (See [1, 3, 4].) Since their proofs are very similar, only the proof of Theorem 3.4 will be presented here. (Consult the remarks at the end of the proof of Theorem 3.4.)

By the Maximum-Modulus Principle every function f belonging to $P(K)$ may be extended to a function in $P(\hat{K})$. This identification is made in the following theorem.

THEOREM 3.2. *Let*

1. *K be a compact set in the plane and $\{G_i\}$ be the sequence of components of* int $\hat{K}$,

2. *f belong to $P(K)$ and $\mathscr{U}$ denote the uniformly closed algebra generated in $C(K)$ by $\bar{f}$ and the polynomials.*

If $f|_{\partial G_i} \neq$ constant for all i then $\mathscr{U} = C(K)$.

DEFINITION 3.3. Let K be a compact set in the plane. We say that f is a minimal function on K if $f \in R(K)$ and for every complex number c

$$R(L_c) = C(L_c)$$

where $L_c = \{z \in K \mid f(z) = c\}$.

Before we present some examples of minimal functions, we show why they are interesting.

THEOREM 3.4. *Let K_0 be a compact subset in the plane, $\{G_i\}$ denote the sequence of bounded components of $C\backslash K_0$, and let*

$$K_1 = K_0 \cup \left(\bigcup_{i\in I} G_i\right)$$

where I is an arbitrary subset of the positive integers.

If f is a minimal function on K_1 and $\mathscr{U}$ is the uniformly closed algebra generated in $C(K_0)$ by $\bar{f}$ and $R(K_1)$ then $\mathscr{U} = C(K_0)$.

Proof. Let F be a maximal set of antisymmetry of $\mathscr{U}$. Now f and $\bar{f}$ belong to $\mathscr{U}$ so there exists a constant c such that

$$F \subset \{z \in K_0 \,|\, f(z) = c\} \subset \{z \in K_1 \,|\, f(z) = c\} .$$

By Bishop's Theorem it suffices to show $C(F) = \mathscr{U}|_F$.

Since f is minimal on K_1 we clearly have $R(F) = C(F)$. Therefore it suffices to show $\mathscr{U}|_F \supset R(F)$. Since $\mathscr{U}|_F$ is closed in $C(F)$ the last statement will follow if it can be demonstrated that $r \in \mathscr{U}|_F$ for all rational functions r with poles off F. Before we prove the latter the following observation is made.

Claim one. f minimal on K_1 implies f is not constant on any component of int K_1.

Suppose the contrary, say there exists a component G of int K_1 such that $f(G) = c_0$. If $z \in G$ then for every neighborhood U of z it follows that $L_{c_0} \cap \bar{U}$ has nonempty interior. Hence $R(L_{c_0} \cap \bar{U}) \neq C(L_{c_0} \cap \bar{U})$ because every function belonging to the former set is analytic on int $(L_{c_0} \cap \bar{U})$. This contradicts the fact that f is minimal on K_1.

Now fix a rational function r with poles off F. Let the sequence of components of $C\backslash F$ be denoted by $\{U_i\}$ where U_0 denotes the unbounded component and let the sequence of $C\backslash K_1$ be denoted by $\{\mathscr{O}_j\}$ where $\mathscr{O}_0$ denotes the unbounded component. Because $C\backslash K_1 \subset C\backslash F$, for each j there exists a unique i_j such that $\mathscr{O}_j \subset U_{i_j}$. We wish to show that the map $\mathscr{O}_j \to U_{i_j}$ is an onto map from $\{\mathscr{O}_j\}$ to $\{U_i\}$. Therefore we want to prove.

Claim two. For each i there exists a j such that $\mathscr{O}_j \subset U_i$.

Suppose that some U_i does not contain any $\mathscr{O}_j$. Then because $C\backslash K_1 = \bigcup_j \mathscr{O}_j$ and $\mathscr{O}_0 \cup U_0$ it follows that $U_i \subset K_1$. But $\partial U_i \subset F$ and so f is constant on some component of int K_1 which contradicts claim one.

Hence, each U_i contains a point of the complement of K_1. Since

r is analytic on some open set F_1 and since $\mathscr{U}$ contains the rational functions with poles of K_1, Runge's Theorem gives $r \in \mathscr{U}|_F$. This completes the proof.

Remarks. (1) In the last section of this paper an example is given to show that (notation as above) if $\mathscr{U}'$ is the uniformly closed algebra generated in $C(K_0)$ by $\bar{f}$ and $P(K_1)$ then $\mathscr{U}'$ may fail to be $C(K_0)$. (2) To prove Theorem 3.1 observe that any maximal set of antisymmetry of the uniformly closed algebra generated by $\mathscr{F}$ must consist of only finite number of points by the conditions on f. Therefore, since $\mathscr{F}$ contains the polynomials, each maximal set of antisymmetry must consist of only one point. (3) To prove Theorem 3.2 observe that each maximal set of antisymmetry F must be nowhere dense and $C\backslash F$ must be connected. Therefore $C(F) = P(F)$ by Lavrentiev's Theorem. (Consult [3, p. 48].)

This section of the paper is concluded by giving some examples of minimal functions. Let f be an entire function which is not constant. Then $f \in R(K)$ for any compact set K and for any constant c the level set $L_c = \{z \in K | f(z) = c\}$ consists of a finite number of points and therefore f is minimal on K.

If $f \in P(K)$ and f is not constant on any components of int $\hat{K}$ it is easy to show f is minimal on K. (In fact, one gets $P(L_c) = C(L_c)$ for any level set L_c.) Consequently, (by the first claim in the proof of Theorem 3.4) if $f \in P(K)$ then a necessary and sufficient condition that f be minimal on $\hat{K}$ is that f is not constant on any component of int $\hat{K}$.

LEMMA 3.5. *Let K be a compact subset in the plane satisfying either of the following conditions:*

(a) *The planar measure of ∂K is equal to zero,*

(b) *$R(K)$ is a Dirichlet algebra.* (See [3, p. 34].)

If $f \in R(K)$ and f is not constant on any component of int K *then f is minimal on K.*

Proof. First case. $f \in R(K)$ and ∂K has planar measure zero. Fix a constant c and the level set $L_c = \{z \in K | f(z) = c\}$. Then

$$L_c = \{L_c \cap \operatorname{int} K\} \cup \{L_c \cap \partial K\} . \tag{3.6}$$

$\{L_c \cap \operatorname{int} K\}$ is countable by the hypothesis on f and $\{L_c \cap \partial K\}$ has planar measure zero by the hypothesis on ∂K. Therefore, the planar measure of L_c is zero and by the Hartog-Rosenthal Theorem [3] we have $R(L_c) = C(L_c)$.

Second case. Again fix a constant c and write L_c as in (3.6). Since every point of $\{L_c \cap \operatorname{int} K\}$ is an isolated point (because f is not constant on any component of int K) each such point is clearly a peak point for $R(L_c)$. Since $R(K)$ is a Dirichlet algebra every point in $\{L_c \cap \partial K\}$ is a peak point of $R(K)$ and hence a peak point of $R(L_c)$. Thus every point of L_c is a peak point for $R(L_c)$ so by [3, p. 54]

$$R(L_c) = C(L_c) .$$

4. Answers to question 2.2.

THEOREM 4.1. *Let S be a subnormal operator on $\mathscr{H}$ and N be its* mne. *on $\mathscr{K}$. If f is analytic on an open set $G \supset \sigma(S)$ such that f is not constant on any component of G then $f(N)$ is the* mne. *of $f(S)$.*

Proof. Let $\mathscr{M}$ denote the closure of the linear manifold

$$\left\{\sum_{j=0}^{n} f(N)^{*j} x_j \,\middle|\, x_j \in \mathscr{H},\ n \in N\right\} .$$

It suffices to show $\mathscr{M} = \mathscr{K}$.

Observe that N, $f(N)$ and $f(N)^*$ all leave $\mathscr{M}$ invariant. Therefore any operator T in the norm closed algebra $\mathscr{A}$ generated by these operators and the identity leaves $\mathscr{M}$ invariant. By the Spectral Theorem the algebra $\mathscr{A}$ is * isometrically isomorphic to the uniformly closed algebra $\mathscr{U}$ in $C(\sigma(N))$ generated by $\bar{f}$, f, z and 1. By Theorem 3.1, $\bar{z} \in \mathscr{U}$, hence $N^* \in \mathscr{A}$. Therefore $N^*\mathscr{M} \subset \mathscr{M}$. Since $\mathscr{H} \subset \mathscr{M} \subset \mathscr{K}$, by minimality $\mathscr{M} = \mathscr{K}$. This finishes the proof.

As an immediate consequence of the last proof we have

COROLLARY 4.2. *If N is a normal operator, $f \in \mathscr{A}(\sigma(N))$ and f is not constant on any component of its domain then the commutative C^* algebra generated by N and the identity is equal to the commutative algebra generated by $f(N)$, $f(N)^*$, N and the identity.*

Just as Theorem 4.1 follows directly from Theorem 3.1 the following two theorems follow directly from the other two approximation theorems in section three.

THEOREM 4.3. *Let S be a subnormal operator on $\mathscr{H}$ and $\mathscr{N}$ be its* mne. *on $\mathscr{K}$. Let $\{G_i\}$ denote the sequence of components of* int $\widehat{\sigma(S)}$. *If $f \in P(\sigma(S))$ and f is not constant on any G_i then $f(N)$ is the* mne. *of $f(S)$.*

THEOREM 4.4. *Let S be a subnormal operator on $\mathscr{H}$ with* mne.

N on $\mathscr{K}$. If f is a minimal function on $\sigma(S)$ then $f(N)$ is the mne. *of $f(S)$.*

REMARK. The proof of Theorem 4.4 follows exactly like the proof of Theorem 4.1 except one replaces Theorem 3.1 with Theorem 3.4 using $K_0 = \sigma(N)$ and $K_1 = \sigma(S)$.

Using Theorem 4.3 it is possible to describe the mne. of $f(S)$ for an arbitrary function f belonging to $P(\sigma(S))$. More precisely; let the sequence of components of int $\widehat{\sigma(S)}$ be denoted by $\{\mathscr{O}_i\}_{i=0}^{\infty}$ (if the sequence is finite the procedure is the same) and let I_f denote the set of positive integers such that if $i \in I_f$ then $f|_{\mathscr{O}_i} = \text{constant} = \alpha_i$, and if $i \notin I_f$ then $f|_{\mathscr{O}_i} \neq \text{constant}$.

Using [8, p. 296] we can find closed subspaces $\mathscr{H}_i$ $i = 0, 1, 2, \cdots$ such that $\mathscr{H}_i \perp \mathscr{H}_j$ for $i \neq j$, $\sum_0^\infty \oplus \mathscr{H}_i = \mathscr{H}$, each $\mathscr{H}_i$ reduces S and if $S_i = S|_{\mathscr{H}_i}$ then $S = S_0 \oplus (\sum_1^\infty \oplus S_i)$ with respect to $\mathscr{H} = \sum_0^\infty \oplus \mathscr{H}_i$. Moreover we can choose the $\mathscr{H}_i$'s such that (a) S_0 is normal and $\sigma(S_0) \subset \partial\widehat{\sigma(S)}$ and (b) $\bar{\mathscr{O}}_i$ is a spectral set for S_i.

Without loss of generality we can assume that the mne. of S is $N = S_0 \oplus (\sum_1^\infty \oplus N_i)$ with respect to $\mathscr{K} = \mathscr{H}_0 \oplus (\sum_1^\infty \oplus \mathscr{K}_i)$ where N_i (on $\mathscr{K}_i$) is the mne. of S_i (on $\mathscr{H}_i$) for $i = 1, 2 \cdots$.

Claim. $f(N_i)$ is the mne. of $f(S_i)$ for all $i \notin I_f$. First observe that $f(N_i)$ and $f(S_i)$ are defined because $f \in P(\sigma(S)) \subset P(\sigma(S_i))$.

Now fix a complex number c and its associated level set $L_c = \{z \in \sigma(S_i) \mid f(z) = c\} \subset \{z \in \bar{\mathscr{O}}_i \mid f(z) = c\} \equiv B$.

Since f is not constant on $\mathscr{O}_i$ and $R(\widehat{\sigma(S)}) = P(\sigma(S))$ is a Dirichlet algebra, by the same methods used in the proof of Lemma 3.5, it follows that $R(B) = C(B)$ which implies $R(L_c) = C(L_c)$. This in turn implies that f is minimal on $\sigma(S_i)$. This establishes the claim.

It is now clear that the mne. of $f(S)$ is $f(S_0) \oplus (\sum_{i \notin I_f} \oplus f(N_i)) \oplus (\sum_{i \in I_f} \oplus \alpha_i)$ with respect $\mathscr{H}_0 \oplus (\sum_{i \notin I_f} \oplus \mathscr{K}_i) \oplus (\sum_{i \in I_f} \oplus \mathscr{H}_i)$.

What is the mne. of $f(S)$ for an arbitrary function f belonging to $R(\sigma(S))$? If $R(\sigma(S))$ is a Dirichlet algebra the question has a natural solution. Proceed exactly as above except let $\{\mathscr{O}_i\}_1^\infty$ be the sequence of components of int $\sigma(S)$. Since $R(\sigma(S))$ is a Dirichlet algebra the nontrivial Gleason parts of $R(\sigma(S))$ are precisely the $\mathscr{O}_i$'s.

Now we can use Mlak's results [6] to find subspaces $\mathscr{H}_i$ whose relationships to the $\mathscr{O}_i$'s are precisely like the case above except that $\sigma(S_0) \subset \partial\sigma(S)$. The calculation above now carries over.

The general case (no assumption on $R(\sigma(S))$) still remains open.

5. Some consequences.

Recall that an operator $T \in \mathscr{B}(\mathscr{K})$ is reductive if and only if every invariant subspace of T reduces T.

COROLLARY 5.1. *Let N be a normal operator and $f \in P(\sigma(N))$ such that f is not constant on any component of* int $\widehat{\sigma(N)}$. *If $f(N)$ is a reductive operator then N is a reductive operator.*

Proof. If N (on $\mathscr{K}$) is not a reductive operator then N must be the mne. of some (nonnormal) subnormal S (on $\mathscr{H}$). By Theorem 4.3 $f(N)$ is the mne. of $f(S)$, so $\mathscr{H}$ is a nonreducing invariant subspace for $f(N)$.

COROLLARY 5.2. *If f is a minimal function on $\sigma(S)$ and $f(S)$ is normal then S is normal.*

A subnormal operator S is called completely subnormal if S has no nonzero reducing subspace on which it is normal.

LEMMA 5.3. *Let N (on $\mathscr{K}$) be the* mne. *of S (on $\mathscr{H}$). S is completely subnormal if and only if N^* is the* mne. *of $N^*|_{\mathscr{H}^\perp}$.*

Proof. Suppose N^* is not the mne. of $N^*|_{\mathscr{H}^\perp}$. ($N^*\mathscr{H}^\perp \subset \mathscr{H}^\perp$ because $N\mathscr{H} \subset \mathscr{H}$.) Then there exists a closed subspace $\mathscr{M}$ such that $\mathscr{H}^\perp \subset \mathscr{M} \subsetneqq \mathscr{K}$, $N^*\mathscr{M} \subset \mathscr{M}$ and $N\mathscr{M} \subset \mathscr{M}$. Therefore $0 \neq \mathscr{M}^\perp \subset \mathscr{H}$ and $N|_{\mathscr{M}^\perp} = S|_{\mathscr{M}^\perp}$ is normal which implies S is not completely subnormal. By reversing the above argument the proof is completed.

COROLLARY 5.4. *If S is completely subnormal and f is a minimal function on $\sigma(S)$ then $f(S)$ is completely subnormal.*

Proof. By Lemma 5.3 it suffices to show $f(N)^*$ is the mne. of $f(N)^*|_{\mathscr{H}^\perp}$. It is easy to show that $\overline{\sigma(S)} = \sigma(N^*|_{\mathscr{H}^\perp})$.

Now define a function g on $\sigma(N^*|_{\mathscr{H}^\perp})$ by $g(z) = \overline{f(\bar{z})}$ for $z \in \overline{\sigma(S)}$. Because f is minimal on $\sigma(S)$, g is minimal on $\overline{\sigma(S)}$. Therefore $g(N^*) = f(N)^*$ is the mne. of $f(N)^*|_{\mathscr{H}^\perp}$.

6. An example. The purpose of this section is to give an example of a compact set K and a minimal function φ on K such that the uniformly closed algebra $\mathscr{U}$ generated by $\bar{\varphi}$ and $P(K)$ is not $C(K)$. (See Theorem 3.4.)

Let F be the cornucupia (that is, the closed unit disk, $\bar{D}$, with a ribbon which winds around from the outside and clusters on the boundary of the disk, ∂D). (For a picture, see [3, p. 152].) Let $K = F$ with the interior of the disk deleted. That is, $K = \{\text{ribbon}\} \cup \partial D$. Since the interior of K is simply connected we may choose a conformal map φ from int K onto the open unit disk D.

By [7, p. 280] we extend φ to be a one-to-one and continuous function on $\Lambda \cup \operatorname{int} K$, where Λ equals the boundary of the ribbon with ∂D delated. Since Λ is connected, $\varphi(\Lambda)$ is an arc on the unit circle.

Let $\partial D \backslash \varphi(\Lambda) = [\alpha, \beta]$. (It is clear that $\partial D \backslash \varphi(\Lambda)$ is a closed arc.) Since φ^{-1} has radial limits almost everywhere, because φ^{-1} is a bounded analytic function in the open disk, it can be shown that $[\alpha, \beta]$ consists of exactly one point, say $[\alpha, \beta] = \{\lambda\}$. We now exend the definition of φ to all of F by defining φ on $\bar{D}$ to be equal to λ where $\{\lambda\} = \partial D \backslash \varphi(\Lambda)$. A straightforward sequence argument shows φ is continuous on F, and by construction φ is analytic on $\operatorname{int} F$. Therefore by Mergelyan's Theorem [3, p. 48] $\varphi \in P(K)$. It now follows that φ is minimal on K.

What is the uniform algebra $\mathscr{U}$ generated in $C(K)$ by $\bar{\varphi}$ and the polynomials? It is easy to show that the maximal sets on antisymmetry for $\mathscr{U}$ are precisely the singleton sets $\{w\}$ for all w in $K \backslash \partial D$ together with the set ∂D. It then follows that a function f is an element of $\mathscr{U}$ if and only if $f \in C(K)$ and f can be extended to a function $\hat{f} \in C(\hat{K})(=C(F))$ such that $\hat{f}|_{\bar{D}} \in P(\bar{D})$.

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