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PRESERVATION OF LOCAL PROPERTIES AND CHAIN CONDITIONS IN COMMUTATIVE GROUP RINGS

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This note gives necessary and sufficient conditions for a commutative group ring to be locally Noetherian, locally Cohen-Macaulay, locally Gorenstein, or locally regular, and sufficient conditions for it to satisfy the chain condition or the second chain condition for prime ideals.

The purpose of this note is to extend certain results from [1] (and, as there, all rings are commutative with identity). The object in that paper was the description of a certain *UFD*, but the authors' results included necessary and sufficient conditions for $S[G]$ to be locally Noetherian for G an abelian group of finite torsion-free rank and S either a field or the ring of integers. They also gave conditions for a localization of $S[G]$ to be Cohen-Macaulay or regular for such coefficient rings S. Their proof is quite general; we use it here to give necessary and sufficient conditions for $S[G]$ to be locally Noetherian, locally Cohen-Macaulay, locally Gorenstein, or locally regular, for a general coefficient ring. It was also shown in [1] that $S[G]$ satisfies the second chain condition when S is a field and G is an abelian group of finite torsion-free rank; and Brewer suggested that this result might also extend to more general coefficient rings. We prove the result here for S a Noetherian Hilbert domain satisfying the second chain condition, and we see by example that the hypotheses "Noetherian" and "Hilbert" are both needed.

Our first lemma is Lemma 2 in [1].

LEMMA 1. *If G is an abelian group of finite torsion-free rank and F is a free subgroup of G such that G/F is torsion and $(G/F)_p$ is finite for some prime p, then G_p is finite and $G = G_p \oplus H$ where H contains a free subgroup F_1 such that H/F_1 is torsion and $(H/F_1)_p = 0$.*

We also need two facts about the Cohen-Macaulay property in commutative group rings.

LEMMA 2. *Let S be a Cohen-Macaulay ring, G an abelian group of finite torsion-free rank, and M a prime ideal in $S[G]$. If $S[G]_M$ is Noetherian, then $S[G]_M$ is Cohen-Macaulay.*

Proof. Let F be a free subgroup of G such that G/F is torsion, and set $M_0 = M \cap S[F]$. Since $S[G]$ is integral over $S[F]$, rank $M \leqq$ rank M_0. Since $S[G]_M$ is a faithfully flat extension of $S[F]_{M_0}$, a regular $S[F]_{M_0}$-sequence is a regular $S[G]_M$-sequence, so grade $MS[G]_M \geqq$ grade $M_0S[F]_{M_0}$. Now $S[F]_{M_0}$ is a localization of $S[X_1, \cdots, X_t]$ where $t = \text{rank } F$, so it is Cohen-Macaulay. Thus

$$\begin{aligned}\text{rank } MS[G]_M &\geqq \text{grade } MS[G]_M \geqq \text{grade } M_0S[F]_{M0} \\ &= \text{rank } M_0S[F]_{M_0} = \text{rank } M_0 \geqq \text{rank } M = \text{rank } MS[G]_M .\end{aligned}$$

LEMMA 3. *Let (S, P) be a Cohen-Macaulay ring of type r (in the sense of* [3], *Definition* 1.20), *and let G be a finitely generated abelian group. Let M be a prime ideal of $S[G]$ lying over P. Then $S[G]_M$ is also Cohen-Macaulay of type r.*

Proof. Since G is finitely generated, $S[G]$ is Noetherian, so $S[G]_M$ is Noetherian, and hence Cohen-Macaulay by Lemma 2. Let F be a free group and $F \rightarrow G$ a group epimorphism. Let $g_1, \cdots, g_r$ be a basis for the kernel of $F \rightarrow G$, and denote by M' the inverse image of M under the induced epimorphism $S[F] \rightarrow S[G]$. Now

$$S[F]_{M'}/PS[F]_{M'} = (S/P)[F]_{\overline{M'}} ,$$

a localization of a polynomial ring over a field, and hence of type 1. By [3], Satz 1.24, $S[F]_{M'}$ has type r. Now $X^{g_{i+1}} - 1$ is regular in $S[F]/(X^{g_1} - 1, \cdots, X^{g_i} - 1) = S[F/(g_1, \cdots, g_i)]$, and each $X^{g_i} - 1$ is in M', so $X^{g_1} - 1, \cdots, X^{g_r} - 1$ is a regular $S[F]_{M'}$-sequence. By [3], 1.22, $S[G]_M = S[F/(g_1, \cdots, g_r)]_M = S[F]_{M'}/(X^{g_1} - 1, \cdots, X^{g_r} - 1)_{M'}$ also has type r.

Let S be a commutative ring, G an abelian group of finite torsion-free rank, and $\tilde{M}$ a prime ideal of $S[G]$. Write $P = \tilde{M} \cap S$ and pick a free subgroup F of G such that G/F is torsion. The ideal $\tilde{M}$ contains at most one prime integer p, as represented in the prime subring of S. (If $\tilde{M}$ contains no prime integer, the last part of each statement below is vacuous.)

PROPOSITION 1. *The ring $S[G]_{\tilde{M}}$ is:*

(1) *Noetherian if S_p is Noetherian and $(G/F)_p$ is finite.*

(2) *Cohen-Macaulay of type r if S_P is Cohen-Macaulay of type r and $(G/F)_p$ is finite.*

(3) *regular is S_P if regular, $(G/F)_p$ is finite, and $G_p = 0$.*

Proof. (1) If $p \in \tilde{M}$, write $G = G_p \oplus H$ as in Lemma 1, and assume $F \subseteq H$ and $(H/F)_p = 0$; otherwise let $H = G$. Write $R = S[H]$ and $R_0 = S[F]$, and set $M = \tilde{M} \cap R$ and $M_0 = \tilde{M} \cap R_0$. We show first that $MR_M = M_0R_M$: For $m \in M$, $m \in T = S[E]$ for some finitely generated subgroup E of H containing F. Since E/F is finite, we can write $R_0 \subseteqq R_1 \subseteqq \cdots \subseteqq R_s = T$ where $R_j = S[E_j]$ and E_j is generated by E_{j-1} and $h_j \in E(E_0 = F)$. Set $M_j = M \cap R_j$. The order k of h_j in E/F is prime to p, so $k(X^{h_j})^{k-1}$ is a unit in $(R_j)_{M_j}$; and $(X^{h_j})^k \in R_{j-1}$, so by [5], Theorem (38.6), $M_{j-1}(R_j)_{M_j} = M_j(R_j)_{M_j}$. Thus $m/1 \in M_0(R_s)_{M_s} = M_0T_{M\cap T} \subseteqq M_0R_M$.

Let I be a finitely generated ideal of R and $r/1 \in \bigcap_{q=0}^{\infty}(IR_M + M^qR_M)$. Pick a new finitely generated subgroup E of H containing F such that r and a set of generators for I lie in $S[E] = T$. Since $T_{M\cap T}$ is Noetherian, and R_M is faithfully flat over it, and $R_0 \subseteqq T \subseteqq R$, we have $r/1 \in \bigcap_{q=0}^{\infty}[(I\cap T) + (M\cap T)^q]R_M \cap T_{M\cap T} = \bigcap_{q=0}^{\infty}[(I \cap T) + (M\cap T)^q]T_{M\cup T} = (I\cap T)T_{M\cap T} \subseteqq (I\cap T)R_M = IR_M$. With $I = 0$, this shows $\bigcap_{q=0}^{\infty} M^qR_M = 0$, so that R_M is a "local ring which may not be Noetherian", and for arbitrary I it shows that every finitely generated ideal of R_M is closed in the MR_M-adic topology. By [5], Theorem (31.8), R_M is Noetherian. Now $S[G]$ is a finitely generated R-module, so $(R\backslash M)^{-1}S[G]$ is a finitely generated R_M-module and hence is Noetherian; so $S[G]_{\tilde{M}}$ is Noetherian.

(2) By Lemma 2, $S[G]_{\tilde{M}}$ is Cohen-Macaulay. Now in the notation of (1), $(R_0)_{M_0}$ has type r by Lemma 3. Since $M_0R_M = MR_M$, Satz 1.24 of [3], applied to $(R_0)_{M_0} \subseteqq R_M$ yields the fact that R_M has type r. If $G = H$, we are finished. If $G = G_p \oplus H$, we can apply Lemma 3 to $S[G]_{\tilde{M}}$ as a localization of $R_M[G_p]$.

(3) In the notation of (1), $R = S[G]$ and $M = \tilde{M}$. Now $(R_0)_{M_0}$ is a localization of $S_P[X_1, \cdots, X_t]$ where $t = \operatorname{rank} F$, and so is regular; and $M_0R_M = MR_M$ (together with the fact that R_M is a faithfully flat extension of $(R_0)_{M_0}$) shows that R_M is regular.

The proof of (1) in Proposition 1 (and of (1) in the theorem below) is the proof of Theorem A in [1]. We repeat it here to establish notation for the other parts, and to make the minor changes needed to prove this proposition. Lady has pointed out the similarity of this argument to that used in the proof of the lemma in the middle of page 95 in [8].

Letting $\tilde{M}$ vary over the maximal ideals of $S[G]$ yields one direction of the theorem below. For the reversal of one implication we shall need:

LEMMA 4. *Let S be a ring and G be an abelian group. Let P be a prime ideal of S and H be a finite subgroup of G. Denote by M the* (prime) *ideal of $S[G]$ generated by P and the elements $X^g - 1$, $g \in G$. Then $\sum_{g\in H} X^g$ is nonzero in $S[G]_M$.*

Proof. For any element $f = \sum_{g \in G} a_g X^g$ of $S[G]$, $(\sum_{g \in H} X^g)f = \sum_{g \in G}(\sum_{g' \in g+H} a_{g'})X^g$, which is 0 only if $\sum_{g' \in g+H} a_{g'} = 0$ for each coset $g + H$ of H in G. This condition implies that $\sum_{g \in G} a_g = 0$, an element of P, so $f \in M$. It follows that $\operatorname{ann}(\sum_{g \in H} X^g) \subseteq M$.

THEOREM. *Let S be a commutative ring with identity and Ω be the set of prime integers which are nonunits in S. Let G be an abelian group of finite torsion-free rank and F a free subgroup of G such that G/F is torsion. Then $S[G]$ is:*

(1) *Locally Noetherian if and only if S is locally Noetherian and, for each p in Ω, $(G/F)_p$ is finite.*

(2) *Locally Cohen-Macaulay if and only if S is locally Cohen-Macaulay and for each p in Ω, $(G/F)_p$ is finite.*

(3) *Locally Gorenstein if and only if S is locally Gorenstein and, for each p in Ω, $(G/F)_p$ is finite.*

(4) *Locally regular if and only if S locally regular and, for each p in Ω, $(G/F)_p$ is finite and $G_p = 0$.*

Moreover, if S is locally Cohen-Macaulay and P is a prime ideal in $S[G]$ for which $S[G]_P$ is Noetherian, then $S[G]_P$ is Cohen-Macaulay.

Proof. Sufficiency of the conditions follows from Proposition 1 (including (3), since a Gorenstein ring is a Cohen-Macaulay ring of type 1). For the necessity, let P be a maximal ideal of S, and denote by M the maximal ideal of $S[G]$ generated by P and all elements of the form $X^g - 1$, $g \in G$. Then $S[G]_M$ is a faithfully flat extension of S_P, so if $S[G]_M$ is Noetherian, or Cohen-Macaulay, or Gorenstein, or regular, then S_P has the same property. Thus we need only verify the assertions about the group.

If $(G/F)_p$ is infinite for some p in Ω, then let P be a maximal ideal of S containing p and form M as in the last paragraph. We show $MS[G]_M$ is not finitely generated: Let I be any finitely generated ideal contained in M, and let H be a finitely generated subgroup of G containing F for which a set of generators for I is contained in $S[H]$. Since H is finitely generated, G/H contains an element $g + H$ of order p. Since $I \subseteq M$, the image of I in $S[G]/(\{X^g - 1 \colon g \in H\}) = S[G/H]$ is 0, but since $\operatorname{ann}(X^{g+H} - 1) = (\sum_{n=0}^{p-1} X^{ng+H})$ is contained in the image of M, we have that the image of any f in $S[G]\backslash M$ does not annihilate $X^{g+H} - 1$, so $f(X^g - 1) \notin I$, so $X^g - 1 \notin IS[G]_M$. Thus $IS[G]_M \neq MS[G]_M$.

Assume $G_p \neq 0$ for some p in Ω, and pick a maximal ideal P of S containing p; form M as above. Pick an element g of G_p of order p and consider $(X^g - 1)(\sum_{n=0}^{p-1} X^{ng}) = 0$. Neither factor is 0, even in

$S[G]_M$, so $S[G]_M$ is not a domain, and thus cannot be regular.

It may be possible to prove a version of the theorem's "Moreover" assertion with "Gorenstein" replacing "Cohen-Macaulay" but another proof will be required, since the condition of faithful flatness seems not to transfer enough properties to the larger ring.

We turn now to chain conditions on commutative group rings. The key facts used are drawn from the work of Ratliffe.

LEMMA 5. *Let R be a ring which satisfies the second chain condition and whose total quotient ring is 0-dimensional, and let S be an integral ring extension which is torsion-free as an R-module. Then S also satisfies the second chain condition.*

Proof. By [7], Corollary 2.17, this will follow if we can show that every minimal prime ideal in S meets R in a minimal prime; but if P is a minimal prime in S and $P \cap R$ is not minimal in R, then $\{tx: t$ is a regular element of R, $x \in S \backslash P\}$ is a multiplicatively closed set in S missing 0 but properly containing $S \backslash P$, a contradiction.

LEMMA 6. *Let R be a Hilbert ring and $x \in R$, not nilpotent. Then:*

(1) *R_x, the ring of quotients of R with denominators powers of x, is Hilbert; and*

(2) *the maximal ideals of R_x are the extensions of maximal ideals of R which do not contain x.*

Proof. A prime in R_x has the form PR_x where P is prime in R and $x \notin P$. Now P is the intersection of maximal ideals M_λ of R; it suffices to show $PR_x = \bigcap_\lambda (M_\lambda) R_x$, and one inclusion is clear. Let $a/x^n \in \bigcap_\lambda (M_\lambda) R_x$; then for each λ there is a positive integer k_λ for which $x^{k_\lambda} a \in M_\lambda$. If $x \notin M_\lambda$, then $a \in M_\lambda$ so xa is in each M_λ. Thus $xa \in P$, so $a/x^n = xa/x^{n+1} \in PR_x$. For (2), suppose PR_x is maximal. The prime P in R is an intersection of maximals, one of which misses x and hence survives in R_x, so P is maximal.

PROPOSITION 2. *Let R be a Noetherian domain and G an abelian group of finite torsion-free rank. If R satisfies the chain condition, then so does $R[G]$. If R is also Hilbert and satisfies the second chain condition, then $R[G]$ satisfies the second chain condition.*

Proof. Let F be a free subgroup of G such that G/F is torsion, and set $t = \operatorname{rank} F$. If R is a Noetherian domain satisfying the chain condition, then by [6], Theorem 3.6, $R[X_1, \cdots, X_t]$ satisfies the chain condition. By [5], Theorem (34.1), the chain condition passes to the localization $R[X_1, \cdots, X_t]_{X_1 \cdots X_t} = R[F]$ and then to the integral extension $R[G]$.

As noted in [5], page 123, in a domain of finite altitude the second chain condition is equivalent to the chain condition and equidimensionality. If R is a Noetherian Hilbert domain satisfying the second chain condition, then each stage in

$$R \subseteqq R[X_1] \subseteqq \cdots \subseteqq R[X_1, \cdots, X_t] \subseteqq R[F] \subseteqq R[G]$$

satisfies the chain condition, so it suffices to show that all but $R[G]$ are equidimensional. For the polynomial extensions, let M be a maximal ideal in $R[X]$; then $M \cap R$ is maximal in R and $(M \cap R)[X] \neq M$, so $\text{alt}\, R[X] \geqq \text{rank}\, M \geqq \text{rank}\, (M \cap R)[X] + 1 \geqq \text{rank}\, (M \cap R) + 1 = \text{alt}\, R + 1 = \text{alt}\, R[X]$. Lemma 6 shows that $R[F]$ is equidimensional, and Lemma 5 yields the result.

To see that the hypotheses of "Noetherian" and "Hilbert" are needed, we give examples of one-dimensional domains R for which equidimensionality fails in $R[G]$, even when G is infinite cyclic:

(1) A discrete (rank one) valuation domain V is Noetherian but not Hilbert, so $V[X]$ contains a maximal ideal lying over 0 in V. This ideal must have rank one, since only two primes in a chain in $V[X]$ can lie over the same prime in V; and it cannot contain X, so it survives in $V[X, X^{-1}]$. On the other hand, if P is the maximal ideal in V, then $P[X] + (X - 1)$ is a rank 2 maximal which survives in $V[X, X^{-1}]$.

(2) Let D be the Hilbert domain of Example 1 in [2], and let P and Q be the ideals described. The maximal ideals $P[Z] + (Z - 1)$ and $Q[Z] + (Z - 1)$ of $D[Z]$ both survive in $D[Z, Z^{-1}]$, but one has rank 3 and the other rank 2.

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