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NONOSCILLATION THEOREMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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The asymptotic behavior of nonoscillatory solutions of a class of nth order nonlinear functional differential equations with deviating argument is investigated. Sufficient conditions are provided which ensure that all nonoscillatory solutions (or all bounded nonoscillatory solutions) of the equations under consideration approach zero as the independent variable tends to infinity. The criteria obtained prove to apply to equations with advanced argument as well as to equations with retarded argument.

1. Introduction. We consider the nth order functional differential equation with deviating argument

$$(1)\quad (r_{n-1}(t)(r_{n-2}(t)(\cdots(r_2(t)(r_1(t)y'(t))')'\cdots)')')' + a(t)f(y(g(t))) = b(t)\ ,$$

where $a(t)$, $b(t)$, $g(t)$, $r_1(t)$, $\cdots$, $r_{n-1}(t)$ are real-valued and continuous on $[\tau, \infty)$ and $f(y)$ is real-valued and continuous on $(-\infty, \infty)$. The following conditions are assumed to hold throughout the paper:

$$(2)\quad \begin{aligned} &\text{(a)} \quad \lim_{t\to\infty} g(t) = \infty\ ;\\ &\text{(b)} \quad yf(y) > 0 \ \text{for}\ y \neq 0\ ;\\ &\text{(c)} \quad r_i(t) > 0 \ \text{and}\ \lim_{t\to\infty} \rho_i(t) = 0,\ \text{where}\\ &\qquad \rho_i(t) = \int_t^\infty \frac{\rho_{i-1}(s)}{r_i(s)}ds,\ i = 1, \cdots, n-1,\ (\rho_0(t) \equiv 1)\ . \end{aligned}$$

We note that the condition (2c) is satisfied if

$$(3)\quad \int_\tau^\infty \frac{dt}{r_i(t)} < \infty\ , \qquad i = 1, \cdots, n-1\ .$$

We restrict our consideration to those solutions $y(t)$ of (1) which exist on some ray $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)|: t_0 \leqq t < \infty\} > 0$$

for any $t_0 \in [T_y, \infty)$. Such a solution is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory.

In the oscillation theory of ordinary differential equations one of the important problems is to find sufficient conditions in order that all (bounded) nonoscillatory solutions of (1) tend to zero as $t \to \infty$. Since the work of Hammett [3] this problem has been the subject of a considerable amount of study and a number of results have been obtained. See, for example, Graef and Spikes [1], Grimmer [2], Kartsatos [4], Kusano and Onose [5], Londen [6], Singh [7], [8] and Singh and Dahiya [9].

The purpose of the present paper is to proceed further to add some new results to this problem. First, in the case where $a(t)$ is oscillatory, we present conditions in order that all bounded nonoscillatory solutions of (1) tend to zero as $t \to \infty$. Secondly, in the case where $a(t)$ is nonnegative, we provide conditions which force all nonoscillatory solutions of (1) to approach zero as $t \to \infty$. Incidentally, our results serve to strengthen recent results of Kartsatos [4], who gave conditions under which every (bounded) nonoscillatory solution of (1) satisfying (3) tends to a finite limit as $t \to \infty$.

2. Nonoscillation theorems. We begin with two lemmas that will be needed in the proof of our main results.

LEMMA 1. *Consider the differential equation*

$$u' - \frac{\rho'(t)}{\rho(t)}u + \frac{\rho'(t)}{\rho(t)}\phi(t) = 0\,, \tag{4}$$

where $\phi(t)$ *is continuous on* $[T, \infty)$, $\rho(t)$ *is continuously differentiable on* $[T, \infty)$ *and*

$$\rho(t) > 0\,, \quad \rho'(t) < 0\,, \quad \lim_{t\to\infty} \rho(t) = 0\,.$$

Let $u(t)$ *be the solution of* (4) *on* $[T, \infty)$ *satisfying* $u(T) = 0$. *Then,* $\lim_{t\to\infty} \phi(t) = \infty$[or $-\infty$] *implies* $\lim_{t\to\infty} u(t) = \infty$[or $-\infty$].

Proof. The solution $u(t)$ is given by the formula

$$u(t) = -\rho(t)\int_T^t \frac{\rho'(s)}{\rho^2(s)}\phi(s)ds\,, \quad t \geqq T\,.$$

If $\lim_{t\to\infty} \phi(t) = \infty$[or $-\infty$], then it is obvious that

$$\lim_{t\to\infty}\Big(-\int_T^t \frac{\rho'(s)}{\rho^2(s)}\phi(s)ds\Big) = \infty[\text{or } -\infty]\,.$$

Hence, by L'Hospital's rule,

$$\begin{aligned}\lim_{t\to\infty} u(t) &= \lim_{t\to\infty}\left|\Big(-\int_T^t \frac{\rho'(s)}{\rho^2(s)}\phi(s)ds\Big)'\Big/\Big(\frac{1}{\rho(t)}\Big)'\right| \\ &= \lim_{t\to\infty}\phi(t) = \infty[or\ -\infty]\,.\end{aligned}$$

LEMMA 2. *Let* $\sigma(t)$ *be continuous on* $[T, \infty)$ *and let* $v(t)$ *be con-*

tinuously differentiable on $[T, \infty)$. *If the limit* $\lim_{t\to\infty}[\sigma(t)v'(t) + v(t)]$ *exists in the extended real line* $R^{\#}$, *then the limit* $\lim_{t\to\infty} v(t)$ *exists in* $R^{\#}$.

Proof. If the conclusion is false, then there are numbers ξ and η such that

$$\liminf_{t\to\infty} v(t) < \xi < \eta < \limsup_{t\to\infty} v(t) .$$

We are able to select an increasing sequence $\{t_\nu\}_{\nu=1}^{\infty}$ with the following properties:

$$\lim_{\nu\to\infty} t_\nu = \infty , \qquad v'(t_\nu) = 0 , \qquad \nu = 1, 2, \cdots , \tag{5}$$

$$v(t_{2\nu-1}) < \xi , \qquad v(t_{2\nu}) > \eta , \qquad \nu = 1, 2, \cdots . \tag{6}$$

In view of (5) we see that the limit

$$\lim_{\nu\to\infty} [\sigma(t_\nu)v'(t_\nu) + v(t_\nu)] = \lim_{\nu\to\infty} v(t_\nu)$$

exists in $R^{\#}$. However, this is a contradiction, since (6) implies that the sequence $\{v(t_\nu)\}_{\nu=1}^{\infty}$ cannot have a limit in $R^{\#}$.

We are now in a position to state and prove our nonoscillation results. The following notation will be used: $a^+(t) = \max\{a(t), 0\}$, $a^-(t) = \max\{-a(t), 0\}$.

THEOREM 1. *Let the following conditions hold*:

$$\int^{\infty} \rho_{n-1}(t)a^+(t)dt = \infty , \tag{7}$$

$$\int^{\infty} \rho_{n-1}(t)a^-(t)dt < \infty , \tag{8}$$

$$\int^{\infty} \rho_{n-1}(t)\,|\,b(t)\,|\,dt < \infty . \tag{9}$$

Then, all bounded nonoscillatory solutions of (1) *tend to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1). We may suppose without loss of generality that $y(t) > 0$ for $t \geqq t_0$. By (2a) there exists $t_1 \geqq t_0$ such that $g(t) \geqq t_0$ for $t \geqq t_1$. Thus, $y(g(t)) > 0$ for $t \geqq t_1$. We define

$$G_0(t) = y(t) , \quad G_i(t) = r_i(t)G'_{i-1}(t) , \qquad i = 1, \cdots, n-1 , \tag{10}$$

$$u_k(t) = \int_{t_1}^{t} \rho_{n-k-1}(s)G'_{n-k-1}(s)ds , \qquad k = 0, 1, \cdots, n-1 . \tag{11}$$

An integration by parts yields

$$\begin{aligned}
u_{k-1}(t) &= \int_{t_1}^{t} \rho_{n-k}(s)G'_{n-k}(s)ds \\
&= \rho_{n-k}(t)G_{n-k}(t) - \rho_{n-k}(t_1)G_{n-k}(t_1) + \int_{t_1}^{t} \frac{\rho_{n-k-1}(s)}{r_{n-k}(s)}G_{n-k}(s)ds \\
&= \frac{\rho_{n-k}(t)r_{n-k}(t)}{\rho_{n-k-1}(t)}\rho_{n-k-1}(t)G'_{n-k-1}(t) - \rho_{n-k}(t_1)G_{n-k}(t_1) \\
&\quad + \int_{t_1}^{t} \rho_{n-k-1}(s)G'_{n-k-1}(s)ds \\
&= -\frac{\rho_{n-k}(t)}{\rho'_{n-k}(t)}u'_k(t) + u_k(t) - \rho_{n-k}(t_1)G_{n-k}(t_1) .
\end{aligned}$$

This shows that $u_k(t)$ satisfies the differential equation

$$\frac{\rho_{n-k}(t)}{\rho'_{n-k}(t)}u' - u + \phi_k(t) = 0 , \tag{12}$$

or equivalently,

$$u' - \frac{\rho'_{n-k}(t)}{\rho_{n-k}(t)}u + \frac{\rho'_{n-k}(t)}{\rho_{n-k}(t)}\phi_k(t) = 0 , \tag{13}$$

where

$$\phi_k(t) = u_{k-1}(t) + \rho_{n-k}(t_1)G_{n-k}(t_1) .$$

Since $u_k(t_1)=0$ by (11) and since $\rho_{n-k}(t) > 0$, $\rho'_{n-k}(t) < 0$, $\lim_{t\to\infty} \rho_{n-k}(t)=0$ by (2c), we apply Lemma 1 to (13) to conclude that $\lim_{t\to\infty} u_{k-1}(t) = \infty$[or $-\infty$] implies that $\lim_{t\to\infty} u_k(t) = \infty$[or $-\infty$]. Moreover, applying Lemma 2 to (12), we conclude that $\lim_{t\to\infty} u_k(t)$ exists in $R^{\#}$ whenever $\lim_{t\to\infty} u_{k-1}(t)$ exists in $R^{\#}$.

We now multiply both sides of (1) by $\rho_{n-1}(t)$ and integrate it over $[t_1, t]$. Then,

$$\begin{aligned}
&\int_{t_1}^{t} \rho_{n-1}(s)G'_{n-1}(s)ds + \int_{t_1}^{t} \rho_{n-1}(s)a^{+}(s)f(y(g(s)))ds \\
&\quad = \int_{t_1}^{t} \rho_{n-1}(s)b(s)ds + \int_{t_1}^{t} \rho_{n-1}(s)a^{-}(s)f(y(g(s)))ds .
\end{aligned} \tag{14}$$

We distinguish the following two cases:

$$\int_{t_1}^{\infty} \rho_{n-1}(t)a^{+}(t)f(y(g(t)))dt = \infty , \tag{15}$$

$$\int_{t_1}^{\infty} \rho_{n-1}(t)a^{+}(t)f(y(g(t)))dt < \infty . \tag{16}$$

Suppose (15) holds. In view of (8), (9) and the boundedness of $y(t)$

the right-hand side of (14) tends to a finite limit as $t \to \infty$, so that from (14) we see that $\lim_{t\to\infty} u_0(t) = -\infty$. Hence, by Lemma 1 applied to (13) with $k = 1$, we have $\lim_{t\to\infty} u_1(t) = -\infty$. Applying Lemma 1 again to (13) with $k = 2$, we find $\lim_{t\to\infty} u_2(t) = -\infty$. Repeating the same argument, we conclude that $\lim_{t\to\infty} u_{n-1}(t) = -\infty$, which implies that $\lim_{t\to\infty} y(t) = -\infty$. This, however, contradicts the assumption that $y(t)$ is positive. Consequently, (15) is impossible. Now, letting $t \to \infty$ in (14) and using (16), we see that $\lim_{t\to\infty} u_0(t)$ is finite. From Lemma 2 applied to (12) with $k = 1$ it follows that $\lim_{t\to\infty} u_1(t)$ exists in $R^\#$. This limit must be finite, since $\lim_{t\to\infty} u_1(t) = -\infty$ would imply $\lim_{t\to\infty} y(t) = -\infty$, a contradiction to the positivity of $y(t)$, and $\lim_{t\to\infty} u_1(t) = \infty$ would imply $\lim_{t\to\infty} y(t) = \infty$, a contradiction to the boundedness of $y(t)$. Continuing in this way, we conclude that $\lim_{t\to\infty} u_{n-1}(t)$ is finite. Therefore, $\lim_{t\to\infty} y(t)$ exists as a finite number. On the other hand, in view of (2b), (7) and (16) it is easy to verify that

$$\liminf_{t\to\infty} y(g(t)) = \liminf_{t\to\infty} y(t) = 0 .$$

Thus it follows that $\lim_{t\to\infty} y(t) = 0$, and the proof is complete.

EXAMPLE 1. Consider the equation

$$(t(t(t^2y'(t))')')' + y(\gamma t) = \gamma^{-1}t^{-1} , \quad t > 0 , \tag{17}$$

where γ is a positive constant (possibly greater than 1). We have $\rho_1(t) = \rho_2(t) = \rho_3(t) = t^{-1}$ and see that all conditions of Theorem 1 are satisfied. Hence, all bounded nonoscillatory solutions of (17) tend to zero as $t \to \infty$. In fact, $y(t) = t^{-1}$ is a solution of (17) having this property.

In the following theorem it will be shown that the conclusion of Theorem 1 still holds if the roles of $a^+(t)$ and $a^-(t)$ are interchanged.

THEOREM 2. *All bounded nonoscillatory solutions of* (1) *tend to zero as* $t \to \infty$ *if the following conditions are satisfied*:

$$\int^\infty \rho_{n-1}(t)a^+(t)dt < \infty , \tag{18}$$

$$\int^\infty \rho_{n-1}(t)a^-(t)dt = \infty , \tag{19}$$

$$\int^\infty \rho_{n-1}(t)\,|\,b(t)\,|\,dt < \infty . \tag{20}$$

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1) such that $y(g(t)) > 0$ for $t \geqq t_1$. A parallel argument holds if $y(g(t)) < 0$

for $t \geqq t_1$. Define the functions $G_i(t)$ and $u_k(t)$ by the formulas (10) and (11). Assume that

$$\int_{t_1}^{\infty} \rho_{n-1}(t)a^-(t)f(y(g(t)))dt = \infty .$$

Then, letting $t \to \infty$ in (14) and using (19), (20) and the boundedness of $y(t)$, we obtain $\lim_{t\to\infty} u_0(t) = \infty$, so that applying Lemma 1 to (13) with $k = 1$, we see that $\lim_{t\to\infty} u_1(t) = \infty$. Repeated application of this argument shows that $\lim_{t\to\infty} u_{n-1}(t) = \infty$, which implies that $\lim_{t\to\infty} y(t) = \infty$. But this contradicts the fact that $y(t)$ is bounded. Consequently, we must have

$$\int_{t_1}^{\infty} \rho_{n-1}(t)a^-(t)f(y(g(t)))dt < \infty .$$

The rest of the proof now proceeds exactly as in the second half of the proof of Theorem 1. The details are therefore omitted.

EXAMPLE 2. Consider the equation

$$(e^t y'(t))''' - e^t y^3(\log t) = t^{-3}e^t , \quad t \geqq 0 , \tag{21}$$

which has a nonoscillatory solution $y(t) = e^{-t}$ tending to zero as $t \to \infty$. It is easily verified that the conditions of Theorem 2 are satisfied with $\rho_1(t) = \rho_2(t) = \rho_3(t) = e^{-t}$. It follows that all bounded nonoscillatory solutions of (21) approach zero as $t \to \infty$.

Finally, we examine the equation (1) in which $a(t)$ is nonnegative and present conditons under which all nonoscillatory solutions are necessarily bounded and approach zero as $t \to \infty$.

THEOREM 3. *Let the condition* (3) *hold. Suppose that* $a(t) \geqq 0$, $\liminf_{y\to\infty} f(y) > 0$ *and* $\limsup_{y\to-\infty} f(y) < 0$. *If*

$$\int^{\infty} \rho_{n-1}(t)a(t)dt = \infty , \tag{22}$$

$$\int^{\infty} |b(t)|dt < \infty , \tag{23}$$

then all nonoscillatory solutions of (1) *tend to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). We may suppose that $y(g(t)) > 0$ for $t \geqq t_1$. Define $G_i(t)$ and $u_k(t)$ by (10) and (11). We shall first show that $y(t)$ is bounded above. From (1) we obtain

$$G_{n-1}(t) - G_{n-1}(t_1) + \int_{t_1}^{t} a(s)f(y(g(s)))ds = \int_{t_1}^{t} b(s)ds . \tag{24}$$

Since the first integral of (24) is positive and, by (23), the second integral is bounded, there exists a constant K_{n-1} such that

$$G_{n-1}(t) = r_{n-1}(t)G'_{n-2}(t) \leqq K_{n-1} \quad \text{for} \quad t \geqq t_1 .$$

Dividing the above inequality by $r_{n-1}(t)$ and integrating from t_1 to t, we get

$$G_{n-2}(t) - G_{n-2}(t_1) \leqq K_{n-1}\int_{t_1}^{t} \frac{ds}{r_{n-1}(s)} \quad \text{for} \quad t \geqq t_1 ,$$

which shows, in view of (3), that there exists a constant K_{n-2} such that

$$G_{n-2}(t) = r_{n-2}(t)G'_{n-3}(t) \leqq K_{n-2} \quad \text{for} \quad t \geqq t_1 .$$

Applying the above argument repeatedly, we have

$$G_{n-3}(t) \leqq K_{n-3}, \cdots, G_1(t) \leqq K_1, G_0(t) \leqq K_0 \quad \text{for} \quad t \geqq t_1 ,$$

where $K_{n-3}, \cdots, K_1, K_0$ are constants. It follows that $y(t)$ is bounded above for $t \geqq t_1$.

From this point on, we argue as in the proof of Theorem 1 on the basis of the relation

$$(25) \quad \int_{t_1}^{t} \rho_{n-1}(s)G'_{n-1}(s)ds + \int_{t_1}^{t} \rho_{n-1}(s)a(s)f(y(g(s)))ds = \int_{t_1}^{t} \rho_{n-1}(s)b(s)ds .$$

Noting that on account of (23) the right-hand side of (25) tends to a finite limit as $t \to \infty$, we can deduce from (25) that

$$(26) \qquad \int_{t_1}^{\infty} \rho_{n-1}(t)a(t)f(y(g(t)))dt < \infty ,$$

since otherwise we could use Lemma 1 to obtain $\lim_{t\to\infty} u_k(t) = -\infty$ for $k = 0, 1, \cdots, n-1$, which implies $\lim_{t\to\infty} y(t) = -\infty$, a contradiction. Next, using (25), (26) and applying Lemma 2, we can show that $\lim_{t\to\infty} u_k(t)$ is finite for each $k = 0, 1, \cdots, n-1$. Thus, $\lim_{t\to\infty} y(t)$ exists and is finite. On the other hand, from (22) and (26) we see that $\liminf_{t\to\infty} y(t) = 0$. Therefore, we conclude that $y(t)$ tends to zero as $t \to \infty$. This completes the proof.

EXAMPLE 3. Consider the equation

$$(27) \qquad (t^2(t^2(t^2y'(t))')')' + t^4y^3(\gamma t) = \gamma^{-6}t^{-2} , \quad t > 0 ,$$

where γ is a positive constant. In this case, we have $\rho_1(t) = t^{-1}$, $\rho_2(t) = (1/2)t^{-2}$, $\rho_3(t) = (1/6)t^{-3}$. Since all assumptions of Theorem 3 are satisfied, every nonoscillatory solution of (27) approaches zero as $t \to \infty$. This equation has a nonoscillatory solution $y(t) = t^{-2}$.

EXAMPLE 4. Consider the equation

$$(e^t(e^t(e^t y'(t))')')' + e^{3t}y(t+\theta) = (24 + e^{-4\theta})e^{-t}\,, \quad t \geqq 0\,, \tag{28}$$

where θ is a constant. This equation possesses $y(t) = e^{-4t}$ as a nonoscillatory solution tending to zero as $t \to \infty$. It is easy to verify that $\rho_1(t) = e^{-t}$, $\rho_2(t) = (1/2)e^{-2t}$, $\rho_3(t) = (1/6)e^{-3t}$, and the conditions of Theorem 3 are satisfied. Therefore, all other nonoscillatory solutions of (28) also tend to zero as $t \to \infty$.

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