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OSCILLATION PROPERTIES OF CERTAIN SELF-ADJOINT DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

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Assuming oscillation, a connection between the decreasing and increasing solutions of

(1) $$(ry'')'' = py$$

is established. With this result, it is shown that if $r \equiv 1$ and p positive and monotone the decreasing solution of (1) is essentially unique. It is also shown that if $p > 0$ and $r \equiv 1$ the decreasing solution tends to zero.

It will also be assumed that p and r are positive and continuous and at times continuously differentiable on $[a, +\infty)$. By an oscillatory solution of (1) will be meant a solution $y(x)$ such that there is a sequence $\{x_n\}_{n=1}^{\infty}$ diverging to $+\infty$ such that $y(x_n) = 0$ for every n. Equation (1) will be called oscillatory if it has an oscillatory solution.

Equation (1) has been studied previously by Ahmad [1], Hastings and Lazer [3], Leighton and Nehari [8] and Keener [7].

Hastings and Lazer [3] have shown that if $p > 0$, $r \equiv 1$ and $p' \geqq 0$ then (1) has two linearly independent oscillatory solutions which are bounded on $[a, +\infty)$. They further show that if $\lim_{t\to\infty} p(t) = +\infty$ then all oscillatory solutions tend to zero. Our result will show that there is a nonoscillatory solution which goes to zero "faster" than the oscillatory ones.

Keener [7] shows the existence of a solution y of (1) such that $\operatorname{sgn} y = \operatorname{sgn} y'' \neq \operatorname{sgn} y' = \operatorname{sgn}(ry'')'$. Under the additional hypothesis that $\liminf p(t) \neq 0$ he shows that $y(t) \to 0$ as $t \to \infty$. We will give a condition for $y(t) \to 0$ where $\liminf p(t)$ can be zero.

Ahmad [1] shows that if (1) is nonoscillatory then every solution z of (1) with the properties of y above satisfy $z = cy$ for some constant c.

The following lemmas due to Leighton and Nehari [8] will be basic in our investigation.

LEMMA 1. *If y is a solution of* (1) *with $y(c) \geqq 0$, $y'(c) \geqq 0$, $y''(c) \geqq 0$ and $(r(c)y''(c))' \geqq 0$ but not all zero for $c \geqq a$ then $y(x)$, $y'(x)$, $y''(x)$ and $(r(x)y''(x))'$ are positive for $x > c$.*

LEMMA 2. *If y is a solution of* (1) *with $y(c) \geqq 0$, $y''(c) \geqq 0$, $y'(c) \leqq 0$ and $(r(c)y''(c))' \leqq 0$ but not all zero for $c \geqq a$ then $y(x) > 0$, $y''(x) > 0$, $y'(x) < 0$ and $(r(x)y''(x))' < 0$ for $x \in [a, c)$.*

We will also use the following theorem of Keener [7].

THEOREM 1. *There exists a solution* $w(x)$ *of* (1) *which has the following property*:

$$\text{(P)}\qquad \begin{aligned} &w(x)w'(x)w''(x)[r(x)w''(x)]' \neq 0;\\ &\operatorname{sgn} w(x) = \operatorname{sgn} w''(x) \neq \operatorname{sgn} w'(x) = \operatorname{sgn}[r(x)w''(x)]' ;\\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } a \leqq x . \end{aligned}$$

We will first show a connection between the decreasing solution of (1) given by Theorem 1 and the solution that tends to ∞ given by Lemma 1. We will use the fact that if y_1, y_2 and y_3 are solutions of (1) then $r(x)W(y_1, y_2, y_3\colon x) = r(x)\det(y_i^{j-1}(x))$ $(i, j = 1, 2, 3, 4)$ is a solution of (1). Further we have

LEMMA 3. *If* y_1, y_2, y_3, y_4 *is a basis for the solution space of* (1) *then* $W_{123} = rW(y_1, y_2, y_3)$, $W_{124} = rW(y_1, y_2, y_4)$, $W_{134} = rW(y_1, y_3, y_4)$ *and* $W_{234} = rW(y_2, y_3, y_4)$ *is a basis for the solution space of* (1).

Proof. Let

$$A = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ ry_1'' & ry_2'' & ry_3'' & ry_4'' \\ (ry_1'')' & (ry_2'')' & (ry_3'')' & (ry_4'')' \end{vmatrix}$$

Then

$$\operatorname{adj} A = \begin{vmatrix} (rW_{234}'')' & -rW_{234}'' & W_{234}' & -W_{234} \\ -(rW_{134}'')' & rW_{134}'' & -W_{134}' & W_{134} \\ (rW_{124}'')' & -rW_{124}'' & W_{124}' & -W_{124} \\ -(rW_{123}'')' & rW_{123}'' & -W_{123}' & W_{123} \end{vmatrix}$$

$$= \begin{vmatrix} (rW_{234}'')' & rW_{234}'' & W_{234}' & W_{234} \\ (rW_{134}'')' & rW_{134}'' & W_{134}' & W_{134} \\ (rW_{124}'')' & rW_{124}'' & W_{124}' & W_{124} \\ (rW_{123}'')' & rW_{123}'' & W_{123}' & W_{123} \end{vmatrix} .$$

Thus since $\det A \neq 0$, $\det \operatorname{adj} A \neq 0$. Consequently W_{123}, W_{124}, W_{134} and W_{234} is a basis for the solution space of (1).

LEMMA 4. *Let* y_1, y_2, y_3, y_4 *be a basis for the solution space of* (1). *Then there is a basis for the solution space of* (1), z_1, z_2, z_3, z_4 *such that* $W_{123} = rW(z_1, z_2, z_3) = k_1y_1$, $W_{124} = rW(z_1, z_2, z_4) = k_2y_2$, $W_{134} =$

$rW(z_1, z_3, z_4) = k_3y_3$ and $W_{234} = rW(z_2, z_3, z_4) = k_4y_4$ where $k_i \neq 0$, $i = 1, 2, 3, 4$ is a constant.

Proof. Let u_1, u_2, u_3, u_4 be a basis for the solution space of (1). Then $rW(u_1, u_2, u_3)$, $rW(u_1, u_2, u_4)$, $rW(u_1, u_3, u_4)$, $rW(u_2, u_3, u_4)$ is also a basis for the solution space of (1) by Lemma 3. Thus each y_i is a linear combination of the rW's. Suppose $y = c_1rW(u_1, u_2, u_3) + c_2rW(u_1, u_2, u_4) + c_3rW(u_1, u_3, u_4) + c_4rW(u_2, u_3, u_4)$ where $c_1 \neq 0$. Letting $v_1 = c_1u_1 + c_4u_4$, $v_2 = c_2u_2 - c_2u_4$, $v_3 = c_1u_3 + c_2u_4$ and $v_4 = u_4$, we have $W(v_1, v_2, v_3) = c_1^2[c_1W(u_1, u_2, u_3) + c_2W(u_1, u_2, u_4) + c_3W(u_1, u_3, u_4) + c_4W(u_2, u_3, u_4)]$, $W(v_1, v_2, v_4) = c_1^2W(u_1, u_2, u_4)$, $W(v_1, v_3, v_4) = c_1^2W(u_1, u_3, u_4)$, $W(v_2, v_3, v_4) = c_1^2W(u_2, u_3, u_4)$. Repeating the argument three times gives the desired result.

LEMMA 5. *Let z be a nonoscillatory solution of* (1). *Then the solution space of*

$$z(ry'')' - z'ry'' + z''ry' - (rz'')'y = 0 \tag{2}$$

is a three dimensional subspace of (1). *Further, if z satisfies the conditions of Lemma* 1 *or Theorem* 1 *then* (2) *is oscillatory if and only if* (1) *is oscillatory.*

Proof. Using Lemma 4, choose solutions y_1, y_2, y_3 of (1) such that $kz = rW(y_1, y_2, y_3)$, where $k \neq 0$. Then

$$\begin{vmatrix} y_1 & y_2 & y_3 & y \\ y_1' & y_2' & y_3' & y' \\ ry_1'' & ry_2'' & ry_3'' & ry'' \\ (ry_1'')' & (ry_2'')' & (ry_3'')' & (ry'')' \end{vmatrix} = 0$$

is equivalent to (2). Thus, the first part of the lemma follows. It follows from Lemma 1 that if z satisfies the conclusion of the lemma and if y is a solution of (2) such that $y(d) = y'(d) = 0$, $r(d)y''(d) = 1$ where $d > c$, then $y(x) > 0$ for $x > d$, or using the definition of Hanan [2], (2) is C_{II}. In the same way it follows from Lemma 2 that if y is a solution of (2) where z satisfies (P) such that $y(d) = y'(d) = 0$, $r(d)y''(d) = 1$ then $y(x) > 0$ for $x \in [a, d)$, i.e. (2) is C_I [2]. Writing (2) is the form

$$(ry''/z)' + rz''y'/z^2 - (rz'')'y/z^2 = 0\ , \tag{3}$$

we have by [4, Theorem 3, p. 338] that (3) is $C_I(C_{II})$ if and only if

$$[(ry'/z)' + rz''y'/z^2]' = -(rz'')'y/z^2 \tag{4}$$

is $C_{II}(C_I)$. It then follows, using the methods of Hanan [2] that (3) is oscillatory if and only if (4) is oscillatory. Since z satisfies (2), choose a basis for the solution space of (2) of the form z, u_1, u_2. Then $zu_1' - u_1 z'$ and $zu_2' - u_2 z'$ satisfy (4) and

$$(5) \qquad (ry'/z^2) + [2rz''/z^3]y = 0 .$$

But Leighton and Nehari [8, p. 335, 3.4] show that (5) is oscillatory if and only if (1) is oscillatory. Thus the result follows.

THEOREM 2. *Suppose* (1) *is oscillatory. If there exist two linearly independent solutions* n_1 *and* n_2 *of* (1) *which satisfy* (P), *then there is a* $c \geqq a$ *and an oscillatory solution* u *of* (1) *such that* $u + N$ *is oscillatory, where* N *is the solution defined by* $N(c) = N'(c) = N''(c) = 0$, $(r(c)N''(c))' = 1$.

Proof. Consider the equation

$$(6i) \qquad n_i(ry'')' - n_i'ry'' + n_i''ry' - (rn_i'')y = 0 , \qquad i = 1, 2 .$$

By Lemma 5, each of the equations (6) are oscillatory and C_I. Since n_1 and n_2 are linearly independent, we can choose $c \geqq a$ such that $n_1'(c)n_2(c) - n_2'(c)n_1(c) \neq 0$. Let u_i be the solution of (6i) defined by $u_i(c) = u_i'(c) = 0$, $r(c)u_i'' = 1$ for $i = 1, 2$. Since (6i) is C_I and $u_i(c) = 0$, it follows that u_1 and u_2 are oscillatory solutions of (1). But $u_1(c) - u_2(c) = u_1'(c) - u_2'(c) = u_1''(c) - u_2''(c) = 0$, $(r(c)u_1''(c))' - (r(c)u_2''(c))' = n_1'(c)/n_1(c) - n_2'(c)/n_2(c) \neq 0$. Thus $u_1 - u_2$ is a multiple of N and the result follows.

THEOREM 3. *Suppose* (1) *is oscillatory. If there is a* $c \geqq a$ *and an oscillatory solution* u *of* (1) *such that* $u + N$ *is oscillatory, where* N *is the solution of* (1) *defined by* $N(c) = N'(c) = N''(c) = 0$, $(r(c)N''(c))' = 1$ *then* (1) *has a basis for the solution space with all oscillatory elements.*

Proof. Let z be a solution of (1) that satisfies (P). Then (2) is C_I and oscillatory. Thus there is a basis for the solution space of (2), say $\{u_1, u_2, u_3\}$, with all oscillatory elements [5]. Since N does not satisfy (2), there is a constant $0 < k < 1$ such that $u + kN$ is not in the solution space of (2). Since $u + N$ is oscillatory, $u + kN$ is oscillatory. Thus $\{u + kN, u_1, u_2, u_3\}$ is a basis for the solution space of (1).

THEOREM 4. *Suppose* (1) *has a basis for its solution space with all oscillatory elements. Then there are two linearly independent* *solutions* n_1 *and* n_2 *of* (1) *which satisfy* (P).

Proof. Suppose $\{y_1, y_2, y_3, y_4\}$ is a basis for the solution space of (1) with all oscillatory elements. By Lemma 4 there is a basis $\{z_1, z_2, z_3, z_4\}$ of (1) such that $W_{123} = k_1 y_1$, $W_{124} = k_2 y_2$, $W_{134} = k_3 y_3$, $W_{234} = k_4 y_4$ where $k_i \neq 0$ for $i = 1, 2, 3, 4$. Since y_1 is oscillatory, there is a sequence $\{x_i\} \to \infty$ such that $y_1(x_i) = 0$ for every i. Since $W_{123} = k_1 y_1$, for every x_i there are constants c_{i_j} for $j = 1, 2, 3$ such that $c_{i_1}^2 + c_{i_2}^2 + c_{i_3}^2 = 1$ and

$$u_i \equiv c_{i_1} z_1 + c_{i_2} z_2 + c_{i_3} z_3$$

has a triple zero at x_i. Since $\{c_{i_j}\}_{i=1}^{\infty}$ are bounded for $i = 1, 2, 3$, we can assume without loss of generality that

$$\lim_{i \to \infty} c_{i_j} = c_j \quad \text{for} \quad j = 1, 2, 3 \, .$$

Hence using Lemma 2 and an argument such as in [7, p. 281]

$$W_1 = c_1 z_1 + c_2 z_2 + c_3 z_3$$

satisfies (P). In the same way there are constants d_{i_j}, $i = 2, 3, 4,$; $j = 1, 2, 3$, such that

$$\begin{aligned} W_2 &\equiv d_{2_1} z_1 + d_{2_2} z_3 + d_{2_3} z_4 \\ W_3 &\equiv d_{3_1} z_1 + d_{3_2} z_2 + d_{3_3} z_4 \\ W_4 &\equiv d_{4_1} z_2 + d_{4_2} z_3 + d_{4_3} z_4 \end{aligned}$$

satisfy the (P). Clearly at least two of W_1, W_2, W_3, W_4 are linearly independent.

We will now use the above theorems to prove the following results for

$$(6) \qquad y^{iv} = p(x) y \, .$$

THEOREM 5. *Suppose* (6) *is oscillatory,* $p \in C'[a, +\infty)$ *and* p *is monotone. Then there is a unique solution of* (6) (up to constant multiples) *which satisfies* (P). *Further, a basis for the solution space of* (1) *has at most three oscillatory elements.*

Proof. Suppose there are two solutions of (6) that satisfy (P) and are linearly independent. Then by Theorem 1, there is a $c \geqq a$ and an oscillatory solution u of (6) such that $u + N$ is oscillatory, where N is the solution defined by $N(c) = N'(c) = N''(c) = 0$, $N'''(c) = 1$. By Lemma 1, $N(x)$, $N'(x)$, $N''(x)$, and $N'''(x)$ are positive for $x > c \geqq a$. Thus N, N' and N'' are unbounded. Mutliplying (6) by y' where y

is a solution of (6) and integrating from a to x, we obtain

$$\begin{aligned} G[y(x)] &= y''^2(x) - 2y'(x)y'''(x) + p(x)y^2(x) \\ &= G[y(a)] + \int_a^x p'(t)y^2(t)dt \, . \end{aligned}$$

Assuming that $p'(x) \leqq 0$, $G[y(x)]$ is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence of maximum points of $u''(x)$. Then $u''^2(x_n) \leqq u''^2(x_n) + p(x_n)u^2(x_n) = G[u(x_n)]$. But since $u + N$ is oscillatory and N'' is unbounded, u''^2 is unbounded, contradicting the boundedness of $G[y(x)]$. The second part of the conclusion follows from Theorem 4.

If $p'(x) \geqq 0$, Lazer and Hastings [3] have shown that all oscillatory solutions are bounded. The results then follow from the above theorems.

Whether or not the conclusion of Theorem 5 is true without the monotone condition on p is an open question.

We conclude with the following observation.

THEOREM 6. *If $n(x)$ is a solution of* (6) *satisfying the conditions of Theorem* 1 *where* (6) *is oscillatory, then* $\lim_{n\to\infty} n(x) = 0$

Proof. Equation (6) is oscillatory if and only if

$$(y'/n^2)' + (2n''/n^3)y = 0 \tag{7}$$

is oscillatory. But, as in [6] it can be shown that $\lim_{x\to\infty} x^2 n''(x) = 0$. Thus if $\lim_{x\to\infty} n(x) = c > 0$ (7) is nonoscillatory.

REFERENCES

1. S. Ahmad, *On the oscillation of solutions of a class of linear fourth order differential equations*, Pacific J. Math., **43** (1970), 289-299.
2. M. Hanan, *Oscillation criteria for third-order linear differntial equations*, Pacific J. Math., **11** (1961), 919-944.
3. S. P. Hastings and A. C. Lazer, *On the asymptotic behavior of solutions of the differential equation* $y^{(4)} = p(t)y$, Czechoslovak Math. J., **18** (1968), 224-229.
4. G. D. Jones, *Oscillation properties of* $y'' + p(x)y = f(x)$, Accademia Nazionale dei Lincei, **5** (1974), 337-341.
5. ———, *Oscillation properties of third order differential equations*, Rocky Mountain J. Math., **3** (1972), 507-517.
6. ———, *An asymptotic property of solutions of* $y''' + py' + qy = 0$, Pacific J. Math., **48** (1973), 135-138.
7. M. S. Keener, *On solutions of certain self-adjoint differential equations of fourth order*, J. Math. Anal. Appl., **33** (1971), 278-305.
8. W. Leighton and Z. Nehari, *On the oscillations of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc., **89** (1958), 325-377.

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