

## INTERPOLATING SEQUENCES FOR FUNCTIONS SATISFYING A LIPSCHITZ CONDITION

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Let  $D$  be the unit disk in  $C$ ,  $\text{Lip}(D)$  the space of functions,  $f$ , holomorphic in  $D$ , continuous on  $\bar{D}$  and satisfying a Lipschitz condition:

$$|f(z) - f(w)| \leq M |z - w| \quad \forall z, w \in \bar{D}$$

If  $S = \{a_i\}_{i=1}^\infty \in D$  is a discrete sequence with no accumulation points in  $D$ , let  $\text{Lip}(S)$  be the set of functions,  $g$ , defined on  $S$  satisfying

$$|g(a_i) - g(a_j)| \leq M |a_i - a_j| \quad \forall i, j = 1, 2, \dots$$

We say  $S$  is a  $\text{Lip}(D)$  interpolating ( $LI$ ) sequence if the restriction mapping from  $\text{Lip}(D)$  to  $\text{Lip}(S)$  given by  $f \rightarrow f|_S$  is surjective. Our aim is to describe some of the properties of such sequences and to give some examples. Specifically we show that an  $LI$  sequence must either be a uniformly separated sequence, or the union of two such sequences which approach one another as they tend to  $\partial D$ .

If  $f \in \text{Lip}(D)$  let  $M_f = \text{Inf} \{M: |f(z) - f(w)| \leq M |z - w| \forall z, w \in \bar{D}\}$ .  $M_f$  is a pseudo-norm on  $\text{Lip}(D)$ . If  $E$  is a closed subset of  $\bar{D}$ , let  $\text{Lip}_E(D) = \{f \in \text{Lip}(D): f|_E \equiv 0\}$ .  $M_f$  becomes a norm on  $\text{Lip}_E(D)$  which makes it a Banach space. Let  $H^\infty(D)$  be the space of bounded analytic function on  $D$ , for  $f \in H^\infty(D)$ , let  $\|f\| = \text{Sup} \{|f(z)|: z \in D\}$ .  $H^\infty(D)$  is related to  $\text{Lip}(D)$  by the following:

- (1)  $f \in \text{Lip}(D)$  iff  $f' \in H^\infty(D)$  in which case  $M_f = \|f'\|$ .

If  $S \subset \bar{D}$  is a set with no interior, define

$$\text{Lip } S = \{f: S \longrightarrow C: \exists M \text{ s.t. } |f(z) - f(w)| \leq M |z - w| \forall z, w \in S\}.$$

If  $E$  is a closed subset of  $\bar{S}$ , let  $\text{Lip}_E(S) = \{f \in \text{Lip}(S): f(z) \rightarrow 0 \text{ as } z \rightarrow E \text{ in } S\}$ . If  $f \in \text{Lip}(S)$  define  $M_f = \text{Inf} \{M: |f(z) - f(w)| \leq M |z - w| \forall z, w \in S\}$ . Again we see that  $M_f$  makes  $\text{Lip}_E(S)$  into a Banach space. We will say  $S$  is a  $\text{Lip}(D)$  interpolating ( $LI$ ) set if the restriction map,  $R: f \rightarrow f|_S$  is a surjection from  $\text{Lip}(D)$  onto  $\text{Lip}(S)$ . Similarly if  $E \subset \bar{S}$  is closed,  $S$  will be called a  $\text{Lip}_E(D)$  interpolating ( $L_EI$ ) set if  $R$  is a surjection from  $\text{Lip}_E(D)$  onto  $\text{Lip}_E(S)$ . Clearly if  $S$  is an  $LI$  set, then  $\bar{S}$  is an  $LI$  set and  $S$  is an  $L_EI$  set for all closed  $E \subset \bar{S}$ . If  $E$  is finite, then  $S$  is an  $LI$  set iff  $S$  is an  $L_EI$  set.

PROPOSITION 1. *If  $S$  is an  $LI$  set, there exists a constant  $m_s > 0$*

such that for every  $F \in \text{Lip}(S)$  there exists  $f \in \text{Lip}(D)$  such  $M_f \leq m_S M_F$  and  $f|_S = F$ .

*Proof.* Since  $\text{Lip}_{|z_0|} S$  and  $\text{Lip}_{|z_0|} D$  are Banach spaces, the open mapping theorem gives such an  $m_S$  for  $L_{|z_0|} I$  sets. If  $F \in \text{Lip}(S)$  then  $F - F(z_0) \in \text{Lip}_{|z_0|} S$  and  $M_{F-F(z_0)} = M_F$ . Therefore there exists  $g \in \text{Lip}_{|z_0|}(D)$   $M_g \leq m_S M_F$  and  $g|_S = F - F(z_0)$ . Let  $f = g + F(z_0)$ .

Taylor and Williams [5] and (independently) Korenblum [4] characterized the zero sets of  $\text{Lip}(D)$ , showing that there exists  $f \in \text{Lip}(D)$  such that  $\bar{S} = \{z \in D: f(z) = 0\}$  if and only if

$$(2) \quad \int_0^{2\pi} \log(d(e^{i\theta}))d\theta > -\infty \text{ where } d(e^{i\theta}) \text{ is the} \\ \text{(Euclidean) distance from } e^{i\theta} \text{ to } \bar{S}$$

and

$$(3) \quad S \cap D = \{z_i\}_{i=1}^\infty \text{ where } \sum_{i=1}^\infty 1 - |z_i| < \infty .$$

**PROPOSITION 2.** *If  $S$  is LI then  $\exists f \in \text{Lip}(D)$  such that  $f \not\equiv 0$  but  $f(S) \equiv 0$ . Hence  $S$  satisfies (2) and (3).*

*Proof.* Let  $z_0 \in \bar{D} \setminus \bar{S}$  then the distance,  $d(z_0, \bar{S})$ , between  $z_0$  and  $S$  must be bounded away from zero. Consequently  $F(z) = 1/(z - z_0) \in \text{Lip}(S)$ . So there exists  $g \in \text{Lip}(D)$  s.t.  $g(z) = 1/(z - z_0)$  for all  $z \in S$ . Clearly if  $f(z) = 1 - (z - z_0)g(z)$ , then  $f \in \text{Lip}(D)$ ,  $f(z_0) = 1$ , while  $f(S) \equiv 0$ .

We are interested primarily in the case where  $S = \{a_i\}_{i=1}^\infty$  is a sequence in  $D$ , and from now on we will always assume  $S$  is such a sequence. We relate LI sequences to interpolating sequences for  $H^\infty(D)$ .

If  $z, w \in D$ , we define the pseudo-hyperbolic distance,  $\rho(z, w) = |(z - w)/(1 - z\bar{w})|$ . If  $S = \{a_i\}_{i=1}^\infty$ , let  $B_S$  be the Blaschke product with zeros precisely at  $a_i = 1, 2 \dots$ .  $S$  is interpolating for  $H^\infty(D)$  if the evaluation mapping from  $H^\infty(D)$  to  $l^\infty$  given by  $f \rightarrow \{f(a_i)\}_{i=1}^\infty$  is surjective. A well known theorem of L. Carleson [1] is that  $S$  is interpolating for  $H^\infty(D)$  iff it is uniformly separated (US), i.e., there exists a constant  $m$  such that

$$\prod_{\substack{i \neq j \\ i=1}}^\infty \rho(a_i, a_j) > \frac{1}{m} \text{ for all } j .$$

We call the constant  $m$  in the above definition, a US bound on  $S$ .

**THEOREM 1.** *If  $S$  is LI then either*

(a)  *$S$  is US or*

(b)  *$S$  is not US but  $S = S_1 \cup S_2$  where*

$S_1 = \{b_i\}_{i=1}^\infty, S_2 = \{c_i\}_{i=1}^\infty$  *are both US and  $\text{Inf}_i \{\rho(b_i, c_i): b_i \neq c_i\} = 0$ .*

**REMARKS.** In case (b) we allow for the possibility that  $b_i = c_i$  for some (possibly infinitely many) but not all indices  $i$ .

**Lemma 1.** *Suppose  $T = \{d_j\}_{j=1}^\infty$  and suppose there exists a non-zero function  $f \in \text{Lip}(D)$  which vanishes on  $T$  then*

$$|f(z)| \leq M_f |z - d_j| |B_{T \setminus \{d_j\}}(z)| \quad \forall j = 1, 2, \dots$$

*Proof.*  $f \in H^\infty(D)$  so  $f = FB_T$  where  $F \in H^\infty(D)$ . Now fix  $j$ .  $|f(z)| = |f(z) - f(d_j)| \leq M_f |z - d_j| \forall z \in D$  i.e.,

$$\left| \frac{F(z)}{1 - \bar{d}_j z} B_{T \setminus \{d_j\}}(z) \right| \leq M_f \quad \forall z \in D.$$

Since  $F(z)/(1 - \bar{d}_j z) \in H^\infty(D)$  and  $|B_{T \setminus \{d_j\}}(z)| = 1$  almost everywhere on  $\partial D$  (here we are using (2)) it follows that

$$\left| \frac{F(z)}{1 - \bar{d}_j z} \right| \leq M_f \quad \forall z \in D$$

i.e.  $|f(z)| \leq M_f |B_T(z)| |1 - \bar{d}_j z| = M_f |B_{T \setminus \{d_j\}}(z)| |z - d_j|$ .

*Proof of Theorem.* If  $S$  is LI, Proposition 1 implies the existence of a constant  $m_S$  and function  $f_1, f_2, \dots, \in \text{Lip}(D)$  such that  $M_{f_i} \leq m_S$  and

$$f_i(a_j) = \begin{cases} 0 & \text{if } j \neq i \\ \text{Inf} \{ |a_i - a_k| : \forall k. k \neq i \} & \text{if } i = j. \end{cases}$$

Now fix  $i$ , and let  $T = S \setminus \{a_i\}$  then Lemma 1 gives

$$(4) \quad |f_i(a_i)| \leq m_S |B_{T \setminus \{a_j\}}(a_i)| |a_i - a_j| \quad \text{for all } j \neq i.$$

Choose  $a_j$  so that  $|a_i - a_j| \leq 2 |f_i(a_i)|$ . Then by (4)

$$(5) \quad 1/2m_S \leq \prod_{\substack{k \neq i, j}} \rho(a_i, a_k).$$

If  $\text{Inf} \{\rho(a_i, a_k): i \neq k\} = \varepsilon > 0$ , then by (5),

$$\prod_{\substack{j=1 \\ j \neq i}}^\infty \rho(a_i, a_j) \geq \frac{\varepsilon}{2m_S}$$

so that  $S$  is US.

Otherwise  $\text{Inf} \{ \rho(a_i, a_j) : \forall i, j, i \neq j \} = 0$ . In that case  $S$  is clearly not  $US$ .

If  $i$  and  $j$  are such that  $\rho(a_i, a_j) < 1/(2m_s)$  then (5) implies  $\rho(a_i, a_k) \geq 1/(2m_s)$ ,  $\rho(a_j, a_k) \geq 1/(2m_s)$  for  $k \neq i, j$ . Consequently the relation,  $a_i \sim a_j$  iff  $\rho(a_i, a_j) < 1/(2m_s)$  is an equivalence relation on  $S$  which partitions  $S$  into a collection,  $C$ , of equivalence classes of no more than two elements each. Let

$$S_1 = \{ a_{j_1} : \exists j_2 \geq j_1 \text{ s.t. } \{ a_{j_1}, a_{j_2} \} \in C \}$$

$$S_2 = \{ a_{j_2} : \exists j_1 \leq j_2 \text{ s.t. } \{ a_{j_1}, a_{j_2} \} \in C \} .$$

Then for  $\alpha = 1, 2$ , if  $a_{j_0} \in S_\alpha$ , (5) yields

$$\prod_{\substack{a_k \in S_\alpha \\ a_k \neq a_{j_0}}} \rho(a_k, a_{j_0}) \geq \frac{1}{4m_s^2} .$$

Consequently  $S_1$  and  $S_2$  are both  $US$ . Renaming the points:  $S_1 = \{ b_i \}_{i=1}^\infty$ ,  $S_2 = \{ c_i \}_{i=1}^\infty$  so that  $\{ b_i, c_i \} \in C$ , we get the desired decomposition.

There remains the question of whether any  $LI$  sequences exist. For example, it is possible to construct a  $US$  sequence, converging to only one point on  $D$ , but violating (2). Hence  $US$  need not imply  $LI$ . Theorem 2 and its corollary give some conditions under which  $US$  does imply  $LI$ .

For fixed  $t$ ,  $0 < t < 1$ , define the nontangential wedge,  $W_t = \{ z \in D : (1 - |z^2|)/|1 - z^2| > t \}$ .  $W_t$  is the region lying between two distinct circles with centers on the imaginary axis, which intersect one another at  $\pm 1$ . Let  $\hat{W}_t = D \cup [(\bar{D} \setminus \bar{W}_t)^*]$  where  $(-)$  means closure and  $*$  means reflection across the unit circle. Let  $\sigma(z) = (1 - z)/(1 + z)$ .  $\sigma$  maps  $D$  into the right half-plane. It takes  $W_t$  onto the region lying to the right of both lines,  $y = \pm(\sqrt{1 - t^2}/t)x$ , and it takes  $\hat{W}_t \setminus \bar{W}_t$  onto the region lying between the two lines.

**PROPOSITION 3.** *Suppose  $0 < t < 1$  and  $S = \{ a_n \}_{n=1}^\infty \subset W_t$  is  $US$ . Then if  $g(z) = (1 - z^2)B_S(z)$ ,  $g \in \text{Lip}(D)$ .*

We require the following elementary facts:

**Lemma 2.** *If  $\Omega \subset C$  is open, and  $f \in H^\infty(\Omega)$ , then for  $z \in \Omega$ ,  $|f'(z)| \leq \|f\|/d(z, \partial\Omega)$ , where  $d(z, \partial\Omega)$  is the Euclidean distance from  $z$  to  $\partial\Omega$ , and  $\|f\| = \text{Sup} \{ |f(z)| : z \in \Omega \}$ .*

This is proved by applying the Cauchy Integral Formula around circles centered at  $z$  with radii  $\leq d(z, \partial\Omega)$ .

**LEMMA 3** (Theorem 1 of [2]). *If  $S$  is  $US$  with  $US$  bound  $1/\delta$ ,*

then there exists a constant  $\beta$ ,  $0 < \beta < 1$ , depending only on  $\delta$ , such that if  $\rho(z, a_n) > \delta/2$  for all  $n$ ,  $|B_s(z)| \geq \beta$ .

LEMMA 4. For any  $t$ ,  $0 < t < 1$ , and any  $\varepsilon$ ,  $0 < \varepsilon < 1$ , there exists  $s > 0$  such that  $U_s(W_t) = \{z \in D: \exists w \in W_t \text{ with } \rho(z, w) < \varepsilon\} \subset W_s$ .

*Proof of Proposition 3.* Assume  $S$  has US bound,  $1/\delta$ . We will show  $g' \in H^\infty(D)$ . Upon differentiating  $g$ , we see that it will be sufficient to show  $(1 - z^2)B_s'(z)$  is bounded in  $D$ .

Take  $s$  as in Lemma 4, so that  $U_s(W_t) \subset W_s$ , whereby  $|B_s(z)| \geq \beta$  for  $z \in D \setminus W_s$  by Lemma 3. Now if  $z \in C \setminus \bar{S}$ ,  $B_s(z^*) = (B_s(z))^*$  (\* still means reflection across  $\partial D$ ). Hence by the definition of  $\hat{W}_s$ ,  $|B_s(z)| \leq 1/\beta$  for  $z \in \hat{W}_s \setminus D$ . Since  $|B_s(z)| \leq 1$  for  $z \in D$ , it follows that  $B_s \in H^\infty(\hat{W}_s)$ , and, by Lemma 2,  $|B_s'(z)| \leq 1/(\beta d(z, \partial \hat{W}_s))$  for all  $z \in \hat{W}_s$ . The proposition will be proved once we show  $d(z, \partial \hat{W}_s) \geq k|1 - z^2|$  for all  $z \in D$  and some constant,  $k$ .

If  $z \in D$ , let  $w_z \in \partial \hat{W}_s$  be such that  $|w_z - z| = d(z, \partial \hat{W}_s)$ . Clearly, if  $\text{Re}(z) \geq 0$ , then  $\text{Re}(w_z) \geq 0$ . Recalling the map  $\sigma(z) = (1 - z)/(1 + z)$ , we see that

$$2|z - w_z| \geq |\sigma(z) - \sigma(w_z)| \geq \sin \theta |\sigma(z)|$$

where  $\theta$  is the angle between the lines  $y = \pm(\sqrt{1 - s^2}/s)x$  and the imaginary axis. In fact,  $\sin \theta = \sqrt{1 - s^2}$ . Therefore, if  $z \in D$  and  $\text{Re}(z) \geq 0$ ,

$$2d(z, \partial \hat{W}_s) \geq \sqrt{1 - s^2} |\sigma(z)| \geq (1/4)\sqrt{1 - s^2} |1 - z^2|.$$

Since  $\hat{W}_s$  and  $D$  are symmetric about  $\text{Re}(z) = 0$  the desired inequality holds for all  $z \in D$ .

COROLLARY. If  $S_1, S_2 \subset W_t$  and  $S_1$  and  $S_2$  are both US then

$$(1 - z^2)B_{S_1 \cup S_2}(z) \in \text{Lip}(D).$$

THEOREM 2. Suppose  $0 < t < 1$ ,  $S = \{a_n\}_{n=1}^\infty \subset W_t$ , and  $S$  is US. Then

(a)  $S$  is LI.

(b) There exists a sequence  $S_1 = \{b_n\}_{n=1}^\infty$  such that  $S_1$  is US,  $S \cup S_1$  is LI, and  $\rho(a_n, b_n) \rightarrow 0$  as  $n \rightarrow \infty$

(c) If  $h \in \text{Lip}(S)$  and  $\{\lambda_n\}_{n=1}^\infty \in l^\infty$  then there exists  $f \in \text{Lip}(D)$  such that  $f|S = h$  and  $f'(a_n) = \lambda_n$  for all  $n$ .

The first of the following two lemmas is a straightforward calculation with infinite products; the second is due to J. P. Earle.

LEMMA 5. Suppose  $S = \{a_n\}_{n=1}^\infty \subset D$  is US with US bound  $1/\delta$ . If  $\rho(a_n, z_n) < 1/3\delta$ , then  $\{z_n\}_{n=1}^\infty$  is US, and there exists a constant,  $C_\delta$  such that  $|B_S(z_n)| \geq C_\delta \rho(a_n, z_n)$ .

LEMMA 6 (see [3]). Suppose  $S = \{a_n\}_{n=1}^\infty \subset D$  is US, then there exists a constant  $\eta_0 > 0$  such that for any  $\eta$ ,  $0 < \eta < \eta_0$ , and any sequence  $\{\lambda_n\}_{n=1}^\infty \in l^\infty$ , there exists a (US) sequence,  $S' = \{z_n\}_{n=1}^\infty \subset D$  and a complex constant,  $Q$ , such that  $\rho(a_n, z_n) < \eta$  and  $QB_{S'}(a_n) = \lambda_n$  for all  $n$ .

*Proof of Theorem 2.* Assume  $S$  has US bound  $1/\delta$ .

a) It is sufficient to show  $S$  is  $L_{\pm 1}I$ . So let  $h \in \text{Lip}_{\pm 1} S$ . This implies  $|h(a_n)|/|1 - a_n^2| \leq M_h$ . If  $\eta < \text{Min}\{1/3\delta, \eta_0\}$  (where  $\eta_0$  is as in Lemma 6), there exists a complex number  $Q$  and a US sequence  $S' = \{c_n\}_{n=1}^\infty$  such that  $\rho(a_n, c_n) < \eta$  and  $QB_{S'}(a_n) = h(a_n)/(1 - a_n^2)$  for all  $n$ . By Lemma 4 and proposition 3, if  $f(z) = Q(1 - z^2)B_{S'}(z)$ ,  $f \in \text{Lip}(D)$  and  $f|S = h$ . Therefore,  $S$  is  $LI$ .

b) Define  $b_n$  as follows. If  $1 - |a_n| < \delta/3$ , choose  $b_n$  so that  $|b_n - a_n| \leq 1/2(1 - |a_n|)^2$ . Otherwise, let  $b_n = a_n$ .

$$(6) \quad \rho(a_n, b_n) \leq |b_n - a_n|/(1 - |a_n|) < 1 - |a_n|.$$

Moreover,  $|1 \pm b_n| \geq |1 \pm a_n| - |a_n - b_n| \geq 1/2(1 - |a_n|)$ .

So  $|1 - b_n^2| \geq 1/4(1 - |a_n|)^2 \geq 12|a_n - b_n|$ .

Consequently,

$$(7) \quad \frac{|b_n - a_n|}{\rho(a_n, b_n)|1 - b_n^2|} = \frac{|1 - |b_n|^2 + b_n(\bar{b}_n - \bar{a}_n)|}{|1 - b_n^2|} \leq 3.$$

Now (6) and Lemmas 4 and 5 imply  $S_1 = \{b_n\}_{n=1}^\infty$  is US,  $S_1 \subset W_s$  for some  $s > 0$ , and  $\rho(a_n, b_n) \rightarrow 0$ , so  $S \cup S_1$  is not US.

If  $h \in \text{Lip}_{\pm 1}(S \cup S_1)$ , by part (a), we have  $F \in \text{Lip}(D)$  such that  $F|S = h|S$ . Let  $h_1 = h - F|(S \cup S_1)$ . Then  $|h_1(b_n)| = |h_1(b_n) - h_1(a_n)| \leq M_{h_1}|a_n - b_n|$ . Lemma 5 and (7) give  $|h_1(b_n)|/|(1 - b_n^2)B_S(b_n)| \leq 3M_{h_1}/C_\delta$ . By Lemmas 4, 5, and 6, there is a constant  $Q'$  and a US sequence,  $S_3$  contained in a wedge, such that  $Q'B_{S_3}(b_n) = h_1(b_n)/(1 - b_n^2)B_S(b_n)$  for all  $n$ . Let  $f(z) = F(z) + Q'(1 - z^2)B_{S_3 \cup S}(z)$ . Then  $f \in \text{Lip}(D)$  and  $f|(S \cup S_1) = h$ .

c) We note that  $B'_S(a_n) = B_{S \setminus \{a_n\}}(a_n)/(1 - |a_n|^2)$  so that if  $\{\gamma_n\}_{n=1}^\infty \in l^\infty$  and  $\Gamma = \text{Sup}_n |\gamma_n|$ , then  $|\gamma_n/(1 - |a_n|^2)B'_S(a_n)| \leq \Gamma/\delta$ . Applying Lemmas 4, 5, and 6 and the corollary to Proposition 3, we have a constant  $Q''$ , a US sequence  $S''$  (contained in a wedge) such that if  $g(z) = Q''(1 - z^2)B_{S \cup S''}(z)$ ,  $g \in \text{Lip}(D)$ ,  $g(a_n) = 0$ , and  $g'(a_n) = \gamma_n$  for all  $n$ .

Finally, if  $h \in \text{Lip}(S)$  and  $\{\lambda_n\}_{n=1}^\infty \in l^\infty$ , by part (a) there exists

$G \in \text{Lip}(D)$  such that  $G|_S = h$ . Letting  $\gamma_n = \lambda_n - G'(a_n)$ , and obtaining  $g$  as above, we see that  $f = G + g$  performs the desired interpolation.

We note that for  $0 < t < 1$ , if  $\Phi_t = \sigma((\sigma(z))^\alpha)$ , where  $\alpha = (2/\pi) \arccos(t)$ ,  $\Phi_t: D \rightarrow W_t$  and  $\Phi_t: \hat{W}_s \rightarrow D$  for an appropriately chosen  $s$ .

**COROLLARY.** *Suppose  $S = \{a_n\}_{n=1}^\infty \subset D$  and there exists  $t, 0 < t < 1$ , such that  $\Phi_t(S) = \{\Phi_t(a_n)\}_{n=1}^\infty$  is US. Then  $S$  is LI.*

*Proof.* The hypothesis guarantees that  $\partial D \cap \bar{S} \subset \{\pm 1\}$  and  $S$  is US. Let  $h \in \text{Lip}_{\pm 1} S$ , then  $h(a_n)/(1 - a_n^2) \in l^\infty$  so there exists a constant  $Q$  and a US sequence of points,  $S'$ , near  $\Phi_t(S)$  such that  $QB_{S'}(\Phi_t(a_n)) = h(a_n)/(1 - a_n^2)$ . A straight forward calculation, or an appeal to Lemma 2, shows that if  $f(z) = (1 - z^2)QB_{S'} \circ \Phi_t(z)$ , then  $f \in \text{Lip}(D)$ . Clearly  $f(a_n) = h(a_n)$ .

**REMARKS.** (1) Call any region in  $D$  which lies between two circular arcs that intersect twice on  $\partial D$  a nontangential wedge. Any such wedge can be mapped into one of the form  $W_t$  by a Mobius transformation. Since composition with Mobius transformations preserves  $\text{Lip}(D)$ , it follows that Theorem 2 remains true for sequences contained in a finite union of arbitrary nontangential wedges.

(2) By noting that  $S$  is  $H^\infty$  interpolating for  $\hat{W}_t$  in Theorem 2, and applying Lemma 2, we can eliminate the need for Lemma 6, simplifying (slightly) the proof of Theorem 2. On the other hand, the argument presented here shows that as in the  $H^\infty$  case, Lipschitz interpolation can be performed by (somewhat modified) Blaschke products.

(3) The corollary to Theorem 2 enables one to construct tangential LI sequences by taking a US sequence  $S_1 \subset W_t$  which is contained in no  $W_s$  for  $s > t$ , and letting  $S = \Phi_t^{-1}(S_1)$ .

For certain types of tangential sequences we can obtain conditions b) and c) of Theorem 2.

**DEFINITION.** If  $0 < c < \infty$ , let  $\Omega_c = \{z \in D: (1 - |z|^2)/|1 - z|^2 > c\}$ .  $\Omega_c$  is a disk in  $D$ , tangent to  $\partial D$  at 1 (with radius  $1/1 + c$ ).

**THEOREM 3.** *Suppose  $S = \{a_n\}_{n=1}^\infty \subset \Omega_{c_1} \setminus \Omega_{c_2}$  is US ( $0 < c_1 < c_2$ ). Then*

(a) *If  $S$  is LI, statements b) and c) of Theorem 2 hold. Otherwise,*

(b) *There exists a sequence  $S_1 = \{b_n\}_{n=1}^\infty$  with  $\rho(a_n, b_n) \rightarrow 0$ , such that if  $h \in \text{Lip}_{\bar{S}}(S_1 \cup S)$ , there exists  $f \in \text{Lip}(D)$  such that  $f|_{(S_1 \cup S)} = h$ .*

(c) If  $\{\lambda_n\}_{n=1}^\infty \in l^\infty$  there exists  $f \in \text{Lip}(D)$  such that  $f(a_n) = 0$  and  $f'(a_n) = \lambda_n$  for all  $n$ .

The proof of Theorem 3 is essentially the same as that of parts (b) and (c) of Theorem 2, once we have the following.

LEMMA 4a. For fixed  $c > 0$  and  $\varepsilon > 0$ , there exists  $d$ ,  $0 < d < c$ , such that  $U_\varepsilon(\Omega_c) = \{z \in D: \exists w \in \Omega_c \text{ with } \rho(z, w) < \varepsilon\} \subseteq \Omega_d$ .

PROPOSITION 3a. If  $S_1$  and  $S_2 \subset \Omega_c$  are US then  $(1 - z)^2 B_{S_1 \cup S_2}(z) \in \text{Lip}(D)$ .

*Proof.* It is sufficient to show  $(1 - z)^2 B_{S_j}(z) \in \text{Lip}(D)$  for  $j = 1, 2$ . Lemmas 3 and 4a enable us to take  $d < c$  such that  $|B_{S_j}(z)| > \beta$  for  $z \in D \setminus \Omega_d$ . Since composition with Mobious transformation preserves  $\text{Lip}(D)$ , we may as well assume that  $d = 1$ . If  $\hat{\Omega}_1$  is the half-plane,  $\{z: \text{Re}(z) < 1\}$ , it is clear that  $\hat{\Omega}_1 \setminus \bar{D} = (D \setminus \Omega_1)^*$ . Simple arithmetic shows that for  $z \in D$ ,  $d(z, \partial \hat{\Omega}_1) \geq 1/2 |1 - z|^2$ . The arguments of Proposition 3 can now be applied.

Finally, we note that the problem of describing  $LI$  sequences can be broken into two parts: describing  $LI$  sets on  $\partial D$ , and describing sequences,  $S$ , which are  $L_{\bar{S} \cap \partial D} I$  sequences.

PROPOSITION 4. Let  $S = \{a_n\}_{n=1}^\infty \subset D$  and let  $E = \bar{S} \cap \partial D$ . Then  $S$  is an  $LI$  sequence iff  $S$  is an  $L_E I$  sequence and  $E$  is an  $LI$  set.

*Proof.* Suppose  $S$  is  $LI$ . Then  $S$  is clearly  $L_E I$ . Let  $F \in \text{Lip}(E)$ . By a theorem of Valentine (see [6]),  $F$  can be extended to  $G \in \text{Lip}(\bar{S})$ . Hence there exists  $g \in \text{Lip}(D)$  such that  $g|_S = G|_S$ . Since  $g$  is uniformly continuous, it is clear that  $g|_E = F$ .

Conversely, if  $S$  is an  $L_E I$  sequence, and  $E$  is an  $LI$  set, let  $h \in \text{Lip}(S)$ .  $h$  can be extended to  $F \in \text{Lip}(\bar{S})$ . Hence there exists  $g \in \text{Lip}(D)$  such that  $g|_E = F|_E$ . If  $H = h - g|_S$ , then  $H \in \text{Lip}_E(S)$  and there exists  $G \in \text{Lip}(D)$  such that  $G|_S = H$ .  $(G + g) \in \text{Lip}(D)$  and  $(G + g)|_S = h$ .

REMARKS. (1) Theorems 1 and 2 should be compared with the results of S. A. Vinogradov [7] which deal with nontangential sequences for functions with derivatives in  $H^1(D)$ . In this case,  $US$  is necessary and sufficient for the appropriate interpolation problem.

(2) We are indebted to the referee whose remarks resulted in greatly simplified proofs of Propositions 3 and 3a. He also pointed out the existence of tangential  $LI$  sequences.



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