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REPRESENTATIONS BY SPINOR GENERA

J. S. HSIA

If f and g are two nonsingular quadratic forms with rational integral coefficients such that f represents g integrally over every p-adic fields and also over the reals, then it is a well-known classical result that the genus $\mathrm{Gen}(f)$ of f represents g. This paper considers the question of how many spinor genera in the genus of f will represent g, when f and g are integral forms defined over some fixed domain of algebraic integers and when $\dim(f) - \dim(g) \geqq 2$.

Unless otherwise mentioned the following general assumptions will be understood throughout this paper: F is an algebraic number field with R as its ring of algebraic integers, V and W are finite dimensional regular quadratic spaces over F with $\dim V - \dim W = d \geqq 2$, L and K are respectively R-lattices on V and W, and S is the set of all discrete spots on F. All unexplained notations and terminologies are from [6]. Suppose now that $L_{\mathfrak{p}}$ represents $K_{\mathfrak{p}}$ for every $\mathfrak{p} \in S$, then it is a well-known result that there is a lattice L' in the genus of L that represents K, provided V represents W (in fact, if W were a subspace of V, this L' may be chosen so as to contain K; see 102: 5, [6]). We introduce the notations $K \rightarrowtail \mathrm{Gen}(L)$, $\mathrm{Spn}(L)$, $\mathrm{Spn}^+(L)$, $\mathrm{Cls}(L)$, $\mathrm{Cls}^+(L)$ to denote respectively that K is representable by a member in the genus, spinor genus, proper spinor genus, class, proper class of L. Thus, in this notation, $L_{\mathfrak{p}}$ represents $K_{\mathfrak{p}}$ locally everywhere at $\mathfrak{p} \in S$ and $W \rightarrowtail V$ is equivalent to $K \rightarrowtail \mathrm{Gen}(L)$, which is, of course, the same as representation by $\mathrm{Gen}^+(L)$. We show here that if $d \geqq 3$ then $K \rightarrowtail \mathrm{Gen}(L)$ implies $K \rightarrowtail \mathrm{Spn}^+(L)$ so that in the indefinite case for L every proper class in the genus represents K. This fact must surely have been known to the specialists although I have not seen it in print and choose to record it here for completeness; its proof is quite standard and does not employ any of the subtler or deeper aspects of the theory. On the other hand, when $d = 2$, the theory is a good deal more intricate. We show that here too in most cases K is representable by every proper spinor genus in the genus of L; the exceptional cases will be pointed out, and there one needs to know the precise results for the local computations of the spinor norms of local integral rotations on $L_{\mathfrak{p}}$; the known facts about these are found in [3] for nondyadic $\mathfrak{p}$, in [1] for unramified dyadic $\mathfrak{p}$, and in [2] for arbitrary dyadic $\mathfrak{p}$ but with $L_{\mathfrak{p}}$ modular. This study was motivated by Kneser's paper [4], and the results as well as the method follow closely along his lines with some refinements necessitated by handling here also the cases where L may not be totally indefinite. Indeed, it may be definite.

Suppose for *all* spots on F we are given that $K_\mathfrak{p} \rightarrowtail L_\mathfrak{p}$, then surely $F_\mathfrak{p}K_\mathfrak{p} \rightarrowtail F_\mathfrak{p}L_\mathfrak{p}$ for every $\mathfrak{p}$. Hence, Hasse-Minkowski implies $W \rightarrowtail V$. Thus, we may as well assume at the outset that W is a regular subspace of V, and K is a lattice in V. Write $V = W \perp U$, and δ the discriminant (in O'Meara's sense) of U. As in Example 102: 5, [6] if we let $T = \{\mathfrak{p} \in S: L_\mathfrak{p} \not\supseteq K_\mathfrak{p}\}$, then T is a finite set. By lattice theory, there is a lattice L' on V such that

$$L'_\mathfrak{p} = \begin{cases} L_\mathfrak{p} & \text{for } \mathfrak{p} \notin T \\ s_\mathfrak{p} L_\mathfrak{p} & \text{for } \mathfrak{p} \in T \,. \end{cases}$$

Here $s_\mathfrak{p} \in O(V_\mathfrak{p})$, and the hypotheses permits us to choose $s_\mathfrak{p}$ so that $s_\mathfrak{p}L_\mathfrak{p} \supseteq K_\mathfrak{p}$. Clearly, $s_\mathfrak{p}$ can be assumed to be in $O^+(V_\mathfrak{p})$. Should $d \geqq 3$, then $\theta(O^+(U_\mathfrak{p})) = F_\mathfrak{p}^\times$ by 91: 6, [6]. Therefore, we may find a rotation $t_\mathfrak{p}$ on $U_\mathfrak{p}$ whose spinor norm $\theta(t_\mathfrak{p}) = \theta(s_\mathfrak{p})$. Extend $t_\mathfrak{p}$ trivially to a rotation on $V_\mathfrak{p}$ and composing it with $s_\mathfrak{p}$, one sees that $t_\mathfrak{p}s_\mathfrak{p}L_\mathfrak{p}$ still contains $K_\mathfrak{p}$. Thus, we may further assume that our original $s_\mathfrak{p}$ belongs to $O'(V_\mathfrak{p})$. This shows that L' belongs to the proper spinor genus of L. Thus, we have: if *$L_\mathfrak{p}$ represents $K_\mathfrak{p}$ at every (finite and infinite) spot $\mathfrak{p}$ and if $d = rk(L) - rk(K) \geqq 3$, then K is represented by every proper spinor genus in the genus of L; in particular, if L is indefinite with respect to S, then K is representable by every proper class in* Gen(L).

REMARK. Specializing this statement to the case when L is indefinite, $rk\,(K) = 1$, $F = \boldsymbol{Q}$, a theorem of Watson's [9] is recaptured. Suppose we permit W to be a degenerate space, say the radical Rad(W) has dimension r. Then, the same result prevails provided we have: $rk(L) - rk(K) = d \geqq 3 + r$. To see this, note that one can embed W in a nonsingular space $\widetilde{W} = \boldsymbol{H} \perp W_{an}$, where W_{an} is the anisotropic kernel of W, and $\boldsymbol{H}$ a hyperbolic space of dimension $2r$.

From here onward we assume that $d = 2$. Clearly, the group J_U of split rotations (adèles) on U may be viewed as a subgroup of J_V; similarly P_V, J'_V, J_L are defined in Chapter X, [6], as are the following subgroups of the idèle group J_F of F: P_F, P_D, J_F^L. We ask the following two basic questions:

(A): *If L represents K properly, is it true that for every $\phi \in J_U$ we have $K \rightarrowtail$* Spn$^+$ $(\phi(L))$?

(B): *What is the group index $[J_V: J_U P_V J'_V J_L]$?*

We shall see below that Question (A) has an affirmative answer (Theorem 1); and Theorem 2 will show that this group index mentioned above is less or equal to two. This means that at least half (if not all) of the proper spinor genera in the genus of L will represent K, provided there is just one lattice in the genus that represents K.

To treat Question (A), suppose $s: K \rightarrowtail L$ is a proper representation of K by L, choose any full lattice N on U (recall that $V = W \perp U$), and set $K' = K \perp N$. If $\phi \in J_U$ is given, clearly, the lattice $\phi(K')$ is just the lattice $K \perp \phi(N)$. If we select our N above to be the lattice $s^{-1}(L) \cap U$, then $s(K') = s(K \perp N) = s(K) \perp s(s^{-1}(L) \cap U) \subseteqq L + L = L$. Therefore, s also induces a proper representation of K' by L. Now, $rk(K') = rk(L)$. We have $s^{-1}(L) \supseteqq K'$ so that $\phi s^{-1}(L) \supseteqq \phi(K')$. Since J'_V contains the commutator subgroup of J_V, $J_V/P_VJ'_V$ is abelian. Hence, $\phi s^{-1} \in \phi P_V \subseteqq \phi P_V J'_V = P_V J'_V \phi$. This means there is a lattice in the proper spinor genus of $\phi(L)$ that represents $\phi(K')$. But, ϕ is trivial on K and therefore, we obtain:

THEOREM 1. *If $K \rightarrowtail \mathrm{Cls}^+(L)$, and $d = rk(L) - rk(K) = 2$, then $K \rightarrowtail \mathrm{Spn}^+(\phi(L))$ for every $\phi \in J_U$, where $FL = FK \perp U$. In particular, if L is indefinite with respect to the defining set S of spots on F, then $\phi(L)$ represents K properly for all such ϕ.*

Put $E = F(\sqrt{-\delta})$, where $\delta = \mathrm{disc}(U)$, and $D = \theta(O^+(V))$. The map from J_V into J_F given by $s = (s_\mathfrak{p}) \mapsto j = (j_\mathfrak{p})$ where $j_\mathfrak{p} \in \theta(s_\mathfrak{p})$ induces a monomorphism $\phi_L: J_V/P_VJ'_VJ_L \rightarrowtail J_F/P_DJ_F^L$ (which is an isomorphism when $rk(L) \geqq 3$). As a preliminary step toward determining the group index in Question (B), we have:

LEMMA. *ϕ_L described above induces an isomorphism:*

$$(*) \qquad \phi_{L,K}: J_V/J_UP_VJ'_VJ_L \twoheadrightarrow J_F/P_DN_{E/F}(J_E)J_F^L .$$

Proof. If $-\delta$ is a square in F, then $E = F$, and the right hand side in $(*)$ is trivial. But then, U is a hyperbolic plane and $\theta(O^+(U_\mathfrak{p})) = F_\mathfrak{p}^\times$. Hence, every split rotation on V may be composed with one U so as to have the resulting element lying inside J'_V, and this implies the left hand side of $(*)$ also is trivial. Therefore, we may suppose that $-\delta$ is a nonsquare. If $s \in J_U$, then $\theta(s_\mathfrak{p}) \in \theta(O^+(U_\mathfrak{p})) = \theta(O^+(\langle 1, \delta\rangle)) = Q(\langle 1, \delta\rangle)F_\mathfrak{p}^{\times 2} = N_{E_\mathfrak{P}/F_\mathfrak{p}}(E_\mathfrak{P}^\times)$ for $\mathfrak{P} | \mathfrak{p}$. Thus, the map which sends J_V into J_F mentioned above also sends J_U into $N_{E/F}(J_E)$ so that the map $\phi_{L,K}: J_V/J_UP_VJ'_VJ_L \rightarrow J_F/P_DN_{E/F}(J_E)J_F^L$ is well induced by ϕ_L. Since $rk(K) > 0$, $rk(L) \geqq 3$ necessarily so that our $\phi_{L,K}$ must be surjective as well by the above discussion. To see the kernel is precisely

$J_UP_VJ'_VJ_L$, consider $s \in J_V$ such that $\phi_{L,K}(\bar{s}) = \bar{1}$. This means that $\theta(s_\mathfrak{p}) = aj_\mathfrak{p}i_\mathfrak{p}F_\mathfrak{p}^{\times 2}$, where $a \in D$, $j = (j_\mathfrak{p}) \in N_{E/F}(J_E)$, and $i = (i_\mathfrak{p}) \in J_F^L$. Write $a = \theta(f)$, $f \in 0^+(V)$, $i_\mathfrak{p} = \theta(\Sigma_\mathfrak{p})$, $\Sigma_\mathfrak{p} \in O^+(L_\mathfrak{p})$. For each $\mathfrak{P}|\mathfrak{p}$, the local norm $N_{\mathfrak{P}|\mathfrak{p}}(E_\mathfrak{P}^\times)$ is either all of $F_\mathfrak{p}^\times$ or it is a subgroup $Q(\langle 1, \delta\rangle)$ of index two in $F_\mathfrak{p}^\times$. Therefore, we can find a local rotation $h_\mathfrak{p}$ on $U_\mathfrak{p}$ such that $\theta(h_\mathfrak{p}) = j_\mathfrak{p}F_\mathfrak{p}^{\times 2}$. Thus, $\theta(s_\mathfrak{p}) = \theta(f)\theta(h_\mathfrak{p})\theta(\Sigma_\mathfrak{p})$ which implies that $s_\mathfrak{p}$ belongs to $h_\mathfrak{p}f\Sigma_\mathfrak{p}\cdot O'(V_\mathfrak{p})$, or equivalently, s belongs to $J_UP_VJ'_VJ_L$, proving the lemma.

This lemma translates the index computation in Question (B) to an equivalent one in terms of idèles, which is usually more manageable, and we now take up this calculation.

LEMMA. $[J_F: P_DN_{E/F}(J_E)] = 2\cdot[F^\times: D]$.

Proof. Here D is characterized as the set of nonzero field elements from F that are positive at all real spots $\mathfrak{p}$ for which the quadratic space $V_\mathfrak{p}$ is definite. See 101:8, [6]. Let R denote the set of such real spots on F. Note that if $\mathrm{Card}(R) = t$, then $F^\times/D$ is a vector space of dimension t over $\boldsymbol{F}_2$ It is well-known that $[J_F: P_FN_{E/F}(J_E)] = 2$; see 65:21, [6]. Therefore, $[J_F: P_DN_{E/F}(J_E)] = 2\cdot[P_FN_{E/F}(J_E): P_DN_{E/F}(J_E)] = 2[P_F: P_F \cap P_DN_{E/F}(J_E)] = 2\cdot[P_F: P_D] = 2\cdot[F^\times: D]$. Only the second last equality requires some explanations. If $x \in F^\times$, and $d \in D$, then x/d belongs to $N_{E/F}(J_E)$ implies, in particular, that at each real spot $\mathfrak{p}$ from R, x/d is a local norm at $\mathfrak{p}$. But $V_\mathfrak{p}$ is anisotropic so that $-\delta$ is a nonsquare at $\mathfrak{p}$. Hence, the local norms at $\mathfrak{p}$ consist of all the positive reals. This means x/d is positive at $\mathfrak{p}$, and so x is positive at $\mathfrak{p}$. Therefore, $x \in D$.

LEMMA. *$J_F^L \subseteq P_DN_{E/F}(J_E)$ if and only if $J_F^L \subseteq N_{E/F}(J_E)$.*

Proof. This is the type of result that is typically bewildering and yet at the same time powerfully evident of the beauty and depth of the arithmetic of global fields. For the proof, clearly it suffices to prove the "only if" part. $L_\mathfrak{p}$ is unimodular almost everywhere. So, let T be the set of discrete spots $\mathfrak{p}$ on F for which $\theta(O^+(L_\mathfrak{p}))$ is not contained in $(N_{E/F}(J_E))_\mathfrak{p}$. So, T is a finite set. If T is not empty, there must be an $x_\mathfrak{p} \in \theta(O^+(L_\mathfrak{p}))$ not lying in $N_{E_\mathfrak{P}/F_\mathfrak{p}}(E_\mathfrak{P}^\times)$ for $\mathfrak{P}|\mathfrak{p}$. This means $x_\mathfrak{p}$ is not represented by the binary quadratic space $\langle 1, \delta\rangle$ over $F_\mathfrak{p}$. Consider the idèle $i = (i_\mathfrak{p})$ where $i_\mathfrak{q} = x_\mathfrak{p}$ at $\mathfrak{q} = \mathfrak{p}$, and $i_\mathfrak{q} = 1$ elsewhere. Surely, j belongs to J_F^L. Hence, by hypotheses, there exists $d \in D$ such that $dj \in N_{E/F}(J_E)$. This means d is a local norm at all the spots $\mathfrak{q} \neq \mathfrak{p}$. Hence, by Hilbert Reciprocity Law, d is also a local norm at $\mathfrak{p}$. On the other hand, $dx_\mathfrak{p}$ is a local norm at $\mathfrak{p}$. So, we arrive at a contradictory conclusion that $x_{\mathfrak{p}}$ is a local norm after all. Therefore, T must be empty and $J_F^L \subseteqq N_{E/F}(J_E)$.

THEOREM 2. $[J_F: P_D N_{E/F}(J_E) J_F^L] \leqq 2$.

Proof. If $D = F^\times$ (i.e. when L is totally indefinite with respect to S), then already $[J_F: P_F N_{E/F}(J_E)] = 2$ in which case the index is two if and only if J_F^L is contained in $N_{E/F}(J_E)$ by lemma. So, we may assume that $D \neq F^\times$.

Let $R = \{\mathfrak{p}_1, \cdots \mathfrak{p}_t\}$ be the set of all real spots on F for which $V_{\mathfrak{p}_i}$ is anisotropic, and let e_i $(1 \leqq i \leqq t)$ be the idèle which has -1 as its component at $\mathfrak{p}_i$ and 1 elsewhere. Clearly, all these e_i's belong to J_F^L but not to $N_{E/F}(J_E)$, and therefore, also not to $P_D N_{E/F}(J_E)$ by Lemma. Define the chain of subgroups of $P_D N_{E/F}(J_E) J_F^L$ by: $G_0 = P_D N_{E/F}(J_E)$, $G_j = I_F(p_j) G_{j-1}$ for $1 \leqq j \leqq t$, where $I_F(\mathfrak{p}_j)$ denotes the group of idèles which have all the components different from $\mathfrak{p}_j$ the value 1. Thus, we have an increasing tower:

$$P_D N_{E/F}(J_E) = G_0 \leqq \cdots \leqq G_t .$$

We assert that all the inclusions are strict. If not, we shall have at some j, $I_F(\mathfrak{p}_j) \subseteqq G_{j-1} = I_F(\mathfrak{p}_1) \cdots I_F(\mathfrak{p}_{j-1}) P_D N_{E/F}(J_E)$. The idèle e_j surely belongs to $I_F(\mathfrak{p}_j)$ and this means for some $d \in D$ and $\eta \in I_F(\mathfrak{p}_1) \cdots I_F(\mathfrak{p}_{j-1}) N_{E/F}(J_E)$ one has $e_j = (d)\eta$. But, at all the spots outside of R, $\eta_{\mathfrak{p}} = d^{-1}$ which implies that d^{-1}, hence also d, is a local norm. Inside of R the element d is positive at each $\mathfrak{p}$ and so is also a local norm. Thus, d is itself a global norm. Therefore, we conclude that e_j belongs to $I_F(\mathfrak{p}_1) \cdots I_F(\mathfrak{p}_{j-1}) N_{E/F}(J_E)$. On the other hand, at $\mathfrak{p}_j$, e_j is negative whereas every element from $I_F(\mathfrak{p}_1) \cdots I_F(\mathfrak{p}_{j-1}) N_{E/F}(J_E)$ has positive component at $\mathfrak{p}_j$. This contradiction proves our assertion.

Since for each $j (1 \leqq j \leqq t)$ we have $[G_j: G_{j-1}] = 2$, we obtain:

$$\begin{aligned} 2^t = [G_t: G_0] &\leqq [P_D N_{E/F}(J_E) J_F^L: P_D N_{E/F}(J_E)] \\ &= [J_F: P_D N_{E/F}(J_E)] \div [J_F: P_D N_{E/F}(J_E) J_F^L] \\ &\leqq [J_F: P_D N_{E/F}(J_E)] = 2^{t+1} . \end{aligned}$$

This proves the theorem.

COROLLARY. *If $D \neq F^\times$ (i.e., V is not totally indefinite with respect to S), then it is not possible for $J_F^L \subseteqq P_D N_{E/F}(J_E)$. In particular, $J_F = P_F N_{E/F}(J_E) J_F^L$, and $[J_F: P_D N_{E/F}(J_E) J_F^L] = 2$ if and only if $P_F \not\subseteqq P_D N_{E/F}(J_E) J_F^L$.*

Suppose $D = F^\times$. Then, $[J_F: P_D N_{E/F}(J_E) J_F^L] = 2$ if and only if

$J_F^L \subseteqq N_{E/F}(J_E)$ by Lemma. In particular, if $\mathfrak{p}$ is any real spot on F, then $\mathfrak{p}$ splits in E and $\delta \underset{\mathfrak{p}}{<} 0$. Hence, δ is a totally negative element. Moreover, at each discrete spot we must have $\theta(O^+(L_\mathfrak{p})) \subseteqq N_{\mathfrak{P}|\mathfrak{p}}(E_\mathfrak{P}^\times)$ for $\mathfrak{P}|\mathfrak{p}$; equivalently, $\theta(O^+(L_\mathfrak{p})) \subseteqq Q(\langle 1, \delta\rangle)$ over $F_\mathfrak{p}$. Note that if $\mathfrak{p}$ does not divide the volume $\mathrm{Vol}(L)$ of L, then $L_\mathfrak{p}$ is unimodular. Hence, as $rk(L) \geqq 3$ here, $\theta(O^+(L_\mathfrak{p})) = U_\mathfrak{p} F_\mathfrak{p}^{\times 2}$ unless $\mathfrak{p}$ is dyadic and the norm generator has *odd* order parity with respect to the weight generator for $L_\mathfrak{p}$. See [2]. In the exceptional cases, the spinor norm group is all of $F_\mathfrak{p}^\times$. By the local theory of quadratic forms, we see that $\mathrm{ord}_\mathfrak{p}(\delta)$ must be even; modulo squares in $F_\mathfrak{p}$, $-\delta$ is a unit of quadratic defect $4R_\mathfrak{p}$. Hence $E_\mathfrak{P}/F_\mathfrak{p}$ must be quadratic unramified. If $\mathfrak{p}$ is an exceptional dyadic prime, then $-\delta \in F_\mathfrak{p}^{\times 2}$. Therefore, *the only ramified primes for E/F must also divide* $\mathrm{Vol}(L)$. In particular, if L is unimodular, E/F must itself be quadratic unramified. Of course, there are number fields F for which every finite (let alone only quadratic) extension is ramified.

Finally, we point out here that the local lattice representation theory is completely determined when: (i) $\mathfrak{p}$ nondyadic, (ii) $\mathfrak{p}$ unramified dyadic, and (iii) $\mathfrak{p}$ arbitrary dyadic but $L_\mathfrak{p}$ modular. For (i) and (ii), see [7]; for (iii), see [8]. Also, see [5] for $\mathfrak{p}$ arbitrary dyadic but $rk(L_\mathfrak{p}) = 2$.

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THE OHIO STATE UNIVERSITY