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SUBSTITUTION IN NASH FUNCTIONS

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Let D be a domain in R^n. In this paper D is assumed to be defined by a finite number of strict polynomial inequalities. A Nash function on D is a real valued analytic function $f(x)$ such that there exists a polynomial $p(z, x_1, \cdots, x_n)$ in $R[z, x_1, \cdots, x_n]$ such that $p(f(x), x) = 0$ for all x in D. Let A_D be the ring of such functions on D. For any real closed field L containing R, use the Tarski-Seidenberg theorem to extend f to a function from a domain D_L (defined by the same inequalities as D), $D_L \subseteqq L^{(n)}$, to L. Now let $\varphi: A_D \rightarrow L$ be a homomorphism. Since $R[x_1, \cdots, x_n] \subset A_D$, $\varphi x = (\varphi x_1, \cdots, \varphi x_n)$ is a well defined point in $L^{(n)}$ and is in D_L. So $f(\varphi x)$ is defined for any f in A_D. In this paper it is shown that $f(\varphi x) = \varphi f$. From this result one can deduce Mostowski's version of the Hilbert Nullstellensatz for A_D.

As for the Nullstellensatz, since D. Dubois [2], and J. J. Risler [8], independently proved the real Nullstellensatz for polynomial rings, there have been various successful attempts to extend the result to other types of rings, for example, [4], [9]. In [5], a partial result was obtained for Nash rings and then, in [7], T. Mostowski proved the Nullstellensatz for Nash rings. There is still a question as to whether the result holds for Nash rings on more general domains than those considered here.

1. Mostowski's theorem.

We first recall some definitions.

DEFINITION 1. A set C contained in R^n is said to be semi-algebraic if it is defined by Boolean operations (finite union, finite intersection, complement) on sets of the form $\{a \in R^n \mid p(a) > 0$, for $p(x)$ in $R[x_1, \cdots, x_n]\}$. That is, C is defined by a finite number of polynomial inequalities.

DEFINITION 2. Let D be a set defined by a finite intersection of sets of the form $\{a \in R^n \mid p(a) > 0\}$. Then $A_D = \{f: D \rightarrow R$ such that f is analytic on D and there exists a polynomial $p(z, x)$ in $R[z, x_1, \cdots, x_n]$ such that for all x in D, $p(f(x), x) = 0\}$. This ring is called the ring of Nash functions on D.

DEFINITION 3. We wish to define certain subrings of $A_D = A$. Namely, let $B_0 = R(x_1, \cdots, x_n) \cap A_D$. Let $B_1 = \bigvee B_0(\sqrt{f})$ for f in B_0 and $f > 0$ on D. Let $B_2 = \bigvee B_1(\sqrt{f})$ for f in B_1 and $f > 0$ on D.

MOSTOWSKI'S THEOREM. *Let D be as above and let C_1 and C_2 be two disjoint closed semi-algebraic sets contained in D. Then there exists a function g in B_2 such that $g(C_1) > 0$ and $g(C_2) < 0$.*

We will give a proof of this result in this section which is similar to Mostowski's proof, but, by proving a stronger version of Thom's lemma (the Separation Lemma below), we are able to simplify the finish of the proof of Mostowski's theorem.

SEPARATION LEMMA. *We start with a finite number of polynomials $f_1(x_1, \cdots, x_n), \cdots, f_s(x_1, \cdots, x_n)$ in $R[x_1, \cdots, x_n]$. Then the roots of the f_i divide up R^n into a union of semi-algebraic sets. By Theorem 2.1 in [5], we can further divide up the sets so $R^n = \bigcup T_i$, a finite disjoint union of connected semi-algebraic sets bounded by the zeros of the f_i's. We now claim we can find:*

(a) *a further finite subdivision of each $T_i = \bigcup T_{ij}$ a disjoint union of semi-algebraic sets,*

(b) *a finite number of polynomials $f_1, \cdots f_s, f_{s+1}, \cdots, f_m$ derivable from the original polynomials, so that*

(1) *$\text{Sign}\, f_k(T_{ij})$ is constant where $\text{Sign}\,(f) = +, -,$ or 0.*

(2) *Given i_1, i_2 so that $\bar{T}_{i_1} \cap \bar{T}_{i_2} = \emptyset$, then for all j_1, j_2 there exists some f_k with either*

$$f_k(T_{i_1 j_1}) \geqq 0 \quad \textit{and} \quad f_k(T_{i_2 j_2}) < 0\,;$$

or

$$-f_k(T_{i_1 j_1}) \geqq 0 \quad \textit{and} \quad -f_k(T_{i_2 j_2}) < 0\,.$$

Proof. We consider the polynomials $f_1, \cdots, f_s$ as polynomials in x_n with coefficients in $R[x_1, \cdots, x_{n-1}]$. We can divide up R^{n-1} into a disjoint union $\bigcup S_j$ of a finite number of connected semi-algebraic sets so that above each S_j, the polynomials $\partial^k f_i/\partial x_n^k$ have a constant number of real roots and none intersect. We let the T_j's be the regions above the S_i either defined as a root of one of the $\partial^k f_i/\partial x_n^k$ or a connected region bounded by adjacent roots. We first check Thom's lemma which asserts that regions above a fixed S_i are separated by the partials of the f_j. It is clearly enough to check this for one $f_j = f$.

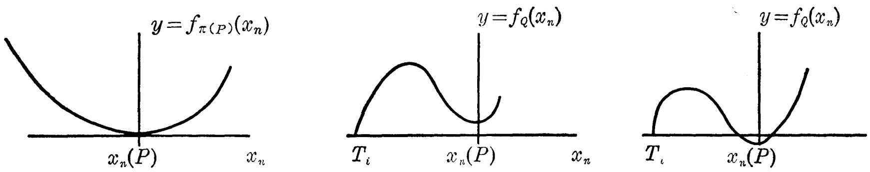
Note, if the regions have a simple root of f between them, then f itself will separate unless there is more than one root of f in between, in which case $\partial f/\partial x_n$ will have a root in between and induction on degree f will handle it. Similarly, a multiple root of f will also be a root of $\partial f/\partial x_n$. If the regions are (1) a simple root of f and (2) a root of one of its derivatives, and they are adjacent, then f separates one way and the corresponding $\partial^k f/\partial x_n^k$ the other way, etc.

The S_i are semi-algebraic sets in R^{n-1} and so they can be separated by induction on n ($n = 1$ is trivial). So if T_i and T_j have projections S_i and S_j which have disjoint closures, the polynomials separating S_i and S_j will also separate T_i and T_j. So the only case left to handle is where $\bar{S}_i \cap \bar{S}_j \neq \emptyset$ but $\bar{T}_i \cap \bar{T}_j = \emptyset$. Since we can assume $f = \sum_{i=0}^{d} a_i x_n^i$ and Sign a_d constant on S_i and on S_j, we may as well assume Sign $a_d \neq 0$ on S_i. We have to consider various cases.

Case 1. $S_j \subseteqq \bar{S}_i$. Let $\pi: R^n \to R^{n-1}$ be the projection $(x_1, \cdots, x_n) \to (x_1, \cdots, x_{n-1})$. Let f_Q denote the polynomial $\sum_{i=0}^{d} a_i(Q) x_n^i$ for Q in R^{n-1}. Choose P in T_j and we wish to find a polynomial which will be $\neq 0$ at P and of opposite sign or 0 on T_i. We have several subcases.

(i) Assume $f_{\pi(P)}$ is not the zero polynomial and that $x_n(P)$ is not a root of $f_{\pi(P)}$. Then, since T_i either is, or is bounded by, a root $x_n = \alpha_i(x_1, \cdots, x_{n-1})$ above S_i, one sees that respectively, either f itself separates P from T_j, or else there is a root of f above S_i between P and T_i. In the second case, if this is a simple root of f and there are not others in between, then f separates. If the root is a multiple root, or if there is more than one in between, then $\partial f/\partial x_n$ will have a root between P and T_i and induction will work.

(ii) Assume $f_{\pi(P)}$ is not the zero polynomial and that $x_n(P)$ is a root of $f_{\pi(P)}$. Note that either $f_{\pi(P)}$ or $(\partial f/\partial x_n)_{\pi(P)}$ changes sign in an interval $(x_n(P) - \delta, x_n(P) + \delta)$ about $x_n(P)$. Thus, for Q near $\pi(P)$ (Q in S_i), f_Q or $(\partial f/\partial x_n)_Q$ will also change sign on this interval. In the first instance, f will have a root $\alpha_{i+1}(x_1, \cdots, x_{n-1})$ above S_i which will have P in its closure. There will then be a root of $\partial f/\partial x_n$ which will be in between these roots and so separate T_i from P. If $\partial f/\partial x_{n(\pi(P))}$ changes sign on the interval about $x_n(P)$, then looking at the graph of $y = f_{\pi(P)}(x_n)$ and the graph of $y = f_Q(x_n)$ for Q near $\pi(P)$, one sees that $\partial f/\partial x_n$ again has a root between T_i and P. Proceed by induction.



(iii) Assume that $f_{\pi(P)}$ is the zero polynomial, i.e., that all

$a_i(\pi(P)) = 0$. Then if some irreducible factor of a_d divides all a_i and vanishes at $\pi(P)$, one can divide this factor out. Otherwise, $\pi(P)$ lies in the intersection of the zeros of two relatively prime polynomials in $R[x_1, \cdots, x_{n-1}]$ derivable from the a_i's. For, either two a_i's have relatively prime factors which vanish at P, or else one of the a_i has an irreducible factor which vanishes at P and is not real. Not real means that the factor does not generate the ideal of its real zeros. This follows from the real nullstellensatz for polynomials (see Theorem 2.1, in [4]). In the factor of a_i which we also call a_i is not real, then its real locus is contained in its singular locus (see Theorem 2.1, in [3]), and so is in the zero set of all the $\partial a_i/\partial x_j$. In this way eventually one will obtain two relatively prime polynomials $u(x_1, \cdots, x_{n-1})$ and $v(x_1, \cdots, x_{n-1})$ which vanish at P. By elimination theory, one can assume that u doesn't involve x_{n-1} and v doesn't involve x_{n-2}. Then, using $u, v, u - v$ and $u + v$, one can define new regions subdividing S_i so that in these regions, which we also call S_i, we have $\pi(P) = \lim Q$ for Q in S_i and $x_i(Q) \to x_i(P)$ $i = 1, 2, \cdots, n-2$. Now let $\beta(x_1, \cdots, x_{n-2}) = \min \alpha(x_1, \cdots, x_{n-1})$ for $(x_1, \cdots, x_{n-1})$ is S_i and where α is a boundary of T_i. And do the same thing for $\max \alpha$. Then, since $f(x_1, \cdots, x_{n-1}, \alpha(x_1, \cdots, x_{n-1})) = 0$, we derive

$$\partial f/\partial x_{n-1}(x_1, \cdots, x_{n-1}, \alpha(x_1, \cdots, x_{n-1})) + \partial f/\partial x_n(x_1, \cdots, x_{n-1}, \alpha(x_1, \cdots, x_{n-1}))\frac{\partial \alpha}{\partial x_{n-1}} = 0 .$$

So either $\beta(x_1, \cdots, x_{n-2})$ is a root of the polynomial obtained from $\partial f/\partial x_{n-1}$ and f by eliminating x_{n-1} or else the minimum occurs on the boundary, in which case one can use elimination theory on f and one of the $u, v, u + v$, or $u - v$ to get a polynomial with root β. In any case, we get a new polynomial involving one less variable than f which will have a root β between T_i and P, at least near P, in S_i. By subdividing S_i again, one can assume that the new polynomial does have a root between T_i and P above S_i. Then the induction can proceed.

Case 2. $S_j \not\subseteq \bar{S}_i$. One must consider the various $S_k \subseteq \bar{S}_i \cap \bar{S}_j$ and see what happens to the roots of f above S_k. If $a_d = 0$ on S_k, then one can use the minimizing techniques of Case 1, (iii) to reduce to the two variable case where a_d or one of its irreducible factors will separate T_i and T_j. When $a_d \neq 0$ on S_k, let $T'_i = T_i \cap \pi^{-1}(S_k)$ and $T'_j = T_j \cap \pi^{-1}(S_k)$. As long as T'_i and T'_j are not adjacent roots, one can find some $\partial^l f/\partial x_n^l$ which is >0 on T'_i and <0 on T'_j and by continuity the same holds on T_i and T_j. If these are adjacent roots, then one chooses the polynomial so it vanishes on T_i and is < 0 on T'_j, etc.

One must subdivide the regions T_i so that the new polynomials have constant sign on these pieces.

MOSTOWSKI'S THEOREM. *Let D be defined by $p_i(x_1, \cdots, x_n) > 0$, $i = 1, \cdots, s$. Then, if C_1 and C_2 are semi-algebraic closed disjoint subsets of D, there exists g in B_2 with $g(C_1) > 0$ and $g(C_2) < 0$.*

Proof. First a remark. Mostowski's proof would show that one could choose g in B_1, but this seems to have no advantage in the applications.

Let $f_1, \cdots, f_t$ be the polynomials defining C_1 and C_2. By the Separation Lemma, we can find more polynomials including the f's and p's, say, $f_1, \cdots, f_u$ and a subdivision $R^n = \bigcup_{i,j} T_{ij}$, $T_i = \bigcup_j T_{ij}$ so that $C_1 = \bigcup T_i$ for i in I_1 and $C_2 = \bigcup T_i$ for i in I_2. Moreover Sign $f_k(T_{ij})$ is constant and for all $\bar{T}_{i_1} \cap \bar{T}_{i_2} = \emptyset$, and all j_1, j_2; there exists f_k with $\pm f_k(T_{i_1 j_1}) \geqq 0$ and $\pm f_k(T_{i_2 j_2}) < 0$.

So choose $T_{11} \subseteqq C_1$ and choose each $\pm f_k$ with $\pm f_k(T_{11}) \geqq 0$. We consider the chosen $\pm f_k$'s as our f_k's. Then, for each $T_{ij} \subseteqq C_2$, there exists f_k with $f_k(T_{ij}) < 0$.

Let $h = \sum_k (|f_k| - f_k)$. Then $h = 0$ on T_{11} and $h > 0$ on C_2. Let

$$\varepsilon(x) = \prod_{i=1}^{s} p_i(x)/(2 + \|x\|^2)^L$$

for $L = \sum_{i=1}^{s} \deg p_i + 1$ and if no p_i, then let the numerator $=1$. Since $h > 0$ on C_2, we can let $\gamma(r) = \min\{h(x) \mid \varepsilon(x) = r, x \in C_2\}$. Then $\gamma(r) > 0$ for $0 < r < 1$ and $\gamma(r)$ is an algebraic function. Thus there exists N so that $\gamma(r) > r^N$ for all r for which $\gamma(r)$ is defined. It follows that $h(x) > \varepsilon(x)^N$ on C_2. We let

$$g_{11}(x) = \sum_{k=1}^{t} \left(\sqrt{f_k^2 + \varepsilon(x)^{2N}/(t+1)^2} - f_k\right) > h(x) .$$

Moreover, $g_{11}(x) < h(x) + \varepsilon(x)^N$, so on T_{11}, we have $g_{11}(x) < \varepsilon(x)^N$. Thus $g_{11}(x) - \varepsilon(x)^N$ is < 0 on T_{11} and > 0 on C_2.

In a similar way, one can find some g_{ij} for each $T_{ij} \subseteqq C_1$ so that $g_{ij} > 0$ on C_2 and $g_{ij} < 0$ on T_{ij}. Each $g_{ij} \in B_1$. Now note that $\sum (|g_{ij}| - g_{ij}) = 0$ on C_2 and > 0 on C_1. Then as above by modifying this function one can obtain

$$g = \sum \left(\sqrt{g_{ij}^2 + \varepsilon(x)^{2M}/M^2} - g_{ij}\right) - \varepsilon(x)^M$$

for some large integer M which will have the desired properties.

2. Substitution in Nash functions. Recall the situation. We have D as in § 1 to be $\{a$ in R^n with $p_i(a) > 0$, $i = 1, \cdots, s\}$. And $A = \{f: D \to R$ with f algebraic and analytic$\}$. Let $\varphi: A \to L$ be a homomorphism of A into a real closed field. Since $A \supset R[x_1, \cdots, x_n]$, φx_i is defined for $i = 1, \cdots, n$. In [5], § 2, it was shown that $f(x) = z$ is equivalent to some elementary statement $A(x, z)$, so one can define a new function $f_L: D_L \to L$, where $D_L = \{(a_1, \cdots, a_n)$ in L^n with all $p_i(a_1, \cdots, a_n) > 0\}$ by setting $f_L(a) = b$ if and only if $A(a, b)$. In [5], loc. cit., it was shown that f_L is a well defined function, and that $\varphi x \in D_L$.

THEOREM 2.1. *With the notation as above,* $f_L(\varphi x_1, \cdots, \varphi x_n) = \varphi f$.

Proof. This goes in several steps and occupies most of this section.

LEMMA 2.2. *We can assume that if* $p_f = \sum_{i=0}^{d} a_i(x) z^i$ *is the irreducible polynomial for* f *over* $R(x_1, \cdots, x_n)$; *then* $a_d(\varphi x) \neq 0$, *and* $\partial p_f / \partial z(\varphi f, \varphi x) \neq 0$.

Proof. Let a be any point of D. For our original f,

$$R[x, f]_{(a, f(a))} = (R[x, z]/(p_f))_{(a, f(a))}$$

is a local ring and is etale over $R[x]_{(a)}$. But by [6], Corollary 7.5, p. 11, this implies that there exists g in $R[x, f]_{(a, f(a))}$ with $p_g(z, x)$ irreducible and $\partial p_g / \partial z(g(a), a) \neq 0$. Let $\alpha(x)$ be the leading coefficient of $p_g(z)$. So $f(x) = q_1(g, x)/q_2(g, x)$ with $g_2(g(a), a) \neq 0$. Let

$$h_a = (\partial p_g / \partial z(g(x), x) q_2(g(x), x) \alpha(x))^2 .$$

Then $h_a \neq 0$ near a, and we can construct such an h_a for every a in D. Let $V_a = V(h_a) =$ zero set of h_a in D. It is clear that $\bigcap V_a = \varnothing$ taking the intersection over all a in D. We claim that there exists a finite number of V_a whose intersection is empty. To prove this we argue as in [5], Lemma 3.1. First choose some h_{a_1} and let W be a connected non-singular piece of $V(h_{a_1})$. Let $a_2 \in W$ and then h_{a_2} can vanish only on a smaller dimensional piece of W. By continuing this process one gets the result.

Let h be the sum of these h_a's. Then $h - \varepsilon(x)^N > 0$ on D for N large enough. So $\sqrt{h - \varepsilon^N} \in A$ and so $\varphi(\sqrt{h - \varepsilon^N})^2 = \varphi(h - \varepsilon^N) > 0$. This implies that $\varphi h_a > 0$ for some a. Take the corresponding g for this h_a. Then $\partial p_g / \partial z(\varphi g, \varphi x) \neq 0$, $q_2(\varphi g, \varphi x) \neq 0$, and $\alpha(\varphi x) \neq 0$.

So suppose we have proved our theorem for g. Then since $f = q_1(g, x)/q_2(g, x)$, we have

$$\varphi f = q_1(\varphi g, \varphi x)/q_2(\varphi g, \varphi x) = q_1(g(\varphi x), \varphi x)/q_2(g(\varphi x), \varphi x) .$$

But $f(b) = q_1(g(b), b)/q_2(g(b), b)$ for all b in R^n with $q_2(g(b)) \neq 0$ so, by the Tarski-Seidenberg principle: [1] or [5], Theorem 1.8, the same holds for all b in L^n. In particular $f(\varphi x) = q_1(g(\varphi x), \varphi x)/q_2(g(\varphi x), \varphi x)$ and this implies that $\varphi f = f(\varphi x)$.

So we now assume that $\partial p_f/\partial z(\varphi f, \varphi x) \neq 0$. Consider $R[x, z]/(p_f)$ and normalize this ring. Let $t_1(z, x), \cdots, t_u(z, x)$ generate the normalization (considered mod p_f). So, as usual, $R[z, x, t]/(p_f, \cdots) \hookleftarrow R[x, z]/(p_f)$ induces $\pi: C^{n+s+1} \rightarrow C^{n+1}$, with the branches of $V(p_f)$ separated in C^{n+s+1}. Of course $C =$ complex numbers.

Note that $t_i(\varphi f, \varphi x)$ is defined since $\partial p_f/\partial z(\varphi f, \varphi x) \neq 0$.

Let

$$C_1 = \{(x, f(x), t_1(f(x), x), \cdots, t_u(f(x), x)) \mid x \in D\}$$

and

$$C_2 = \{(x, z, t_1(z, x), \cdots, t_u(z, x)) \mid p_f(z, x)\} = 0 , \quad x \in D,\ z \neq f(x).$$

Then C_1 and C_2 are closed disjoint semi-algebraic sets in $D \times R^{s+1}$, so by Mostowski's theorem, there exists $g(x, z, t)$ in $B_{2, D\times R^{s+1}}$ with $g(C_1) > 0$ and $g(C_2) < 0$.

Now let $h(x) = g(x, f(x), t_1(f(x), x), \cdots, t_u(f(x), x))$. We have to show that $h(x) \in A$ and that $\varphi h = g(\varphi x, \varphi f, t_1(\varphi f, \varphi x), \cdots, t_u(\varphi f, \varphi x))$. But since each $t_i(f(x), x)$ is integral over $R[x]_{(a)} \forall a \in D$ and analytic on D except for a thin set, $t_i(f(x), x)$ is in fact analytic on D. The rest follows from

LEMMA 2.3. *Let $g \in B_2$ and $h_1, \cdots, h_r \in A_D$ so that $g(h_1, \cdots, h_r)$ is defined and in A_D. Then $\varphi g = g(\varphi h_1, \cdots, \varphi h_r)$.*

Proof. It is enough to show this for g in B_1, as a repeat of the same argument will finish the proof. So let $g = a(x) + b(x)\sqrt{f(x)}$ where $f, a, b \in B_0$ and $f > 0$ on D. Now $f(h_1, \cdots, h_r)$ has

$$\varphi(f(h_1, \cdots h_r)) = f(\varphi h_1, \cdots, \varphi h_r) \quad \text{as} \quad f \in B_0 .$$

Since $\varphi(\sqrt{f(h_1, \cdots, h_s)}^2) = \varphi f(h_1, \cdots, h_s)$, it follows that

$$\varphi(\sqrt{f(h_1, \cdots, h_r)}) = \pm\sqrt{\varphi f(h_1, \cdots, h_s)} .$$

But

$$\varphi(\sqrt[4]{f(h_1, \cdots, h_s)^2}) = \varphi\sqrt{f(h_1, \cdots, h_r)} > 0$$

and so Lemma 2.3 follows.

To finish the proof of Theorem 2.1, note that

($*$) Given (x, z) in $D \times R$ if $p_f(z, x) = 0$ and $g(x, z, t_1(z, x), \cdots, t_u(z, x))$ is defined and > 0; then $f(x) = z$.

By Tarski-Seidenberg, this statement also holds in $D_L \times L$ and since we have $p_f(\varphi f, \varphi x) = 0$ and $g(\varphi x, \varphi f, t_1(\varphi f, \varphi(x), \cdots, t_u(\varphi f, \varphi(x)) > 0$, it follows that $f(\varphi x) = \varphi f$.

We now show that the Nullstellensatz proved by Mostowski is an easy corollary of Theorem 2.1.

THEOREM 2.4 (Mostowski). *Let* $\mathscr{I}$ *be an ideal of* A. *Then* $I(V_D(\mathscr{I})) = \mathscr{I}$ *iff* $\mathscr{I}$ *is a real ideal* (i.e. $\sum \lambda_i^2 \in \mathscr{I}$ implies each $\lambda_i \in \mathscr{I}$).

Proof. First note that A is Noetherian by [5], Theorem 3.4, and so $\mathscr{I} = \mathscr{P}_1 \cap \cdots \cap \mathscr{P}_s$ where each $\mathscr{P}_i$ is a real prime. It will be sufficient then to show that for each i, $I(V_D(\mathscr{P}_i)) = \mathscr{P}_i$. So consider $\mathscr{P}$ a real prime in A and by the Noetherian property of A, we have $\mathscr{P} = (f_1, \cdots, f_t)$ for some $f_1, \cdots, f_t$ in A. Let L be a real closure of the quotient field of $A/\mathscr{P}$. Then we have $\varphi: A \rightarrow A/\mathscr{P} \hookrightarrow L$ where $\varphi =$ the total map.

Now $g \in I(V_D(\mathscr{P}))$ iff; $(*)$ For all $x \in D$, $f_1(x) = 0, \cdots, f_t(x) = 0$; implies $g(x) = 0$. By Tarski-Seidenberg, $(*)$ holds for L. But $\varphi f_i = 0$ for all i so by Theorem 2.1, $f_i(\varphi x) = 0$. This implies, by $(*)$ that $g(\varphi x) = 0$. Again applying Theorem 2.1, we see that $\varphi g = 0$. So $g \in \mathscr{I}$.

THEOREM 2.5 (Mostowski). *Let* $f \in A$, $f \geqq 0$ *on* D. *Then* f *is a sum of squares in* K, *the quotient field of* A.

Proof. If f is not a sum of squares, order K so that $-f > 0$. Then, if L is a real closure of K, one has $\varphi : A \hookrightarrow L$. Since for all x in R^n, $f(x) \geqq 0$; by Tarski-Seidenberg, the same holds for $x \in L^n$. Thus $f(\varphi x) \geqq 0$. By Theorem 2.1, $\varphi f = f(\varphi x) \geqq 0$, but as $\varphi f =$ the image of f in L and L is ordered so $f < 0$, we have a contradiction.

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