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MULTIPLIERS ON A BANACH ALGEBRA WITH A BOUNDED APPROXIMATE IDENTITY

JOHN W. DAVENPORT

Let A be a Banach algebra with a bounded approximate identity $\{e_\alpha \mid \alpha \in \Lambda\}$, and $M(A)$ the multiplier algebra on A. In this paper, we obtain a representation for $M(A)$ such that each multiplier operator appears as a multiplicative operator. The proof makes use of the weak-* compactness of the net $\{Te_\alpha \mid \alpha \in \Lambda\}$ and the algebraic properties of a multiplier.

1. Introduction. In 1951, J. G. Wendel showed that the left centralizers on $L_1(G)$, G a locally compact group, was equivalent to $C_0(G)^*$, the space of regular Borel measures on G. Thus, if T is a centralizer and x is any element in $L_1(G)$ then $Tx = \xi * x$ for some Borel measure ξ. It is also well known that if A is a Banach algebra with an identity element then any multiplier on A is determined by its action on the identity element. In this paper, we show that if A is a Banach algebra with a bounded approximate identity then there exist a continuous isomorphism of A such that each multiplier defined on A is given by point-wise multiplication. In the case that the approximate identity is uniformly bounded by one, the representation is norm preserving. Thus we obtain an isometric isomorphism for all multipliers on $L_1(G)$ and for all multipliers on any B^*-algebra such that the action of a multiplier is given by point-wise multiplication by a fixed element in A.

2. The representation space for $M(A)$.

DEFINITION 2.1. Let A be a Banach algebra and T a mapping from A into A. The map T is a multiplier provided

$$x(Ty) = (Tx)y \qquad (x, y \in A) .$$

Every multiplier turns out to be a continuous function and the set of all multipliers on A under pointwise operations is a commutative subalgebra of $B(A)$, the set of all bounded linear operators on A([5]).

NOTATION 2.2. In this paper, a Banach algebra with a bounded approximate identity will be denoted by A and the multiplier algebra on A will be denoted by $M(A)$. For any Banach algebra X, we denote the weak-* convergence of a net in X^*, the dual space of X, indexed by $\alpha \in \Lambda$, by "$\lim_\alpha^{wk\text{-}*}(\cdot)$". Unless otherwise stated, we denote the bound on the approximate identity by M.

DEFINITION 2.3. Let X be a Banach algebra. The algebra X is said to have a bounded approximate identity provided there exists a net $\{e_\alpha \mid \alpha \in \Lambda\}$ in X and a $M > 0$ such that

2.3.1 $\|e_\alpha\| < M$ $\quad (\alpha \in \Lambda)$

2.3.2 $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$ $\quad (x \in X)$.

DEFINITION 2.4. Let $\{e_\alpha \mid \alpha \in \Lambda\}$ denote the approximate identity on A, and $B_* = \{f \in A^* \mid f \cdot e_\alpha \to f\}$ where $f \cdot a(x) = f(ax)$ for each a, $x \in A$ and $f \in A^*$. The set B_* is a closed subspace of A^* and $B_* = \{f \cdot a \mid f \in A^*, a \in A\}$ ([3]). By defining

2.4.1 $[G, f] = G(f \cdot a)$ $\quad (a \in A, f \in B_*, G \in B_*^*)$

2.4.2 $F \cdot G(f) = F[G, f]$ $\quad (f \in B_*, F, G \in B_*^*)$,

the dual space, B_*^*, becomes a Banach algebra. This follows since the above definitions are the restrictions to B_* of the Arens product on A^{**} which makes A^{**} into a Banach algebra such that if π is the canonical embeding of A into A^{**} then π is an isometric isomorphism ([5]).

LEMMA 2.5. *There exists a norm reducing isomorphism of A into B_*^*.*

Proof. We define $\tau: A \to B_*^*$ by $\tau a(f) = f(a) = \pi a \mid_{B^*}$.

Clearly τ is linear and since $B_* = \{f \cdot a \mid f \in A^*, a \in A\}$, it follows that τ is one-to-one. From $|\tau a(f)| = |f(a)| < \|f\| \cdot \|a\|$, we see that $\|\tau a\| < \|a\|$, for all $a \in A$.

LEMMA 2.6. *Let $\{F_\alpha \mid \alpha \in \Lambda\}$ be a net in B_*^*; $a \in A$; and $F, G \in B_*^*$, then the following properties are satisfied:*

2.6.1 *if* $\lim_\alpha^{wk\text{-}*} F_\alpha = F$ *then* $\lim_\alpha^{wk\text{-}*} F_\alpha \cdot G = F \cdot G$

2.6.2 *if* $\lim_\alpha^{wk\text{-}*} F_\alpha = F$ *then* $\lim_\alpha^{wk\text{-}*} \tau a \cdot F_\alpha = \tau a \cdot F$

2.6.3 *if* $F \cdot \tau a = 0$ *for all* $a \in A$ *or* $\tau a \cdot F = 0$ *for all* $a \in A$ *then* $F = 0$.

Proof. These properties follow from a straightforward application of the definitions of the operations involved.

LEMMA 2.7. *The Banach algebra B_*^* has an identity element which we denote by J.*

Proof. From $\|\tau e_\alpha\| < \|e_\alpha\| < M$, it follows that the net $\{\tau e_\alpha\}$ has a weak-* convergent subnet. Let $J = \lim_\alpha^{wk\text{-}*} \tau e_\alpha$. Since

$$[J, f](x) = J(f \cdot x) = \lim_a \tau e_a(f \cdot x) = \lim_a f(x e_a) = f(x) ,$$

for all $x \in A$, we have that $[J, f] = f$ for all $f \in B_*$. Thus $F \cdot J = F$, for all $F \in B_*^*$. Since $\tau a \cdot F$ is weak-$*$ continuous in F, it also follows that $J \cdot F(f) = \lim_\alpha \tau e_\alpha \cdot F(f) = \lim_\alpha F(f \cdot e_\alpha) = F(f)$ for all $f \in B_*$ and $F \in B_*^*$. Thus $J \cdot F = F$ for all $F \in B_*^*$.

THEOREM 2.8. *Let A be a Banach algebra with a bounded approximate identity $\{e_\alpha \mid \alpha \in \Lambda\}$. Then there exists a map μ from $M(A)$ into B_*^* such that μ is a continuous, algebraic isomorphism of $M(A)$ into B_*^*. Furthermore*

$$\tau(Ta) = (\mu T) \cdot \tau a = \tau a \cdot (\mu T) \qquad (a \in A,\ T \in M(A)) .$$

Proof. Let $T \in M(A)$. Since $\| Te_\alpha \| < \| T \| \cdot M$, the net $\{\tau(Te_\alpha) \mid \alpha \in \Lambda\}$ has a weak-$*$ convergent subnet in B_*^*. If $\{\tau(Te_\beta) \mid \beta \in \Gamma\}$ converges to G and $\{\tau(Te_\alpha) \mid \alpha \in \Lambda\}$ converges to F, each in the weak-$*$ topology; then, for each $f \in B_*$, we have that

$$\begin{aligned} F(f) &= \lim_\alpha \tau(Te_\alpha)(f) = \lim_\alpha \tau(Te_\alpha) \cdot J(f) \\ &= \lim_\alpha \lim_\beta \tau Te_\alpha \cdot \tau e_\beta(f) = \lim_\alpha \lim_\beta (\tau Te_\alpha(e_\beta))(f) \\ &= \lim_\alpha \lim_\beta \tau e_\alpha \cdot \tau Te_\beta(f) = \lim_\alpha \tau e_\alpha \cdot G(f) = G(f) . \end{aligned}$$

Now we define the mapping μ from $M(A)$ to B_*^* by

$$\mu(T) = F = \overset{wk\text{-}*}{\lim_\alpha} \tau(Te_\alpha) \qquad (T \in M(A)) .$$

The previous remarks show that μ is well defined. We first observe that if $F = \mu(T)$, then

$$\tau a \cdot F(f) = \lim_\alpha \tau a \cdot \tau Te_\alpha(f) = \lim_\alpha \tau Ta \cdot \tau e_\alpha(f) = \tau(Ta)(f) .$$

Thus

2.8.1. $\tau a \cdot \mu(T) = \tau(Ta) \qquad (a \in A,\ T \in M(A))$.

By Lemma 2.7, the identity element of B_*^* is the weak-$*$ limit of a subnet of $\{\tau e_\alpha \mid \alpha \in \Lambda\}$. Let $\{\tau e_\beta\}$ denote this subnet. Hence we have

$$\begin{aligned} \mu(T) \cdot \tau a(f) &= \lim_\beta \tau e_\beta \cdot \mu(T) \cdot \tau a(f) = \lim_\beta \tau e_\beta \cdot \mu(T) \cdot \tau a(f) \\ &= \lim_\beta \tau e_\beta \cdot \tau Ta(f) = \tau Ta(f) . \end{aligned}$$

Therefore,

2.8.2. $\mu T \cdot \tau a = \tau Ta \qquad (a \in A,\ T \in M(A))$.

Let $x, y \in A$ and $T \in M(A)$. Then

$$\begin{aligned}\tau x\cdot\mu(TS)\cdot\tau y &= \tau(TSx)y = \tau Sx\cdot\tau Ty = \mu S\cdot\tau x\cdot\mu T\cdot\tau y\\ &= \tau x\cdot\mu S\cdot\mu T\cdot\tau y\end{aligned}$$

and thus by Lemma 2.6, it follows that $\mu(TS) = \mu(S)\cdot\mu(T)$. But C. N. Kellogg [4] proved that $M(A)$ is a closed commutative subalgebra of $B(A)$, the set of all bounded linear operators on A. Thus $\mu(TS) = \mu(ST) = \mu(T)\cdot\mu(S)$ and therefore μ is homomorphic.

If $\mu(T) = \mu(S)$ for some $T, S \in M(A)$ where $\mu(T) = \lim_\alpha^{wk\text{-}*}\tau Te_\alpha$ and $\mu(S) = \lim_\beta^{wk\text{-}*}\tau Se_\beta$ then for each $f \in B_*$, and $a \in A$, we have

$$\begin{aligned}\tau(Ta)(f) &= \lim_\alpha \tau(Ta)\cdot\tau e_\alpha(f) = \lim_\alpha \tau a\cdot\tau Te_\alpha(f)\\ &= \tau a\cdot\mu(T)(f) = \tau a\cdot\mu(S)(f) = \tau a\cdot\lim_\beta \tau(Se_\beta)(f)\\ &= \lim_\beta \tau a\cdot\tau(Se_\beta)(f) = \lim_\beta \tau(Sa)\cdot e_\beta(f) = \tau(Sa)(f)\ .\end{aligned}$$

Since τ is one-to-one, it follows that $Ta = Sa$ for each $a \in A$. Thus μ is one-to-one.

From $\mu(T) = \lim_\alpha \tau Te_\alpha$ and $\|\tau Te_\alpha\| < \|Te_\alpha\| < \|T\|\cdot\|e_\alpha\| < \|T\|\cdot M$, it follows that μ is continuous.

COROLLARY 2.9. *If $M = 1$, then $M(A)$ is isometrically $*$-isomorphic to a subspace of B_*^*.*

Proof. This follows from Theorem 2.8 and the fact that $\|\tau a\| = \|a\|$.

For $A = L_1(G)$, G a nondiscrete locally compact abelian group, the space B_* is the space of uniformily continuous bounded functions on G and B_*^* is the space $M(G)$ of bounded measures of the maximal ideal space of B_*. If G is compact then $M(A) = M(G)$. In the case that A is a B^*-algebra, we have the following result.

COROLLARY 2.10. *If A is a B^*-algebra then $M(A)$ is isometrically $*$-ismorphic to a subspace of A^{**}. Furthermore, if $\mu(T) = F$ for $T \in M(A)$ and $F \in A^{**}$, then*

$$\pi a\cdot F = F\cdot\pi a = \pi Ta \qquad (a \in A)$$

where the above operation is the Arens product on A^{**}.

Proof. D. C. Taylor [7] has shown that $A^* = \{f\cdot a \mid f \in A^*,\ a \in A\} = \{a\cdot f \mid f \in A^*,\ a \in A\}$. Thus $B_* = A^*$ and $B_*^* = A^{**}$. In this case the product operation on $B_*^* = A^{**}$ becomes the Arens product and the involution on A^{**} is given by $F^*(f) = \overline{F(\bar{f})}$ where $\bar{f}(x^*)$ [2]. Since a B^*-algebra possesses an approximate identity uniformly bounded by one, the result follows from Corollary 2.9.

COROLLARY 2.11. *Let A be a B^*-algebra. Then $F \in A^{**}$ belongs to $\mu(M(A))$ if and only if the operator F commutes with πA and $F \cdot \pi a$ is continuous in the weak-$*$ topology on A^* for each $a \in A$.*

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MADISON COLLEGE, HARRISONBURG, VA