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CONCORDANCES OF NONCOMPACT HILBERT CUBE MANIFOLDS

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In this paper two exact sequences are established which are useful in computing $\pi_0\mathscr{C}(M)$, the group of isotopy classes of concordances for a noncompact Hilbert cube manifold M. Roughly speaking, this enables one to study the noncompact case in terms of the compact case. The situation is analogous to Siebenmann's description of groups of infinite simple homotopy types in terms of two exact sequences.

1. Introduction. For any space X we will use $\mathscr{C}(X)$ to denote the space of all *concordances* of X. It is the function space, with the compact-open topology, of all homeomorphisms of $I \times X$ onto itself ($I = [0, 1]$) which are the identity on $\{0\} \times X$. We use $\pi_0\mathscr{C}(X)$ to denote the group of all isotopy classes in $\mathscr{C}(X)$, where the group operation is composition. A *Q-manifold* is a separable metric manifold modeled on the Hilbert cube Q, the countable infinite product of closed intervals. In [3] and [4] the author investigated the group $\pi_0\mathscr{C}(M)$ for M a compact Q-manifold. The main result established there was the following: *Let M be a compact Q-manifold which is written as $R \times Q$, where R is a PL n-manifold.* (It follows from [1] that this can always be done.) *Then $\pi_0\mathscr{C}(M)$ is isomorphic to the direct limit of the sequence*

$$\pi_0\mathscr{C}(R) \xrightarrow{(\times id)_*} {}^*\pi_0\mathscr{C}(R \times I) \xrightarrow{(\times id)_*} \pi_0\mathscr{C}(R \times I^2) \longrightarrow \cdots .$$

Fortunately this direct limit has been studied in [8], and as a result we get the following consequences for Q-manifolds.

A. *If M is a compact Q-manifold, then $\pi_0\mathscr{C}(M)$ depends only on the 3-type of M.*

From this we get.

B. *If M and N are homotopy equivalent compact Q-manifolds, then $\pi_0\mathscr{C}(M)$ is isomorphic to $\pi_0\mathscr{C}(N)$.* (See §2 for a proof which uses only infinite-dimensional techniques.)

C. *If M is a compact Q-manifold, then $\pi_0\mathscr{C}(M)$ is trivial iff each component of M is 1-connected.* (This holds in spite of the recently discovered gap in [9].)

D. *If M is a compact Q-manifold, then $\pi_0\mathscr{C}(M)$ is abelian.* (See §2 for an elementary proof which was suggested to the author by the referee.)

The purpose of this paper is to investigate the group $\pi_0\mathscr{C}(M)$, for M a noncompact Q-manifold. Our main results are Theorems 2 and 3, where we establish two exact sequences for $\pi_0\mathscr{C}(M)$ which relate the noncompact case to the compact case. They are remarkably similar to the exact sequences of [10] used to compute the group of all infinite simple types on a given locally compact polyhedron. Below we will give precise descriptions of these exact sequences and related results.

Recall that a map between spaces (always locally compact, separable and metric) is *proper* provided that preimages of compacta are compact. In analogy with the ordinary homotopy category we obtain the proper homotopy category, where all maps and homotopies are proper. In §2 we show how a proper map $f: M \to N$ between Q-manifolds induces a homomorphism $f_*: \pi_0\mathscr{C}(M) \to \pi_0\mathscr{C}(N)$ and then prove the following result.

THEOREM 1. *$\pi_0\mathscr{C}$ is a covariant proper homotopy functor from the category of Q-manifolds and proper maps to the category of groups and homomorphisms.*

By a proper homotopy functor we mean that proper homotopic maps $f, g: M \to N$ induce the same homomorphisms from $\pi_0\mathscr{C}(M) \to \pi_0\mathscr{C}(N)$. This implies the following result.

COROLLARY 1. *If M, N are Q-manifolds which have the same proper homotopy type, then $\pi_0\mathscr{C}(M)$ is isomorphic to $\pi_0\mathscr{C}(N)$.*

For any Q-manifold M we define $\varprojlim Wh\pi_1(M)$ to be the inverse limit of the inverse system

$$\{Wh\pi_1(M - C) \mid C \subset M \quad \text{compact}\},$$

where Wh is the Whitehead group functor and the homomorphisms are inclusion induced. It is easily seen that $\varprojlim Wh\pi_1(M)$ depends only on the proper homotopy type of M. The following exact sequence is established in §3.

THEOREM 2. *For any Q-manifold M there is an exact sequence*

$$\pi_0\mathscr{C}(M) \xrightarrow{\tau_\infty} \varprojlim Wh\pi_1(M) \longrightarrow Wh\pi_1(M).$$

This result easily gives us examples for which $\pi_0\mathscr{C}(M)$ is nontrivial. For example let $K \subset Q$ be a compact polyhedron with nontrivial Whitehead group and let $M = (Q \times [0, 1]) - (K \times \{0\})$. Then $Wh\pi_1(M) = 0$ and it is not hard to show that $\varprojlim Wh\pi_1(M) \approx Wh\pi_1(K)$. Thus Theorem 2 gives a homomorphism of $\pi_0\mathscr{C}(M)$ onto $Wh\pi_1(K) \neq 0$.

For any Q-manifold M we define $\varinjlim \pi_0\mathscr{C}(M)$ to be the direct limit of the direct system

$$\{\pi_0\mathscr{C}(M_1) \mid M_1 \subset M \text{ is a compact } Q\text{-manifold}\}\ ,$$

where the homomorphisms are inclusion induced. Observe that $\varinjlim \pi_0\mathscr{C}(M)$ is just $\pi_0\mathscr{C}_c(M)$, where $\mathscr{C}_c(M)$ denotes concordances with compact support. In §4 we show that $\varinjlim \pi_0\mathscr{C}$ quite naturally gives us a covariant *homotopy* functor from the category of Q-manifolds and maps to the category of groups and homomorphisms. Consider the inverse system

$$(*)\ \{\varinjlim \pi_0\mathscr{C}(M - C) \mid C \subset M \text{ compact}\}\ ,$$

where the homomorphisms are inclusion induced. We define $\varprojlim \pi_0\mathscr{C}(M)$ to be the inverse limit of $(*)$ and we define $\varprojlim{}^1\pi_0\mathscr{C}(M)$ to be the first derived limit of $(*)$. These are both groups which depend only on the proper homotopy type of M. We refer the reader to §4 for further details of these constructions. The following exact sequence is established in §5.

THEOREM 3. *For any Q-manifold M there is an exact sequence*

$$\varprojlim \pi_0\mathscr{C}(M) \longrightarrow \varinjlim \pi_0\mathscr{C}(M) \longrightarrow \operatorname{Ker}(\tau_\infty) \xrightarrow{\theta} \varprojlim{}^1\pi_0\mathscr{C}(M) \longrightarrow 0\ .$$

There is an important class of Q-manifolds for which $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ is trivial. We say that a space X *is movable at* ∞ provided that for each compactum $A \subset X$ there exists a larger compactum $B \subset X$ such that $X - B$ can be homotoped into any neighborhood of ∞, with the homotopy taking place in $X - A$. It is easy to see that if (K, L) is a finite simplicial pair, then the noncompact polyhedron $K - L$ is movable at ∞. The following result is established in §6.

THEOREM 4. *If M is a Q-manifold which is movable at ∞, then $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ is trivial.*

A closed subset A of a space X is said to be a Z-set in X provided that there exist arbitrarily small maps of X into $X - A$. In §6 we use Theorems 2, 3 and 4 to prove the following result.

THEOREM 5. *If (M, N) is a pair of compact Q-manifolds such that N is a Z-set in M, then there is an exact sequence*

$$\pi_0\mathscr{C}(N) \longrightarrow \pi_0\mathscr{C}(M) \longrightarrow \pi_0\mathscr{C}(M - N) \longrightarrow Wh\pi_1(N) \longrightarrow Wh\pi_1(M) .$$

As an immediate consequence of Theorem 5 we get the following result.

COROLLARY 2. *If $M \subset Q$ is a compact Q-manifold which is a Z-set in Q, then $\pi_0\mathscr{C}(Q - M)$ is isomorphic to $Wh\pi_1(M)$.*

2. The functor $\pi_0\mathscr{C}$. We first prove the assertion made in D of §1.

THEOREM 2.0. *If M is any Q-manifold, then $\pi_0\mathscr{C}(M)$ is abelian.*

Proof. Write $M = N \times [0, 1]$, where N is a Q-manifold. Then $\mathscr{C}(M)$ is naturally homeomorphic to $\mathscr{C}_0(M)$, the subset of $\mathscr{C}(M)$ consisting of all $h \in \mathscr{C}(M)$ which are point-wise fixed on $(I \times N \times \{0\}) \cup (I \times N \times \{1\})$. Let G be the subset of $\mathscr{C}_0(M)$ consisting of all $h \in \mathscr{C}_0(M)$ which are point-wise fixed on $I \times N \times [0, 1/2]$ and let H be the subset of $\mathscr{C}_0(M)$ consisting of all $h \in \mathscr{C}_0(M)$ which are point-wise fixed on $I \times N \times [1/2, 1]$. Clearly G and H are deformation retracts of $\mathscr{C}_0(M)$. Now elements of G and H commute, so $\mathscr{C}_0(M)$ is homotopy-commutative. Hence $\pi_0\mathscr{C}_0(M)$ is abelian.

We will now describe the functor $\pi_0\mathscr{C}$ and prove Theorem 1. Our first step will be to show how a proper map $f: M \to N$ between Q-manifolds induces a homomorphism $f_*: \pi_0\mathscr{C}(M) \to \pi_0\mathscr{C}(N)$. Elements of $\pi_0\mathscr{C}(M)$ will be denoted by $[h]$, where $h \in \mathscr{C}(M)$; that is, $[h]$ denotes the isotopy class of h.

Description of f_*. We are given a proper map $f: M \to N$ of Q-manifolds. We will need an open embedding $i: M \times [0, 2) \to N$ such that $i(M \times [0, 1])$ is closed and such that the map $M \xrightarrow{\times 0} M \times [0, 2) \xrightarrow{i} N$ is proper homotopic to f, where $\times 0(m) = (m, 0)$. One constructs i as follows. First find an embedding $f_1: M \to N$ which is proper homotopic to f and such that $f_1(M)$ is a Z-set in N, and then use the fact that $f_1(M)$ is collared in N. The existence of f_1 and the collaring of $f_1(M)$ follow from [2]. Let $\phi: I \times [0, 1] \to I \times [0, 1]$ be a homeomorphism such that ϕ takes $(\{0\} \times [0, 1]) \cup (I \times \{1\})$ onto $\{0\} \times [0, 1]$ and let $\phi \times id: I \times M \times [0, 1] \to I \times M \times [0, 1]$ be defined so that it is the identity on the M-factor and on $I \times [0, 1]$ it is defined by ϕ. For any $h \in \mathscr{C}(M)$ we define $h_* \in \mathscr{C}(N)$ as follows: $h_* = id$ on $I \times (N - i(M \times [0, 1)))$ and on $I \times i(M \times [0, 1])$ we define h_* by the commuting diagram

$$\begin{array}{ccccc} I \times M \times [0, 1] & \xleftarrow{\phi \times id} & I \times M \times [0, 1] & \xleftarrow{(id \times i)^{-1}} & I \times i(M \times [0, 1]) \\ \Big\downarrow{\scriptstyle h \times id} & & & & \Big\downarrow{\scriptstyle h_*} \\ I \times M \times [0, 1] & \xrightarrow{(\phi \times id)^{-1}} & I \times M \times [0, 1] & \xleftarrow{id \times i} & I \times i(M \times [0, 1]) . \end{array}$$

Then define $f_*([h]) = [h_*]$. It is clear that $f_*: \pi_0\mathscr{C}(M) \to \pi_0\mathscr{C}(N)$ is a homomorphism. The proof of Theorem 1 below contains a proof that f_* is well-defined.

Proof of Theorem 1. We aim to prove that $\pi_0\mathscr{C}$ is a covariant proper homotopy functor. We have divided the proof into several steps.

I. We will first show that the definition of f_* given above depends only on the proper homotopy class of f. Let $i: M \times [0, 2) \to N$ be as above and let $i': M \times [0, 2) \to N$ be an alternate choice for i. Choose any $h \in \mathscr{C}(M)$ and let $h'_* \in \mathscr{C}(N)$ be defined in analogy with the definition of $h_* \in \mathscr{C}(N)$ by replacing i with i'. We must prove that $[h_*] = [h'_*]$. Since the embeddings

$$M \xrightarrow{\times 0} M \times [0, 2) \xrightarrow{i} N$$
$$M \xrightarrow{\times 0} M \times [0, 2) \xrightarrow{i'} N$$

are proper homotopic we can find an isotopy $F_t: N \to N$, $0 \leqq t \leqq 1$, such that $F_0 = id$ and $F_1 i = i'$ on $M \times \{0\}$. This follows from Z-set unknotting [2]. Then $F_1 i(M \times [0, 2))$ and $i'(M \times [0, 2))$ are collars on the same base and it is easy to get an isotopy $G_t: N \to N$ such that $G_0 = id$ and $G_1 F_1 i = i'$ on $M \times [0, 1]$. The construction of G_t is elementary and uses no Q-manifold theory. Define an isotopy $H_t: N \to N$ by

$$H_t = \begin{cases} F_{2t} , & 0 \leqq t \leqq 1/2 \\ G_{2t-1}F_1, & 1/2 \leqq t \leqq 1 . \end{cases}$$

The effect of H_t is to move $i|M \times [0, 1]$ to $i'|M \times [0, 1]$. Now define $h_t \in \mathscr{C}(N)$ by setting $h_t = id$ on $I \times (N - H_t i(M \times [0, 1]))$ and on $I \times H_t i(M \times [0, 1])$ we define h_t by the commuting diagram

$$\begin{array}{ccccc} I\times M\times[0,1] & \xleftarrow{\phi\times id} & I\times M\times[0,1] & \xleftarrow{(id\times H_t i)^{-1}} & I\times H_t i(M\times[0,1]) \\ {\scriptstyle h\times id}\downarrow & & & & \downarrow{\scriptstyle h_t} \\ I\times M\times[0,1] & \xrightarrow{(\phi\times id)^{-1}} & I\times M\times[0,1] & \xrightarrow{id\times H_t i} & I\times H_t i(M\times[0,1])\,. \end{array}$$

Then h_t is an isotopy from h_* to h'_* rel $\{0\}\times N$, hence $[h_*]=[h'_*]$.

II. Now let $f_1\colon M\to N$ and $f_2\colon N\to P$ be proper maps of Q-manifolds. We will prove that $(f_2f_1)_*=(f_2)_*(f_1)_*$. To simplify notation let M be a Z-set in N, with f_1 the inclusion $M\hookrightarrow N$, and let N be a Z-set in P, with f_2 the inclusion $N\hookrightarrow P$. By abuse of notation we may assume that $M\times[0,2)\hookrightarrow N$ is a collaring of $M\equiv M\times\{0\}$ such that $M\times[0,1]$ is closed and similarly let $N\times[0,2)\hookrightarrow P$ be a collaring of $N\equiv N\times\{0\}$ such that $N\times[0,1]$ is closed. Choose any $[h]\in\pi_0\mathscr{C}(M)$. We will prove that $(f_2f_1)_*([h])=(f_2)_*(f_1)_*[h]$.

We first examine $(f_2)_*(f_1)_*([h])$. It is just $[g]$, where g may be chosen so that it is supported on $I\times(M\times[0,1])\times[0,1]$, and on this set it is given by the commutative diagram

$$\begin{array}{ccc} I\times M\times[0,1]^2 & \xleftarrow{\phi^*\times id} & I\times M\times[0,1]^2 \\ {\scriptstyle h\times id}\downarrow & & \downarrow{\scriptstyle g} \\ I\times M\times[0,1]^2 & \xrightarrow{(\phi^*\times id)^{-1}} & I\times M\times[0,1]^2\,, \end{array}$$

where $\phi_*\colon I\times[0,1]^2\to I\times[0,1]^2$ is the composition, $\phi_1\phi_2$, of homeomorphisms ϕ_1 and ϕ_2 defined as follows: $\phi_1\colon I\times[0,1]^2\to I\times[0,1]^2$ is given by $\phi_1(t,u,v)=(\phi(t,u),v)$ and $\phi_2\colon I\times[0,1]^2\to I\times[0,1]^2$ is given by $\phi_2(t,u,v)=(t',u,v')$ where $(t',v')=\phi(t,v)$.

In order to examine $(f_2f_1)_*([h])$ we will need to choose a collaring of M in P. Let $i\colon M\times[0,2)^2\to M\times[0,2)$ be a homeomorphism such that

$$i(M\times[([0,t]\times\{t\})\cup(\{t\}\times[0,t])])=M\times\{t\}\,,$$

for each $t\in[0,2)$. The existence of i follows routinely from the techniques used to prove the Stability Theorem [2]. Then

$$i^{-1}\colon M\times[0,2)\longrightarrow M\times[0,2)^2\hookrightarrow P$$

gives a collaring of M in P. Using this collaring we compute $(f_2f_1)_*([h])=[g']$, where g' may be chosen so that it is supported on $M\times[0,1]^2$, and on this set it is given by the commutative diagram

$$\begin{tikzcd}
I \times M \times [0,1] \arrow[d, "h \times id"'] & I \times M \times [0,1] \arrow[l, "\phi \times id"'] & I \times M \times [0,1]^2 \arrow[l, "id \times i"'] \arrow[d, "g'"] \\
I \times M \times [0,1] \arrow[r, "(\phi \times id^{-1})"] & I \times M \times [0,1] \arrow[r, "(id \times i)^{-1}"] & I \times M \times [0,1]^2 .
\end{tikzcd}$$

So it suffices to prove that $g|I \times M \times [0, 1]^2$ is isotopic to $g'|I \times M \times [0, 1]^2$ rel

$$(\{0\} \times M \times [0, 1]^2) \cup (I \times M \times [0, 1] \times \{1\}) \cup (I \times M \times \{1\} \times [0, 1]) .$$

This is equivalent to proving that the composition

$$(*) \quad (\phi_* \times id)(id \times i)^{-1}(\phi \times id)^{-1}(h \times id)(\phi \times id)(id \times i)(\phi_* \times id)^{-1}$$

is isotopic to $h \times id$ rel $\{0\} \times M \times [0, 1]^2$.

Note that $\phi \times id$ and $\phi_* \times id$ are clearly isotopic to their respective identities. Thus to prove that $(*)$ is isotopic to $h \times id$ it suffices to prove that $(id \times i)^{-1}(h \times id)(id \times i)$ is isotopic to $h \times id$ rel $\{0\} \times M \times [0, 1]^2$, and for this it suffices to prove that $(id \times i)(h \times id)(id \times i)^{-1}$ is isotopic to $h \times id$ rel $\{0\} \times M \times [0, 1]$. It follows from [2] that i can be chosen so that the M-coordinate is moved as little as we please. In fact there exists an isotopy $i_t \colon M \times [0, 1]^2 \to M \times [0, 1]$, $0 \leqq t \leqq 1$, such that $i_0 = i|M \times [0, 1]^2$,

$$i_t(M \times [([0, u] \times \{u\}) \cup (\{u\} \times [0, u])]) = M \times \{u\} ,$$

for each $u \in [0, 1]$, and i_t extends to a proper homotopy

$$\tilde{i}_t \colon M \times [0, 1]^2 \longrightarrow M \times [0, 1] ,$$

$0 \leqq t \leqq 1$, by defining

$$\tilde{i}_1(\{m\} \times [([0, u] \times \{u\}) \cup (\{u\} \times [0, u])]) = \{m\} \times \{u\} ,$$

for all $m \in M$ and $u \in [0, 1]$. Then

$$(id \times \tilde{i}_t)(h \times id)(id \times \tilde{i}_t)^{-1}$$

provides an isotopy of $(id \times i)(h \times id)(id \times i)^{-1}$ to $h \times id$.

III. For the last part of the proof we establish $(id)_* = id$. Choose any Q-manifold M and consider $id \colon M \times [0, 2] \to M \times [0, 2]$. We will prove that $(id)_*([h]) = [h]$, for all $[h] \in \pi_0\mathscr{C}(M \times [0, 2])$. Choose a homeomorphism

$$i \colon M \times [0, 2] \longrightarrow M \equiv M \times \{0\} \subset M \times [0, 2]$$

which is homotopic to the identity on $M \times [0, 2]$ and let $[h] \in \pi_0\mathscr{C}(M \times [0, 2])$ be given. Then $(id)_*([h]) = [g]$, where g may be chosen so that it is supported on $I \times M \times [0, 1]$, and on this set it is given by the commutative diagram

$$\begin{array}{ccccc} I\times M\times [0, 2]\times [0, 1] & \xleftarrow{\phi \times id} & I\times M\times [0, 2]\times [0, 1] & \xleftarrow{(id\times i\times id)^{-1}} & I\times M\times [0, 1] \\ \downarrow{\scriptstyle h \times id} & & & & \downarrow{\scriptstyle g} \\ I\times M\times [0, 2]\times [0, 1] & \xrightarrow{(\phi\times id)^{-1}} & I\times M\times [0, 2]\times [0, 1] & \xrightarrow{id \times i \times id} & I\times M\times [0,1] . \end{array}$$

(Recall that ϕ operates on $I \times [0, 1]$.) Define μ by the commutative diagram

$$\begin{array}{ccc} I \times M \times [0, 2]^2 & \xleftarrow{(id \times i \times id)^{-1}} & I \times M \times [0, 2] \\ \downarrow{\scriptstyle h \times id} & & \downarrow{\scriptstyle \mu} \\ I \times M \times [0, 2]^2 & \xrightarrow{id \times i \times id} & I \times M \times [0, 2] . \end{array}$$

We will prove that $[g] = [\mu]$ and $[\mu] = [h]$. This will fulfill our requirements.

To see that $[g] = [\mu]$ let $\theta_t\colon [0, 2] \to [0, t]$, $1 \leqq t \leqq 2$, be the unique linear homeomorphism such that $\theta_t(0) = 0$ and let g_t, $1 \leqq t \leqq 2$, be the isotopy from $I \times M \times [0, 2]$ to itself defined by

$$g_t = (id \times \theta_t)^{-1} g (id \times \theta_t) .$$

Then $[g] = [g_2] = [g_1]$. Similarly we *get* $[\mu] = [\mu_1]$, where μ_1 is the composition

$$\mu_1 = (id \times \theta_1)^{-1} \mu (id \times \theta_1) .$$

Then to get $[g_1] = [\mu_1]$ we just use the fact that $\phi \times id$ is isotopic to the identity. Thus $[g] = [\mu]$.

Finally we prove that $[\mu] = [h]$. For this we will need to know that i can be chosen so that there exists a proper homotopy $i_t\colon M \times [0, 2] \to M$, $0 \leqq t \leqq 1$, such that $i_0 = i$, i_t is a homeomorphism for $0 \leqq t < 1$, and $i_1\colon M \times [0, 2] \to M$ is the projection map. For i_t we again appeal to [2]. Let $\alpha\colon M\times [0, 2]^2 \to M \times [0, 2]^2$ be the homeomorphism defined by $\alpha(m, t, u) = (m, u, t)$. Then let β_t, $0 \leqq t \leqq 1$, be defined by the commutative diagram

$$\begin{array}{ccccc} I \times M \times [0, 2]^2 & \xleftarrow{id \times \alpha} & I \times M \times [0, 2]^2 & \xleftarrow{(id \times i_t \times id)^{-1}} & I\times M \times [0, 2] \\ \downarrow{\scriptstyle h \times id} & & & & \downarrow{\scriptstyle \beta_t} \\ I \times M \times [0, 2]^2 & \xrightarrow{id \times \alpha} & I \times M \times [0, 2]^2 & \xrightarrow{id \times i_t \times id} & I \times M \times [0, 2] . \end{array}$$

This gives $[\beta_0] = [h]$. To show that $[\beta_0] = [\mu]$ we just use the fact that α is isotopic to the identity. For this we use the fact that M is homeomorphic to $M \times Q$ and any homeomorphism on Q is isotopic to the identity [14].

We now establish Theorem 2.1, a result which will be needed in the sequel. First it will be convenient to establish two lemmas.

LEMMA 2.1. *If M is a Q-manifold, $h \in \mathscr{C}(M)$, and $\times O: M \to M \times [0, 2]$ is given by $\times O(m) = (m, 0)$, then $(\times O)_*([h]) = [h \times id_{[0,2]}]$.*

Proof. Using the definition we have $(\times O)_*([h]) = [g]$, where g is supported on $I \times M \times [0, 1]$ and on this set it is given by $g = (\phi \times id)^{-1}(h \times id)(\phi \times id)$. We must prove that $[g] = [h \times id]$. As in the proof of Theorem 1 let $\theta_t: [0, 2] \to [0, t]$, $1 \leqq t \leqq 2$, be the unique linear homeomorphism such that $\theta_t(0) = 0$. Using θ_t it follows that $[g] = [g_1]$, where

$$g_1 = (id \times \theta_1)^{-1}(\phi \times id)^{-1}(h \times id)(\phi \times id)(id \times \theta_1) .$$

Using the fact that $\phi \times id$ is isotopic to the identity we get

$$[g_1] = [(id \times \theta_1)^{-1}(h \times id)(id \times \theta_1)] = [h \times id] .$$

LEMMA 2.2. *Let M, N be Q-manifolds and assume that $M \times [0, 2]$ is a closed subset of N such that $\mathrm{Bd}\,(M \times [0, 2]) = M \times \{2\}$. Choose $h \in \mathscr{C}(M \times [0, 2])$ such that $h = id$ on $I \times M \times \{2\}$ and define $\tilde{h} \in \mathscr{C}(N)$ which extends h by the identity. Then the inclusion-induced homomorphism $\pi_0\mathscr{C}(M \times [0, 2]) \to \pi_0\mathscr{C}(N)$ sends $[h]$ to $[\tilde{h}]$.*

Proof. It is clear that we may additionally assume that $h = id$ on $I \times M \times [1, 2]$. Define $\mu: I \times M \times [0, 1] \to I \times M \times [0, 1]$ by $\mu = (\phi \times id)h(\phi \times id)^{-1}$. Then $\mu = id$ on $\{0\} \times M \times [0, 1]$. Using Lemma 2.1 and the fact that $\times 0: M \to M \times [0, 1]$ is a proper homotopy equivalence we can find an $f \in \mathscr{C}(M)$ such that $[f \times id] = [\mu]$. Thus $(\phi \times id)^{-1}(f \times id)(\phi \times id)$ is isotopic to $h | I \times M \times [0, 1]$ rel $(\{0\} \times M \times [0, 1]) \cup (I \times M \times \{1\})$. This implies that $i_*([f]) = [h]$, where i is the map $\times 0: M \to M \times [0, 2]$. It j is the inclusion $M \times [0, 2] \hookrightarrow N$, then we also have $(ji)_*([f]) = [\tilde{h}]$. Therefore $j_*([h]) = [\tilde{h}]$.

THEOREM 2.1. *Let M, N be Q-manifolds such that M is closed in N and such that $\mathrm{Bd}\,(M)$ is a bicollared Q-manifold. Choose $[h] \in \pi_0\mathscr{C}(M)$ such that $h = id$ on $I \times \mathrm{Bd}\,(M)$ and define $g \in \mathscr{C}(N)$ which extends h by the identity. Then the inclusion-induced homomorphism $\pi_0\mathscr{C}(M) \to \pi_0\mathscr{C}(N)$ sends $[h]$ to $[g]$.*

Proof. We first note that M and N can be replaced by $M \times [0, 1]$ and $N \times [0, 1]$, where we now have $h \in \mathscr{C}(M \times [0, 1])$ and $g \in \mathscr{C}(N \times [0, 1])$. This easily follows from Lemma 2.1 and the following commutative diagram:

$$\begin{array}{ccc} M \times [0,1] & \hookrightarrow & N \times [0,1] \\ {\scriptstyle \times 0}\uparrow & & \uparrow{\scriptstyle \times 0} \\ M & \hookrightarrow & N \end{array}$$

Define $h' \in \mathscr{C}(M \times [0, 1])$ by setting $h' = (\phi \times id)^{-1}h(\phi \times id)$. Then $h' = id$ on $I \times M \times \{1\}$ and clearly $[h'] = [h]$. We also define $g' \in \mathscr{C}(N \times [0, 1])$ which extends h' by the identity and note that $[g'] = [g]$. It is easy to see that $[h'] = [h'']$ and $[g'] = [g'']$, where $h'' = id$ on $(I \times \mathrm{Bd}(M) \times [0, 1]) \cup (I \times M \times [0, 1])$ and g'' extends h'' by the identity. Let $\tau \in \mathscr{C}(M \times [0, 1/2])$ be defined by $\tau = h'' | I \times M \times [0, 1/2]$. Let i be the inclusion $M \times [0, 1/2] \hookrightarrow M \times [0, 1]$ and let j be the inclusion $M \times [0, 1] \hookrightarrow N \times [0, 1]$. Then Lemma 2.2 implies that $i_*([\tau]) = [h'']$ and similarly $(ji)_*([\tau]) = [g'']$. Therefore $j_*([h'']) = [g'']$.

3. The first exact sequence. The purpose of this section is to establish the exact sequence of Theorem 2, and the first step will be to construct the homomorphism $\tau_\infty: \pi_0\mathscr{C}(M) \to \varprojlim Wh\pi_1(M)$. Before doing this it will be convenient to prove Lemma 3.1 below. The following notation will be useful: (1) If M is a Q-manifold, then $M_1 \subset M$ is said to be *clean* provided that M_1 is a compact Q-manifold and $\mathrm{Bd}(M_1)$ is a bicollared Q-manifold; (2) If M_1 and M_2 are clean in M, then $M_1 \subset\subset M_2$ means that M_1 lies in the interior of M_2.

LEMMA 3.1. *If M is a Q-manifold, $h \in \mathscr{C}(M)$ and $M_1 \subset M$ is clean, then there exists a clean M_2 such that $h(I \times M_1) \subset\subset I \times M_2$ and such that the inclusion*

$$\{0\} \times (M_2 - \mathrm{Int}(M_1)) \hookrightarrow I \times M_2 - h(I \times \mathrm{Int}(M_1))$$

is a homotopy equivalence.

Proof. Choose clean M', M'' and M_2 such that

$$h(I \times M_1) \subset\subset I \times M' \subset\subset h(I \times M'') \subset\subset I \times M_2 \,.$$

We will prove that M_2 fulfills our requirements. By pushing down in the I-direction we can get a homotopy r_t of $I \times M_2 - h(I \times \mathrm{Int}(M_1))$ into itself such that $r_0 = id$, r_1 takes $I \times M_2 - h(I \times \mathrm{Int}(M_1))$ onto

$$(*) \qquad [I \times M' - h(I \times \mathrm{Int}(M_1))] \cup [\{0\} \times (M_2 - \mathrm{Int}(M_1))] \,,$$

and $r_t = id$ on $(*)$ for each t. Similarly let s_t be a homotopy of

$$(**) \qquad h(I \times (M'' - \mathrm{Int}(M_1))) \cup [\{0\} \times (M_2 - \mathrm{Int}(M_1))]$$

into itself such that $s_0 = id$, s_1 takes $(**)$ onto $\{0\} \times (M_2 - \mathrm{Int}(M_1))$, and $s_t = id$ on $\{0\} \times (M_2 - \mathrm{Int}(M_1))$, for each t. Define a homotopy f_t of $I \times M_2 - h(I \times \mathrm{Int}(M_1))$ into itself by

$$f_t = \begin{cases} r_{2t}, & 0 \leqq t \leqq 1/2 \\ s_{2t-1}r_1, & 1/2 \leqq t \leqq 1 . \end{cases}$$

Then f_t gives a strong deformation retraction of $I \times M_2 - h(I \times \mathrm{Int}(M_1))$ onto $\{0\} \times (M_2 - \mathrm{Int}(M_1))$.

Below we define a preliminary version of τ_∞, but first it will be convenient to recall the connection between homeomorphisms on Q-manifolds and simple homotopy theory. It follows from [1] that to each homotopy equivalence $f: M \to N$ between compact Q-manifolds there is a torsion $\tau(f) \in Wh\pi_1(N)$ (the Whitehead torsion of f) which vanishes iff f is homotopic to a homeomorphism. This enables one to do most of the standard results of simple homotopy theory for compact Q-manifolds. This gives us such tools as the Sum Theorem, the formula for the torsion of a composition, etc. We refer the reader to [5] and [6] for further details.

The function $\tau_\infty(h, M_1)$. Choose a Q-manifold M and $M_1 \subset M$ clean. Then to each $h \in \mathscr{C}(M)$ we are going to define an element $\tau_\infty(h, M_1) \in Wh\pi_1(M - \mathrm{Int}(M_1))$. This will be used later on to define the homomorphism τ_∞. Choose a clean M_2 as in Lemma 3.1 and let i be the inclusion

$$\{0\} \times (M_2 - \mathrm{Int}(M_1)) \hookrightarrow I \times M_2 - h(I \times \mathrm{Int}(M_1)) ,$$

which is a homotopy equivalence. Then we have a torsion $\tau(i)$ lying in $Wh\pi_1(I \times M_2 - h(I \times \mathrm{Int}(M_1)))$. Let α be the composition

$$\begin{aligned} Wh\pi_1(I \times M_2 - h(I \times \mathrm{Int}(M_1))) &\longrightarrow Wh\pi_1(I \times M - h(I \times \mathrm{Int}(M_1)) \\ &\longrightarrow Wh\pi_1(M - \mathrm{Int}(M_1)) , \end{aligned}$$

where the first homomorphism is inclusion induced and the second is induced by the map from $I \times M - h(I \times \mathrm{Int}(M_1))$ to $M - \mathrm{Int}(M_1)$ which sends $h(t, x)$ to x. Then we define

$$\tau_\infty(h, M_1) = \alpha(\tau(i)) \in Wh\pi_1(M - \mathrm{Int}(M_1)) .$$

The following result establishes some basic properties of $\tau_\infty(h, M_1)$.

LEMMA 3.2. (1) *$\tau_\infty(h, M_1)$ is well-defined.*

(2) *If M_1, M_1' are clean and $M_1 \subset\subset M_1'$ then $\tau_\infty(h, M_1)$ is the image of $\tau_\infty(h, M_1')$ under the inclusion-induced homomorphism $Wh\pi_1(M - \mathrm{Int}(M_1')) \to Wh\pi_1(M - \mathrm{Int}(M_1))$.*

(3) *For any* $h, h' \in \mathscr{C}(M)$ *we have* $\tau_\infty(h'h, M_1) = \tau_\infty(h', M_1) + \tau_\infty(h, M_1)$.

(4) *If* $[h] = [h']$, *then* $\tau_\infty(h, M_1) = \tau_\infty(h', M_1)$.

Proof. (1) Choose M_2, i and α as in the definition of $\tau_\infty(h, M_1)$ and let $M_2' \subset M$ be clean such that $M_2 \subset\subset M_2'$. Then

$$i'\colon \{0\} \times (M_2' - \mathrm{Int}(M_1)) \hookrightarrow I \times M_2' - h(I \times \mathrm{Int}(M_1))$$

is also a homotopy equivalence and in analogy with the definition of $\tau_\infty(h, M_1) = \alpha(\tau(i))$ we could also define $\tau_\infty(h, M_1) = \alpha'(\tau(i'))$, where α' is the composition

$$\begin{aligned} &Wh\pi_1(I \times M_2' - h(I \times \mathrm{Int}(M_1)) \longrightarrow Wh\pi_1(I \times M - h(I \times \mathrm{Int}(M_1)) \\ &\quad \longrightarrow Wh\pi_1(M - \mathrm{Int}(M_1)) . \end{aligned}$$

We must prove that $\alpha(\tau(i)) = \alpha'(\tau(i'))$. For this it will suffice to prove that $\tau(i')$ is the inclusion-induced image of $\tau(i)$ in $Wh\pi_1(I \times M_2' - h(I \times \mathrm{Int}(M_1)))$. But this is an easy consequence of the Sum Theorem for torsion.

(2) Again choose M_2, i and α as in the definition of $\tau_\infty(h, M_1)$ and note that (1) implies that we may also select M_2 so that

$$i'\colon \{0\} \times (M_2 - \mathrm{Int}(M_1')) \hookrightarrow I \times M_2 - h(I \times \mathrm{Int}(M_1'))$$

is a homotopy equivalence. Thus $\tau_\infty(h, M_1') = \alpha'(\tau(i'))$, where

$$\begin{aligned} &\alpha'\colon Wh\pi_1(I \times M_2 - h(I \times \mathrm{Int}(M_1'))) \longrightarrow Wh\pi_1(I \times M - h(I \times \mathrm{Int}(M_1'))) \\ &\quad \longrightarrow Wh\pi_1(M - \mathrm{Int}(M_1')) . \end{aligned}$$

It will suffice to prove that $\tau(i)$ is the inclusion-induced image of $\tau(i')$ in $Wh\pi_1(I \times M_2 - h(I \times \mathrm{Int}(M_1)))$. Again this is an easy consequence of the Sum Theorem.

(3) With M_2, i and α as above we can choose a clean M_3 in M such that $h'(I \times M_2) \subset\subset I \times M_3$ and such that the inclusion

$$i'\colon \{0\} \times (M_3 - \mathrm{Int}(M_2)) \hookrightarrow I \times M_3 - h'(I \times \mathrm{Int}(M_2))$$

is a homotopy equivalence. Let j be the inclusion

$$j\colon \{0\} \times (M_3 - \mathrm{Int}(M_1)) \hookrightarrow I \times M_3 - h'h(I \times \mathrm{Int}(M_1)) ,$$

which is also a homotopy equivalence. Using the Sum Theorem we see that $\tau(j)$ is the sum of the inclusion-induced images of $\tau(\tilde{i})$ and $\tau(i')$ in $Wh\pi_1(I \times M_3 - h'h(I \times \mathrm{Int}(M_1)))$, where

$$\tilde{i}\colon \{0\} \times (M_2 - \mathrm{Int}(M_1)) \hookrightarrow h'(I \times M_2) - h'h(I \times \mathrm{Int}(M_1)) .$$

Note that the image of $\tau(j)$ under the composition

$$r: Wh\pi_1(I \times M_3 - h'h(I \times \mathrm{Int}\,(M_1))) \xrightarrow{\beta} Wh\pi_1(M_3 - \mathrm{Int}\,(M_1)) \longrightarrow Wh\pi_1(M - \mathrm{Int}\,(M_1))$$

is $\tau_\infty(h'h, M_1)$, where the first homomorphism is induced by a homotopy inverse of $\times 0$: $M_3 - \mathrm{Int}(M_1) \to I \times M_3 - h'h(I \times \mathrm{Int}(M_1))$ and the second is induced by inclusion. It follows from the Sum Theorem that $\beta(\tau(j)) = s(\tau(\tilde{i})) + t(\tau(i'))$, where

$$s: Wh\pi_1(h'(I \times M_2) - h'h(I \times \mathrm{Int}\,(M_1))) \xrightarrow{\beta_1} Wh\pi_1\,(M_2 - \mathrm{Int}\,(M_1)) \longrightarrow Wh\pi_1(M_3 - \mathrm{Int}\,(M_1))\,,$$

$$t: Wh\pi_1(I \times M_3 - h'(I \times \mathrm{Int}\,(M_2))) \xrightarrow{\beta_2} Wh\pi_1(M_3 - \mathrm{Int}\,(M_2)) \longrightarrow Wh\pi_1(M_3 - \mathrm{Int}\,(M_1))$$

are defined in analogy with r. Let

$$Wh\pi_1\,(I \times M_2 - h(I \times \mathrm{Int}\,(M_1))) \xrightarrow{\beta_3} Wh\pi_1(M_2 - \mathrm{Int}\,(M_1))$$

be defined in analogy with β_1. To conclude that $\tau_\infty(h'h, M_1) = \tau_\infty(h', M_1) + \tau_\infty(h, M_1)$ all we need to do is prove that $\beta_1(\tau(\tilde{i})) = \beta_3(\tau(i))$. This is an easy consequence of the formula for the torsion of a composition and the fact that the torsion of

$$h'|: I \times M_2 - h(I \times \mathrm{Int}\,(M_1)) \longrightarrow h'(I \times M_2) - h'h(I \times \mathrm{Int}\,(M_1))$$

is zero.

(4) Choose M_2, i and α so that $\tau_\infty(h, M_1) = \alpha(\tau(i))$. We can choose M_2 so large such that $h'(I \times M_1) \subset\subset I \times M_2$ and

$$i': \{0\} \times (M_2 - \mathrm{Int}\,(M_1)) \hookrightarrow I \times M_2 - h'(I \times \mathrm{Int}\,(M_1))$$

is a homotopy equivalence. Then $\tau_\infty(h', M_1) = \alpha'(\tau(i'))$, with a choice of α' which is analogous to the choice of α. Let $h_t: I \times M \to I \times M$ be an isotopy such that $h_0 = h$, $h_1 = h'$ and $h_t | \{0\} \times M = id$. We may assume that $h_t(I \times M_1) \subset\subset I \times M_2$, for each t. At this point we have to apply the Isotopy Extension Theorem for Q-manifolds. Since it is also needed at several other places in the paper we state it below. The lsotopy Extension Theorem implies that there exists a homeomorphism $f: I \times M_2 \to I \times M_2$ such that $fh = h'$ on $I \times M_1$ and $f = id$ on $(\{0\} \times M_2) \cup (I \times \mathrm{Bd}\,(M_2))$. Then $f | I \times M_2 - h(I \times \mathrm{Int}\,(M_1))$ induces an isomorphism θ of $Wh\pi_1(I \times M_2 - h(I \times \mathrm{Int}\,(M_1)))$ onto $Wh\pi_1(I \times M_2 - h'(I \times \mathrm{Int}\,(M_1)))$ which takes $\tau(i)$ to $\tau(i')$. This easily follows because $\tau(f) = 0$. We have a commutative diagram

$$\begin{tikzcd}
Wh\pi_1(I\times M_2-h(I\times \operatorname{Int}(M_1))) \arrow[r] \arrow[d, "\theta"', "\approx"] & Wh\pi_1(I\times M-h(I\times \operatorname{Int}(M_1))) \arrow[dr] & \\
 & & Wh\pi_1(M-\operatorname{Int}(M_1)) \\
Wh\pi_1(I\times M_2-h'(I\times \operatorname{Int}(M_1))) \arrow[r] & Wh\pi_1(I\times M-h'(I\times \operatorname{Int}(M_1))) \arrow[ur] &
\end{tikzcd}$$

where the top composition is α and the bottom is α'. Since $\theta(\tau(i)) = \tau(i')$ we have $\tau_\infty(h, M_1) = \tau_\infty(h', M_1)$.

We now state the Isotopy Extension Theorem used in the proof above. It follows from [7] or it can be deduced from [4].

ISOTOPY EXTENSION THEOREM. *Let M be a Q-manifold, $U\subset M$ be open, and let $C\subset U$ be compact. If $g_t\colon I\times U\to I\times M$ is an isotopy of open embeddings such that $g_t|\{0\}\times U = id$, then there exists an ambient isotopy $h_t\colon I\times M\to I\times M$ such that $h_0 = id$, $h_t|\{0\}\times M = id$, and $h_tg_0 = g_t$ on $I\times C$, for each t.*

Description of τ_∞. We now define the homomorphism

$$\tau_\infty\colon \pi_0\mathscr{C}(M) \longrightarrow \varprojlim Wh\pi_1(M)$$

of Theorem 2. Write $M=\bigcup_{i=1}^\infty M_i$, where the M_i's are clean and $M_i\subset\subset M_{i+1}$. This can be done because all Q-manifolds are triangulable [11]. In a natural way we may represent $\varprojlim Wh\pi_1(M)$ by

$$\varprojlim\{Wh\pi_1(M-\operatorname{Int}(M_i))\}_{i=1}^\infty ,$$

which is the subgroup of $\prod_{i=1}^\infty Wh\pi_1(M-\operatorname{Int}(M_i))$ which consists of all $(\tau_1, \tau_2, \cdots)$ such that τ_{i+1} is sent to τ_i by the inclusion-induced homomorphism. This representation of $\varprojlim Wh\pi_1(M)$ is independent of the choice of the M_i's. For any $[h]\in\pi_0\mathscr{C}(M)$ we note that Lemma 3.2(2) implies that $(\tau_\infty(h, M_1), \tau_\infty(h, M_2), \cdots)\in\varprojlim\{Wh\pi_1(M-\operatorname{Int}(M_i))\}_{i=1}^\infty$. Then we define $\tau_\infty([h])$ to be the element of $\varprojlim Wh\pi_1(M)$ represented by $(\tau_\infty(h, M_1), \tau_\infty(h, M_2), \cdots)$. It follows from Lemma 3.2(4) that this definition depends only on $[h]$ and it follows from Lemma 3.2(3) that τ_∞ is a homeomorphism, i.e. $\tau_\infty([h'][h]) = \tau_\infty([h']) + \tau_\infty([h])$. Finally it follows from Lemma 3.2(2) that $\tau_\infty([h])$ is independent of the choice of the M_i's.

Proof of Theorem 2. Recall that we want to establish an exact sequence

$$\pi_0\mathscr{C}(M) \xrightarrow{\tau_\infty} \varprojlim Wh\pi_1(M) \xrightarrow{\phi} Wh\pi_1(M) ,$$

where ϕ is yet to be defined. For this proof we will assume that $M = \bigcup_{i=1}^{\infty} M_i$ is given as above and $\varprojlim Wh\pi_1(M)$ will be represented by

$$\varprojlim \{Wh\pi_1(M - \mathrm{Int}(M_i))\}_{i=1}^{\infty} .$$

We define ϕ by $\phi(\tau_1, \tau_2, \cdots) = \bar{\tau}_1$, the *inclusion-induced image of* τ_1 in $Wh\pi_1(M)$. To see that $\phi\tau_\infty = 0$ choose any $h \in \mathscr{C}(M)$ and consider $\tau_\infty(h, M_1) \in Wh\pi_1(M - \mathrm{Int}(M_1))$. We must show that the inclusion-induced homomorphism

$$(*) \qquad Wh\pi_1(M - \mathrm{Int}(M_1)) \longrightarrow Wh\pi_1(M)$$

sends $\tau_\infty(h, M_1)$ to 0. Assume that M_2 and i are as in the definition of $\tau_\infty(h, M_1)$ and let j be the inclusion $\{0\} \times M_2 \hookrightarrow I \times M_2$, which has zero torsion. By the Sum Theorem we have $\tau(j)$ equal to the inclusion-induced image of $\tau(i)$ in $Wh\pi_1(I \times M_2)$, which suffices to prove that $(*)$ sends $\tau_\infty(h, M_1)$ to 0.

For the other half of the proof we must show that $\mathrm{Ker}(\phi) \subset \mathrm{Im}(\tau_\infty)$. This is a little harder to do since we are trying to realize the elements of $\mathrm{Ker}(\phi)$ geometrically. Choose

$$(\tau_1, \tau_2, \cdots) \in \varprojlim \{Wh\pi_1(M - \mathrm{Int}(M_i))\}_{i=1}^{\infty}$$

such that $\phi(\tau_1, \tau_2, \cdots) = 0$. We must construct an element $h \in \mathscr{C}(M)$ such that $\tau_\infty([h]) = (\tau_1, \tau_2, \cdots)$.

Our first step will be to show that we can write $M = \bigcup_{i=1}^{\infty} N_i$, where the N_i's are clean and $N_i \subset\subset N_{i+1}$, and choose elements $\mu_i \in Wh_1\pi_1(\mathrm{Bd}(N_i))$ such that

(1) the inclusion-induced homomorphisms, $Wh\pi_1(\mathrm{Bd}(N_i)) \to Wh\pi_1(N_{i+1} - \mathrm{Int}(N_i))$ and $Wh\pi_1(\mathrm{Bd}(N_{i+1})) \to Wh\pi_1(N_{i+1} - \mathrm{Int}(N_i))$, send μ_i and μ_{i+1} to the same element,

(2) the inclusion-induced homomorphism, $Wh\pi_1(\mathrm{Bd}(N_1)) \to Wh\pi_1(N_1)$, sends μ_1 to 0,

(3) if μ'_i denotes the inclusion-induced image of μ_i in $Wh\pi_1(M - \mathrm{Int}(N_i))$, then $(\mu'_1, \mu'_2, \cdots)$ and $(\tau_1, \tau_2, \cdots)$ represent the same element of $\varprojlim Wh\pi_1(M)$.

The construction of $\{N_i\}_{i=1}^{\infty}$ and $\{\mu_i\}_{i=1}^{\infty}$ will follow from successive modifications of $\{M_i\}_{i=1}^{\infty}$ and $\{\tau_i\}_{i=1}^{\infty}$.

To begin we may assume that there are elements $\tau'_i \in Wh\pi_1(M_{i+1} - \mathrm{Int}(M_i))$ such that

(1) τ'_i is sent to τ_i by the inclusion-induced homomorphism $Wh\pi_1(M_{i+1} - \mathrm{Int}(M_i)) \to Wh\pi_1(M - \mathrm{Int}(M_i))$,

(2) the inclusion-induced homomorphisms,

$$Wh\pi_1(M_{i+2} - \mathrm{Int}(M_{i+1})) \longrightarrow Wh\pi_1(M_{i+2} - \mathrm{Int}(M_i))$$

and $Wh\pi_1(M_{i+1} - \mathrm{Int}(M_i)) \to Wh\pi_1(M_{i+2} - \mathrm{Int}(M_i))$, send τ'_{i+1} and τ'_i to the same element,

(3) τ'_1 is sent to 0 by $Wh\pi_1(M_2 - \mathrm{Int}(M_1)) \to Wh\pi_1(M_2)$.

If the elements $\{\tau'_i\}_{i=1}^{\infty}$ do not exist, we can find them by passing to a subsequence of $\{M_i\}_{i=1}^{\infty}$. This follows from the fact that $Wh\pi_1(M - \mathrm{Int}(M_i))$ is naturally the direct limit of

$$\{Wh\pi_1(M_j - \mathrm{Int}(M_i)) \mid j \geqq i + 1\}.$$

Next we show how to modify $\{M_i\}$ and $\{\tau'_i\}$ to get our required $\{N_i\}$ and $\{\mu_i\}$. For each i we will show how to construct N_i and μ_i, but we will leave it as an exercise for the reader to check details. Let $\alpha: M \to M \times [0, 1]$ be a homeomorphism such that $\alpha(M_i) = M_i \times [0, 1]$ and $\alpha(M_{i+1}) = M_{i+1} \times [0, 1]$. Choose a clean $N'_i \subset M \times [0, 1]$ so that N'_i is the union of $M_i \times [0, 1]$ and a subset of $(M_{i+1} - \mathrm{Int}(M_i)) \times [0, 1]$ which consists of the union of all $\{x\} \times [0, a_x]$, for $x \in M_{i+1} - \mathrm{Int}(M_i)$, such that $a_x = 1$ for $x \in \mathrm{Bd}(M_i)$ and $a_x = 0$ iff $x \in \mathrm{Bd}(M_{i+1})$. Thus $\mathrm{Bd}(N'_i) \hookrightarrow I \times (M_{i+1} - \mathrm{Int}(M_i))$ is a homotopy equivalence. Put $N_i = \alpha^{-1}(N'_i)$ and note that $\mathrm{Bd}(N_i) \hookrightarrow M_{i+1} - \mathrm{Int}(M_i)$ is a homotopy equivalence. Thus there exists an element $\mu_i \in Wh\pi_1(\mathrm{Bd}(N_i))$ which is sent to τ'_i by the inclusion-induced homomorphism $Wh\pi_1(\mathrm{Bd}(N_i)) \to Wh\pi_1(M_{i+1} - \mathrm{Int}(M_i))$. Then the reader can easily check that $\{N_i\}$ and $\{\mu_i\}$ fulfill our requirements.

Now that we have $\{N_i\}$ and $\{\mu_i\}$ we must show how to use them to construct our required $h \in \mathscr{C}(M)$. For notation we say that a clean $A \subset I \times M$ intersects $\{0\} \times M$ cleanly provided that

(1) $A \cap (\{0\} \times M)$ is clean in $\{0\} \times M$,

(2) $\mathrm{Bd}(A \cap (\{0\} \times M)) = \mathrm{Bd}(A) \cap (\{0\} \times M)$,

(3) there is a collaring of $A \cap (\{0\} \times M)$ in A which restricts to give a collaring of $\mathrm{Bd}(A) \cap (\{0\} \times M)$ in $\mathrm{Bd}(A)$.

In Lemma 3.3 below we will show that for each i there exists a clean A_i in $I \times (N_i - \mathrm{Int}(N_{i-1}))$ (where $N_0 = \phi$) such that

(1) A_i intersects $\{0\} \times M$ cleanly and $A_i \cap (\{0\} \times M)$ is a collar on $\{0\} \times \mathrm{Bd}(N_i)$,

(2) $I \times \mathrm{Bd}(N_i) \subset \mathrm{Int}(A_i)$ (Interior computed in $I \times N_i$),

(3) $I \times \mathrm{Bd}(N_i) \hookrightarrow A_i$ is a homotopy equivalence,

(4) $A_i \cap (\{0\} \times M) \hookrightarrow A_i$ is a homotopy equivalence,

(5) $\mathrm{Bd}(A_i) \cap (\{0\} \times M) \hookrightarrow \mathrm{Bd}(A_i)$ is a simple equivalence ($\mathrm{Bd}(A_i)$ is computed in $I \times N_i$),

(6) $\mathrm{Bd}(A_i) \hookrightarrow A_i$ is a homotopy equivalence,

(7) $\tau(A_i \cap (\{0\} \times M) \hookrightarrow A_i)$ is equal to the image of μ_i under

the composition

$$Wh\pi_1(\mathrm{Bd}\,(N_i)) \xrightarrow{(\times 0)} Wh\pi_1(A_i \cap (\{0\} \times M)) \longrightarrow Wh\pi_1(A_i)\,,$$

where the second homomorphism is inclusion-induced.

Let A_i^0 denote the projection of $A_i \cap (\{0\} \times M)$ into M, for each i. We will construct an element $h \in \mathscr{C}(M)$ such that for each i, h takes

$$\mathrm{Cl}\,(I \times [((N_i - \mathrm{Int}\,(N_{i-1})) \cup A_{i-1}^0) - A_i^0])$$

to

$$\mathrm{Cl}\,(I \times ((N_i - \mathrm{Int}\,(N_{i-1})) \cup A_{i-1}) - A_i)\,,$$

where $A_0 = \phi$, $A_0^0 = \phi$ and Cl denotes closure. Let us see how $\tau_\infty([h])$ is represented by the element $(\mu_1, \mu_2, \cdots)$. Write $M = \bigcup_{i=1}^\infty P_i$, where $P_i = \mathrm{Cl}\,(N_i - A_i^0)$. Then the P_i's are clean and $P_i \subset\subset P_{i+1}$. Note that the torsion of the homotopy equivalence

$$\{0\} \times (P_{i+1} - \mathrm{Int}\,(P_i)) \hookrightarrow I \times P_{i+1} - h(I \times \mathrm{Int}\,(P_i))$$

equals the image of μ_i under the composition

$$\begin{aligned} Wh\pi_1\,(\mathrm{Bd}\,(N_i)) &\xrightarrow{(\times 0)} Wh\pi_1(A_i \cap (\{0\} \times M)) \longrightarrow Wh\pi_1(A_i) \\ &\longrightarrow Wh\pi_1(I \times P_{i+1} - h(I \times \mathrm{Int}\,(P_i)))\,. \end{aligned}$$

Thus $\tau_\infty(h, P_i)$ equals the inclusion-induced image of μ_i in $Wh\pi_1(M - \mathrm{Int}\,(P_i))$. This implies that $\tau_\infty([h])$ is represented by $(\mu_1, \mu_2, \cdots)$.

Finally we show how to construct h. Repeatedly using the Sum Theorem the reader can easily check that the following inclusions are simple equivalences:

$$\begin{aligned} &\{0\} \times P_1 \hookrightarrow \mathrm{Cl}\,(I \times N_1 - A_1)\,, \\ &\{0\} \times (P_i - \mathrm{Int}\,(P_{i-1})) \hookrightarrow \mathrm{Cl}\,(I \times ((N_i - \mathrm{Int}\,(N_{i-1})) \cup A_{i-1}) - A_i)\,. \end{aligned}$$

Then we can easily construct our required h by (1) using the fact that simple equivalences between compact Q-manifolds are homotopic to homeomorphisms and (2) Z-set unknotting, which enables us to fit various compact pieces together.

LEMMA 3.3. *Let M be a compact Q-manifold and choose $\tau \in Wh\pi_1(M)$. Then we can find a clean $A \subset I \times M \times [0, 1)$ such that*

(1) *A intersects $\{0\} \times M \times [0, 1)$ cleanly and $A \cap (\{0\} \times M \times [0, 1))$ is a collar on $\{0\} \times M \times \{0\}$,*

(2) *$I \times M \times \{0\} \subset \mathrm{Int}\,(A)$,*

(3) *$I \times M \times \{0\} \hookrightarrow A$ is a homotopy equivalence,*

(4) *$A \cap (\{0\} \times M \times [0, 1)) \hookrightarrow A$ is a homotopy equivalence,*

(5) *$\mathrm{Bd}(A) \cap (\{0\} \times M \times [0, 1)) \hookrightarrow \mathrm{Bd}(A)$ is a simple equivalence,*

(6) *$\mathrm{Bd}(A) \hookrightarrow A$ is a homotopy equivalence,*

(7) *$\tau(A \cap (\{0\} \times M \times [0, 1)) \hookrightarrow A)$ equals the image of τ under the composition*

$$Wh\pi_1(M) \xrightarrow{(\times 0)_*} Wh\pi_1(A \cap (\{0\} \times M \times [0, 1))) \longrightarrow Wh\pi_1(A)\ .$$

Proof. Our first step will be to prove that there exists a clean $N \subset M \times [0, 1)$ such that

(1) $M \times \{0\} \subset \mathrm{Int}(N)$,

(2) $M \times \{0\} \hookrightarrow N$ is a homotopy equivalence,

(3) $\mathrm{Bd}(N) \hookrightarrow N$ is a homotopy equivalence,

(4) $\tau(\mathrm{Bd}(N) \hookrightarrow N) + \tau(M \times \{0\} \hookrightarrow N) = 0$,

(5) $\tau(M \times \{0\} \hookrightarrow N)$ equals the image of τ under the composition

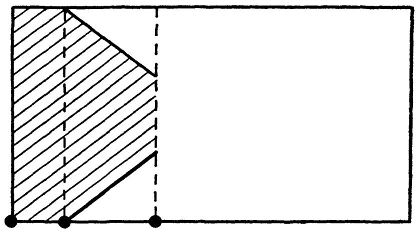
$$Wh\pi_1(M) \xrightarrow{(\times 0)_*} Wh\pi_1(M \times \{0\}) \longrightarrow Wh\pi_1(N)\ .$$

To begin let $f: M \to M_1$ be a homotopy equivalence such that $\tau(f) = f_*(\tau)$ and M_1 is a compact Q-manifold. (Here $f_*: Wh\pi_1(M) \to Wh\pi_1(M_1)$ is induced by f.) By taking the mapping cylinder of f and thickening it we get a compact Q-manifold P_1 containing M, M_1 as Z-sets such that $M_1 \hookrightarrow P_1$ is a simple equivalence and such that $\tau(M \hookrightarrow P_1) = (M \hookrightarrow P_1)_*(\tau)$. The thickening that is required is provided by West's Mapping Cylinder Theorem [13]. Next let $g: M_2 \to M_1$ be a homotopy equivalence such that $\tau(g) = f_*(-\tau)$ and M_2 is a compact Q-manifold. Again we thicken the mapping cylinder of g and obtain a compact Q-manifold P_2 containing M_1 and M_2 as Z-sets such that $M_1 \hookrightarrow P_2$ is a simple equivalence and $\tau(M_2 \hookrightarrow P_2) = (M_1 \hookrightarrow P_2)_* f_*(-\tau)$. Define $N' = P_1 \bigcup_{M_1} P_2$, the Q-manifold formed by sewing P_1 to P_2 along M_1. Then we see that $\tau(M \hookrightarrow N') + \tau(M_2 \hookrightarrow N') = 0$ and $\tau(M \hookrightarrow N') = (M \hookrightarrow N')_*(\tau)$. By the above comments it is now clear that we can find a compact Q-manifold P_3 containing M_2 as a Z-set such that $M_2 \hookrightarrow P_3$ is a homotopy equivalence and if $N'' = N' \bigcup_{M_2} P_3$, then $\tau(M \hookrightarrow N'') = 0$. Thus there exists a homeomorphism $u: N'' \to M \times [0, 1]$ such that $u(M) = M \times \{0\}$ and $u(N') \subset M \times [0, 1]$. Then $N = u(N')$ fulfills our requirements. (Note that in order to get $u(M) = M \times \{0\}$ and $u(N') \subset M \times [0, 1]$ we have to use Z-set unknotting.)

Finally we show how to get our required A from N. Choose $\varepsilon > 0$ so that $M \times [0, \varepsilon] \subset \mathrm{Int}(N)$ and let A be a clean set carved out of $I \times N$ which is the union of all $\{x\} \times [a_x, b_x]$, where

(1) $[a_x, b_x] = [0, 1]$ if $x \in M \times [0, \varepsilon]$,
(2) $[a_x, b_x] \subset (0, 1)$ if $x \in N - M \times [0, \varepsilon]$.

In the picture below the shaded region represents A. It is obtained from $I \times N$ by "poking in" in the I-direction.



The reader can easily check that A fulfills out requirements.

4. **The functors** $\varinjlim \pi_0\mathscr{E}$, $\varprojlim \pi_0\mathscr{E}$ **and** $\varprojlim{}^1\pi_0\mathscr{E}$. In §2 we introduced $\pi_0\mathscr{E}$, a covariant proper homotopy functor from the category of Q-manifolds and proper maps to the category of groups and homomorphisms. In this section we will use the restriction of $\pi_0\mathscr{E}$ to the category of compact Q-manifolds and (ordinary) maps to define functors $\varinjlim \pi_0\mathscr{E}$, $\varprojlim \pi_0\mathscr{E}$ and $\varprojlim{}^1 \pi_0\mathscr{E}$. There are used in the exact sequence of Theorem 3.

1. **The functor** $\varinjlim \pi_0\mathscr{E}$. We will first describe $\varinjlim \pi_0\mathscr{E}$, a covariant homotopy functor from the category of Q-manifolds and maps to the category of abelian groups. It is defined as follows. For any Q-manifold M let $\varinjlim \pi_0\mathscr{E}(M)$ denote the direct limit of the direct system

$$\{\pi_0\mathscr{E}(M_1) \mid M_1 \subset M \text{ is a compact } Q\text{-manifold}\}\ ,$$

where the homomorphisms are inclusion-induced. It follows from D of §1 that $\varinjlim \pi_0\mathscr{E}(M)$ is abelian and we write it multiplicatively. If $f: M \to N$ is a map of Q-manifolds and $M_1 \subset M$ is a compact Q-manifold, then we can choose a compact Q-manifold $N_1 \subset N$ such that $f(M_1) \subset N_1$. This induces a homomorphism

$$\pi_0\mathscr{E}(M_1) \xrightarrow{(f \mid M_1)_*} \pi_0\mathscr{E}(N_1) \longrightarrow \varinjlim \pi_0\mathscr{E}(N)\ ,$$

where the last arrow follows from the direct limit construction. This homomorphism is independent of the choice of N_1. If M_2 is a larger compact Q-manifold in M, then we get a similarly-defined homomorphism $\pi_0\mathscr{C}(M_2) \to \varinjlim \pi_0\mathscr{C}(N)$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_0\mathscr{C}(M_2) \arrow[dr] & \\ & \varinjlim \pi_0\mathscr{C}(N) . \\ \pi_0\mathscr{C}(M_1) \arrow[uu] \arrow[ur] & \end{tikzcd}$$

Here the vertical arrow is inclusion-induced. In this manner there is induced a homomorphism $f_*: \varinjlim \pi_0\mathscr{C}(M) \to \varinjlim \pi_0\mathscr{C}(N)$. The following result is easy.

THEOREM 4.1. *$\varinjlim \pi_0\mathscr{C}$ is a covariant homotopy functor from the category of all Q-manifold and* (ordinary) *maps to the category of abelian groups.*

II. **The functor $\varprojlim \pi_0\mathscr{C}$.** We now describe $\varprojlim \pi_0\mathscr{C}$ a covariant proper homotopy functor from the category of Q-manifolds and proper maps to the category of abelian groups. For any Q-manifold M let $\varprojlim \pi_0\mathscr{C}(M)$ denote the inverse limit of the inverse system

$$\{\varinjlim \pi_0\mathscr{C}(M - C) \mid C \subset M \text{ compact}\} ,$$

where the homomorphisms are inclusion-induced. If $f: M \to N$ is a proper map of Q-manifolds and $C \subset N$ is compact, then we get a homomorphism

$$\varprojlim \pi_0\mathscr{C}(M) \longrightarrow \varinjlim \pi_0\mathscr{C}(M - f^{-1}(C)) \xrightarrow{(f|)_*} \varinjlim \pi_0\mathscr{C}(N - C) ,$$

where the first arrow follows from the inverse limit construction. If $C' \subset N$ is a larger compactum, then we get a similarly-defined homomorphism $\varprojlim \pi_0\mathscr{C}(M) \longrightarrow \varinjlim \pi_0\mathscr{C}(N - C')$ such that the following diagram commutes:

$$\begin{tikzcd} & \varinjlim \pi_0\mathscr{C}(N - C') \arrow[dd] \\ \varprojlim \pi_0\mathscr{C}(M) \arrow[ur] \arrow[dr] & \\ & \varinjlim \pi_0\mathscr{C}(N - C) \end{tikzcd}$$

In this manner there is induced a homomorphism $f_*: \varprojlim \pi_0\mathscr{C}(M) \to \varprojlim \pi_0\mathscr{C}(N)$. Again we have an easy result.

THEOREM 4.2. *$\varprojlim \pi_0\mathscr{C}$ is a covariant proper homotopy functor* *from the category of all-Q-manifolds and proper maps to the category of abelian groups.*

III. The functor $\varprojlim{}^1\pi_0\mathscr{E}$. We now describe $\varprojlim{}^1\pi_0\mathscr{E}$, a covariant proper homotopy functor from the category of Q-manifolds and proper maps to the category of abelian groups. For any Q-manifold M we let $\varprojlim{}^1\pi_0\mathscr{E}(M)$ denote the first derived limit of the inverse system

$$\{\varinjlim \pi_0\mathscr{E}(M - C) \mid C \subset M \text{ compact}\} .$$

To calculate $\varprojlim{}^1\pi_0\mathscr{E}(M)$ we proceed as follows. Write $M = \bigcup_{i=1}^\infty C_i$, where the C_i's are compact and $C_i \subset\subset C_{i+1}$, and consider the sequence

$$\varinjlim \pi_0\mathscr{E}(M - C_1) \xleftarrow{p_1} \varinjlim \pi_0\mathscr{E}(M - C_2) \xleftarrow{p_2} \cdots ,$$

where the p_i's are inclusion-induced. Define a homomorphism Δ from $\prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - C_i)$ to itself by

$$\Delta(g_1, g_2, \cdots) = (g_1p_1(g_2^{-1}), g_2p_2(g_3^{-1}), \cdots) .$$

Then we define $\varprojlim{}^1\{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty$ to be the cokernel

$$\prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - C_i)/\mathrm{Im}\,(\Delta) .$$

It is called the *first derived limit* of the inverse sequence

$$\{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty .$$

If $(g_1, g_2, \cdots)$ is an element of $\prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - C_i)$, then we use $\langle g_1, g_2, \cdots\rangle$ for its image in $\varprojlim{}^1\{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty$.

Just as $\varprojlim \pi_0\mathscr{E}(M)$ is represented by $\varprojlim\{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty$, we will represent $\varprojlim{}^1\pi_0\mathscr{E}(M)$ by $\varprojlim{}^1\{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty$. We must prove that this is independent of the choice of the C_i's. Thus let $M = \bigcup_{i=1}^\infty C_i'$ be given, where the C_i''s are compact and $C_i' \subset\subset C_{i+1}'$. We will describe a canonical procedure for constructing an isomorphism from $\varprojlim{}^1\{\varinjlim \pi_0\mathscr{E}(M-C_i)\}_{i=1}^\infty$ onto $\varprojlim{}^1\{\varinjlim \pi_0\mathscr{E}(M-C_i')\}_{i=1}^\infty$. Write $M = \bigcup_{i=1}^\infty D_i$, where the D_i's are compact and $D_i \subset\subset D_{i+1}$, and (1) some subsequence of $\{D_i\}_{i=1}^\infty$ equals a subsequence of $\{C_i\}_{i=1}^\infty$ and (2) some subsequence of $\{D_i\}_{i=1}^\infty$ equals a subsequence of $\{C_i'\}_{i=1}^\infty$. Let $\{C_{i_n}\}_{n=1}^\infty$ be a subsequence of $\{C_i\}_{i=1}^\infty$, where $i_1 < i_2 \cdots$. It will suffice to construct an isomorphism of $\varprojlim{}^1\{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty$ onto

$\varprojlim^1 \{\varinjlim \pi_0\mathscr{E}(M - C_{i_n})\}_{n=1}^\infty$.

Define

$$\phi: \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - C_i) \longrightarrow \prod_{n=1}^\infty \varinjlim \pi_0\mathscr{E}(M - C_{i_n})$$

by $\phi(g_1, g_2, \cdots) = (h_{i_1}, h_{i_2}, \cdots)$, where

$$\begin{aligned} h_{i_1} &= g_{i_1} g_{i_1+1} \cdots g_{i_2-1} , \\ h_{i_2} &= g_{i_2} g_{i_2+1} \cdots g_{i_3-1} , \\ &\vdots \end{aligned} \quad .$$

For convenience we have omitted writing down the appropriate compositions of the p_i's. Specifically this means that in the term $g_{i_1} g_{i_1+1} \cdots g_{i_2-1}$, the multiplication all takes place in $\varinjlim \pi_0\mathscr{E}(M - C_i)$, and g_j actually represents $p_{i_1+1} p_{i_1+2} \cdots p_{j-1}(g_j)$. The reader can easily check that ϕ induces an isomorphism

$$\hat{\phi}: \varprojlim^1 \{\varinjlim \pi_0\mathscr{E}(M - C_i)\}_{i=1}^\infty \longrightarrow \varprojlim^1 \{\varinjlim \pi_0\mathscr{E}(M - C_{i_n})\}_{n=1}^\infty .$$

Now let $f: M \to N$ be a proper map of Q-manifolds and write $N = \bigcup_{i=1}^\infty C_i$, where the C_i's are compact and $C_i \subset\subset C_{i+1}$. Then $M = \bigcup_{i=1}^\infty f^{-1}(C_i)$, where the $f^{-1}(C_i)$'s are compact and $f^{-1}(C_i) \subset\subset f^{-1}(C_{i+1})$. Define

$$\alpha: \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - f^{-1}(C_i)) \longrightarrow \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(N - C_i)$$

by $\alpha(g_1, g_2, \cdots) = (\alpha_1(g_1), \alpha_2(g_2), \cdots)$, where α_i represents the homomorphism

$$(f \mid M - f^{-1}(C_i))_*: \varinjlim \pi_0\mathscr{E}(M - f^{-1}(C_i)) \longrightarrow \varinjlim \pi_0\mathscr{E}(N - C_i) .$$

It is easy to see that the following diagram commutes:

$$\begin{array}{ccc} \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - f^{-1}(C_i)) & \xrightarrow{\alpha} & \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(N - C_i) \\ \Delta \downarrow & & \downarrow \Delta \\ \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(M - f^{-1}(C_i)) & \xrightarrow{\alpha} & \prod_{i=1}^\infty \varinjlim \pi_0\mathscr{E}(N - C_i) . \end{array}$$

Thus α induces a homomorphism $f_*: \varprojlim^1 \pi_0\mathscr{E}(M) \to \varprojlim^1 \pi_0\mathscr{E}(N)$. We leave it to the reader to check that f_* is independent of the choice of the C_i's. In analogy with Theorems 4.1 and 4.2 we get the following result.

THEOREM 4.3. *$\varprojlim^1 \pi_0\mathscr{E}$ is a covariant proper homotopy functor from the category of Q-manifolds and proper maps to the category of abelian groups.*

There is an important case in which the first derived limit construction vanishes. An inverse sequence of groups, $\{G_i, p_i\}_{i=1}^{\infty}$, satisfies the *Mittag-Leffler condition* provided that for each i there exists a $j > i$ such that the compositions

$$G_j \xrightarrow{p_{j-1}} G_{j-1} \xrightarrow{p_{j-2}} \cdots \xrightarrow{p_i} G_i\ ,$$
$$G_k \xrightarrow{p_{k-1}} G_{k-1} \xrightarrow{p_{k-2}} \cdots \xrightarrow{p_i} G_i$$

have identical images, for all $k \geqq j$. We will need the following result. For a proof see [12].

THEOREM 4.4. *If $\{G_i, p_i\}_{i=1}^{\infty}$ is an inverse sequence of abelian groups which satisfies the Mittag-Leffler condition, then $\varprojlim^1 \{G_i, p_i\}_{i=1}^{\infty}$ is trivial.*

5. **The second exact sequence.** The purpose of this section is to establish the exact sequence of Theorem 3, and the first step will be to construct the homomorphism $\theta\colon \mathrm{Ker}\,(\tau_\infty) \to \varprojlim^1 \pi_0\mathscr{C}(M)$. The following result will be useful in the construction of θ. Throughout this section we will assume that M is a given Q-manifold and

$$\tau_\infty\colon \pi_0\mathscr{C}(M) \longrightarrow \varprojlim Wh\pi_1(M)$$

is the homomorphism of Theorem 2.

LEMMA 5.1. *If $[h] \in \mathrm{Ker}\,(\tau_\infty)$, then we can write $M = \bigcup_{i=1}^{\infty} M_i$ such that*

(1) *the M_i's are clean,*

(2) $I \times M_1 \subset\subset h(I \times M_2) \subset\subset I \times M_3 \subset\subset h(I \times M_4) \subset\subset \cdots$,

(3) $\{0\} \times (M_{i+1} - \mathrm{Int}\,(M_i)) \hookrightarrow h(I \times M_{i+1}) - (I \times \mathrm{Int}\,(M_i))$ *is a simple equivalence, for i odd,*

(4) $\{0\} \times (M_{i+1} - \mathrm{Int}\,(M_i)) \hookrightarrow I \times M_{i+1} - h(I \times \mathrm{Int}\,(M_i))$ *is a simple equivalence, for i even.*

Proof. It follows from the definition of $\mathrm{Ker}\,(\tau_\infty)$ that for each clean $M_1 \subset M$ there exists a larger clean $M_2 \subset M$ such that $h(I \times M_1) \subset\subset I \times M_2$ and

$$\{0\} \times (M_2 - \mathrm{Int}\,(M_1)) \hookrightarrow I \times M_2 - h(I \times \mathrm{Int}\,(M_1))$$

is a simple equivalence. It will suffice to prove that M_2 can be chosen large enough so that $I \times M_1 \subset\subset h(I \times \mathrm{Int}\,(M_2))$ and so that

$$i\colon \{0\} \times (M_2 - \mathrm{Int}\,(M_1)) \hookrightarrow h(I \times M_2) - (I \times \mathrm{Int}\,(M_1))$$

is a simple equivalence. Once this is done we will have an inductive procedure for constructing our desired sequence $\{M_i\}_{i=1}^{\infty}$.

To see that we can make such a choice of M_2 all we have to do is choose M_2 so that there exist clean M_1', M_1'' such that

$$I \times M_1 \subset\subset h(I \times M_1') \subset\subset I \times M_1'' \subset\subset h(I \times M_2)$$

and such that

$$\{0\} \times (M_1'' - \mathrm{Int}\,(M_1')) \hookrightarrow I \times M_1'' - h(I \times \mathrm{Int}\,(M_1'))$$

is a simple equivalence. We will prove that i is a simple equivalence.

First it is easy to see that i is a homotopy equivalence. This is just like the proof of Lemma 3.1. Using the Sum Theorem we see that the torsion of i is equal to the inclusion-induced image of the torsion of

$$j\colon \{0\} \times (M_2 - \mathrm{Int}\,(M_1'')) \hookrightarrow h(I \times \mathrm{Int}\,(M_1''))$$

in $Wh\pi_1(h(I \times M_2) - (I \times \mathrm{Int}\,(M_1)))$. Again using the Sum Theorem it is easy to see that the inclusion-induced image of the torsion of j in $Wh\pi_1(h(I \times (M_2 - \mathrm{Int}\,(M_1'))))$ is 0. This is all we need.

Description of θ. Choose any $[h] \in \mathrm{Ker}\,(\tau_\infty)$ and write $M = \bigcup_{i=1}^{\infty} M_i$ as in Lemma 5.1. Then we can choose homeomorphisms

$$f_{2i-1}\colon I \times (M_{2i} - \mathrm{Int}\,(M_{2i-1})) \longrightarrow h(I \times M_{2i}) - (I \times \mathrm{Int}\,(M_{2i-1}))\,,$$
$$f_{2i}\colon I \times (M_{2i+1} - \mathrm{Int}\,(M_{2i})) \longrightarrow I \times M_{2i+1} - h(I \times \mathrm{Int}\,(M_{2i}))$$

such that

(1) $f_{2i-1} = id$ on $[\{0\} \times (M_{2i} - \mathrm{Int}\,(M_{2i-1}))] \cup (I \times \mathrm{Bd}\,(M_{2i-1}))$,
(2) $f_{2i-1} = h$ *on* $I \times \mathrm{Bd}\,(M_{2i})$,
(3) $f_{2i} = id$ on $[\{0\} \times (M_{2i+1} - \mathrm{Int}\,(M_{2i}))] \cup (I \times \mathrm{Bd}\,(M_{2i+1}))$,
(4) $f_{2i} = h$ on $I \times \mathrm{Bd}\,(M_{2i})$.

We will use the symbol $_*$ to indicate the amalgamation of homeomorphisms on sets for which there is agreement on the common parts. For example we define $f_{2i-1*}f_{2i} \in \mathscr{C}(M_{2i+1} - \mathrm{Int}\,(M_{2i-1}))$ by

$$f_{2i-1*}f_{2i} = \begin{cases} f_{2i-1}, & \text{on } I \times (M_{2i} - \mathrm{Int}\,(M_{2i-1})) \\ f_{2i}, & \text{on } I \times (M_{2i+1} - \mathrm{Int}\,(M_{2i}))\,. \end{cases}$$

Similarly we define $(f_{2i*}f_{2i+1})^{-1}h \in \mathscr{C}(M_{2i+2} - \mathrm{Int}\,(M_{2i}))$ to be the composition

$$I \times (M_{2i+2} - \mathrm{Int}\,(M_{2i})) \xrightarrow{h} h(I \times (M_{2i+2} - \mathrm{Int}\,(M_{2i}))) \xrightarrow{(f_{2i*}f_{2i+1})^{-1}} I \times (M_{2i+2} - \mathrm{Int}\,(M_{2i}))\,,$$

where $f_{2i}*f_{2i+1}$ is defined in analogy with $f_{2i-1}*f_{2i}$. Note that

$$(f_{2i}*f_{2i+1})^{-1}h = id$$

on $I \times (\mathrm{Bd}(M_{2i}) \cup \mathrm{Bd}(M_{2i+2}))$.

Let $\alpha_{2i-1} = \{f_{2i-1}*f_{2i}\} \in \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_{2i-1}))$ denote the inclusion-induced image of $[f_{2i-1}*f_{2i}] \in \pi_0\mathscr{C}(M_{2i+1} - \mathrm{Int}(M_{2i-1})$ in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_{2i-1}))$ and similarly let $\alpha_{2i} = \{(f_{2i}*f_{2i+1})^{-1}h\}$ denote the inclusion-induced image of $[(f_{2i}*f_{2i+1})^{-1}h]$ in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_{2i}))$. Then we get an element $(\alpha_1, \alpha_2, \cdots)$ of $\prod_{i=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))$ and we let $\theta([h])$ denote the element of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ which is represented by the element $\langle \alpha_1, \alpha_2, \cdots \rangle$ of

$$\varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))\}_{i=1}^{\infty} = \prod_{i=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))/\mathrm{Im}(\Delta) .$$

There are several things which need to be checked in order to conclude that θ is well-defined.

LEMMA 5.2. *$\theta([h])$ is independent of the choice of the f_i's.*

Proof. Let $\{f'_i\}_{i=1}^{\infty}$ be an alternate choice for $\{f_i\}_{i=1}^{\infty}$, thus giving us an alternate $(\alpha'_1, \alpha'_2, \cdots) \in \prod_{i=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))$, where

$$\alpha'_{2i-1} = \{f'_{2i-1}*f'_{2i}\} ,$$
$$\alpha'_{2i} = \{(f'_{2i}*f'_{2i+1})^{-1}h\} .$$

We will prove that $\langle \alpha_1, \alpha_2, \cdots \rangle = \langle \alpha'_1, \alpha'_2, \cdots \rangle$, and for this we must prove that

$$(\alpha_1(\alpha'_1)^{-1}, \alpha_2(\alpha'_2)^{-1}, \cdots) \in \mathrm{Im}(\Delta) .$$

Consider the element $(\beta_1, \beta_2, \cdots) \in \prod_{i=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))$ defined by $\beta_{2i-1} = \{(f'_{2i-1})^{-1}f_{2i-1}\}$ and $\beta_{2i} = \{f_{2i}^{-1}f'_{2i}\}$. We will show that

$$\Delta(\beta_1, \beta_2, \cdots) = (\alpha_1(\alpha'_1)^{-1}, \alpha_2(\alpha'_2)^{-1}, \cdots) .$$

It follows from Theorem 2.1 that the inclusion-induced image of β_{2i} in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_{2i-1}))$ is given by $\{id_*(f_{2i}^{-1}f'_{2i})\}$ and similarly the inclusion-induced image of β_{2i+1} in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_{2i}))$ is given by $\{id_*(f'_{2i+1})^{-1}f_{2i+1}\}$. Thus $\Delta(\beta_1, \beta_2, \cdots) = (\nu_1, \nu_2, \cdots)$, where

$$\nu_{2i-1} = \{(f'_{2i-1})^{-1}f_{2i-1}\}\{id_*(f_{2i}^{-1}f'_{2i})\}^{-1} ,$$
$$\nu_{2i} = \{f_{2i}^{-1}f'_{2i}\}\{id_*(f'_{2i+1}\}^{-1}f_{2i+1}\}^{-1} ,$$

Clearly $\nu_{2i-1} = \{(f'_{2i-1}*f'_{2i})^{-1}(f_{2i-1}*f_{2i})\}$, and since $\pi_0\mathscr{C}(M_{2i+1} - \mathrm{Int}(M_{2i-1}))$

is abelian this equals $\alpha_{2i-1}(\alpha'_{2i-1})^{-1}$. Similar reasoning gives $\nu_{2i} = \alpha_{2i}(\alpha'_{2i})^{-1}$.

LEMMA 5.3. *$\theta([h])$ is independent of the choice of the M_i's.*

Proof. Using the notation of the definition of $\theta([h])$ let $\{M_{i_n}\}_{n=1}^{\infty}$ be a subsequence of $\{M_i\}_{i=1}^{\infty}$, where i_{2n-1} is odd, i_{2n} is even and $i_1 < i_2 < \cdots$. We call $\{M_{i_n}\}_{n=1}^{\infty}$ an *odd-even subsequence* of $\{M_i\}_{i=1}^{\infty}$. By amalgamating the f_i's together we can use the M_{i_n}'s to define $\theta([h])$ as follows. Let

$$\alpha'_{i_1} = \{f_{i_1} * f_{i_1+1} * \cdots * f_{i_3-1}\} \in \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{i_1})) ,$$
$$\alpha'_{i_2} = \{(f_{i_2} * f_{i_2+1} * \cdots * f_{i_4-1})^{-1}h\} \in \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{i_2})) ,$$
$$\vdots$$

and thus get $(\alpha'_{i_1}, \alpha'_{i_2}, \cdots) \in \prod_{n=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{i_n}))$. This means that we are replacing the sequence $f_1, f_2, \cdots$ by the sequence

$$f_{i_1} * f_{i_1+1} * \cdots * f_{i_2-1} ,$$
$$f_{i_2} * f_{i_2+1} * \cdots * f_{i_3-1} .$$
$$\vdots$$

Then we get an element $\langle \alpha'_{i_1}, \alpha'_{i_2}, \cdots \rangle \in \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{i_n}))\}_{n=1}^{\infty}$. Recall from §4 that $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ can be represented in a natural way by $\varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i))\}_{i=1}^{\infty}$ and $\varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{i_n}))\}_{n=1}^{\infty}$. The isomorphism

$$\hat{\phi}\colon \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i))\}_{i=1}^{\infty} \longrightarrow \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{i_n}))\}_{n=1}^{\infty}$$

of §4 takes $\langle \alpha_1, \alpha_2, \cdots \rangle$ to $\langle \alpha_{i_1}\alpha_{i_1+1} \cdots \alpha_{i_2-1}, \alpha_{i_2+1} \cdots \alpha_{i_3-1}, \cdots)$, where we have omitted obvious inclusion-induced homomorphisms. So we have to prove that $\hat{\phi}(\langle \alpha_1, \alpha_2, \cdots \rangle) = \langle \alpha'_{i_1}, \alpha'_{i_2}, \cdots \rangle$.

Again omitting obvious inclusion-induced homomorphisms we have

$$(\alpha'_{i_1}, \alpha'_{i_2}, \cdots) = (\alpha_{i_1}\alpha_{i_1+2} \cdots \alpha_{i_3-2}, \alpha_{i_2}\alpha_{i_2+2} \cdots \alpha_{i_4-2}, \cdots) .$$

So we must prove that

$$\langle \alpha_{i_1}, \alpha_{i_1+1} \cdots \alpha_{i_2-1}, \alpha_{i_2}, \alpha_{i_2+1} \cdots \alpha_{i_3-1}, \cdots \rangle$$
$$= \langle \alpha_{i_1}\alpha_{i_1+2} \cdots \alpha_{i_3-2}, \alpha_{i_2}\alpha_{i_2+2} \cdots \alpha_{i_4-2}, \cdots \rangle .$$

Multiplying one by the inverse of the other we must therefore prove that

$$(\alpha_{i_1+1}\alpha_{i_1+3}\cdots\alpha_{i_2-2}\alpha^{-1}_{i_2+1}\alpha^{-1}_{i_2+3}\cdots\alpha^{-1}_{i_3-2},\ \alpha_{i_2+1}\alpha_{i_2+3}\cdots\alpha_{i_3-2}\alpha^{-1}_{i_3+1}\alpha^{-1}_{i_3+3}\cdots\alpha^{-1}_{i_4-2},\ \cdots)$$

lies in $\operatorname{Im}(\Delta)$. But it is clearly equal to

$$\Delta(\alpha_{i_1+1}\alpha_{i_1+3}\cdots\alpha_{i_2-2},\ \alpha_{i_2+1}\alpha_{i_2+3}\cdots\alpha_{i_3-2},\ \cdots)\,.$$

We have just shown that the definition of $\theta([h])$ is independent of the choice of the M_i's up to passage to an odd-even subsequence. As in §4 this clearly suffices to do the general case.

LEMMA 5.4. *$\theta([h])$ depends only on the isotopy class of h.*

Proof. We must show that if $h' \in [h]$ is used to define $\theta([h])$, just as $\theta([h])$ was defined by using h, then we get the same definition. Let $\{M_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ be chosen as in the definition of $\theta([h])$ given above. It is easy to see that the M_i's can be chosen so that $I \times M_1 \subset\subset h'(I \times M_2) \subset\subset$ and so that

$$\{0\} \times (M_{2i} - \operatorname{Int}(M_{2i-1})) \hookrightarrow h'(I \times M_{2i}) - (I \times \operatorname{Int}(M_{2i-1}))\,,$$
$$\{0\} \times (M_{2i+1} - \operatorname{Int}(M_{2i})) \hookrightarrow (I \times M_{2i+1}) - h'(I \times \operatorname{Int}(M_{2i}))$$

are simple equivalences. For this all we have to do is observe that if

$$\{0\} \times (M_2 - \operatorname{Int}(M_1)) \hookrightarrow h(I \times M_2) - (I \times \operatorname{Int}(M_1))$$

is a simple equivalence, then so is

$$\{0\} \times (M_2' - \operatorname{Int}(M_1)) \hookrightarrow h(I \times M_2') - (I \times \operatorname{Int}(M_1))\,,$$

for any clean $M_2' \supset M_2$. A similar statement holds for the other inclusion, $\{0\} \times (M_3 - \operatorname{Int}(M_2)) \hookrightarrow (I \times M_3) - h(I \times \operatorname{Int}(M_2))$. Thus $\{M_i\}_{i=1}^{\infty}$ can be chosen as indicated above. We will continue to use $\theta([h])$ for the definition above which involved h and $\{M_i\}_{i=1}^{\infty}$, and we will use $\theta([h])'$ for the similarly-worded definition which uses h' and $\{M_i\}_{i=1}^{\infty}$. The next step is to make a choice of homeomorphisms f_i' needed to define $\theta([h])'$.

Since $h' \in [h]$ we have an isotopy $h_t\colon I \times M \to I \times M \operatorname{rel}\{0\} \times M$ such that $h_0 = h$ and $h_1 = h'$. Then we may assume that the M_i's have been selected so that for each t and i,

(1) $h_t(I \times (M_{2i+2} - \operatorname{Int}(M_{2i}))) \subset I \times (\operatorname{Int}(M_{2i+3}) - M_{2i-1})$,
(2) $I \times \operatorname{Bd}(M_{2i+1}) \subset h_t(I \times (\operatorname{Int} M_{2i+2} - M_{2i}))$,
(3) $h_t(I \times \operatorname{Bd}(M_{2i})) \subset I \times (\operatorname{Int}(M_{2i+1}) - M_{2i-1})$,
(4) $h_t(I \times \operatorname{Bd}(M_{2i+2})) \subset I \times (\operatorname{Int}(M_{2i+3}) - M_{2i+1})$.

For each i the Isotopy Extension Theorem gives us an isotopy α_t of $I \times (M_{2i+3} - \operatorname{Int}(M_{2i-1}))$ onto itself rel

$$[\{0\} \times (M_{2i+3} - \operatorname{Int}(M_{2i-1})] \cup [I \times (\operatorname{Bd}(M_{2i-1}) \cup \operatorname{Bd}(M_{2i+3}))]$$

such that $\alpha_0 = id$ and $\alpha_t h = h_t$ on $I \times (M_{2i+2} - \mathrm{Int}(M_{2i}))$. Define $f'_{2i-1} = \alpha_1 f_{2i-1}$ and $f'_{2i+2} = \alpha_1 f_{2i+2}$ and note that $f'_{2i-1*}h'_*f'_{2i+2}$ is isotopic to $f_{2i-1*}h'*f'_{2i+2}$ rel

$$[\{0\} \times (M_{2i+3} - \mathrm{Int}(M_{2i-1}))] \cup [I \times (\mathrm{Bd}(M_{2i-1}) \cup \mathrm{Bd}(M_{2i+3}))] .$$

Again using the Isotopy Extension Theorem we can extend the isotopy $\alpha_t f_{2i-1} | I \times (M_{2i} - \mathrm{Int}(M_{2i-1}))$ and construct an isotopy β_t of $I \times (M_{2i+1} - \mathrm{Int}(M_{2i-1}))$ onto itself rel

$$[\{0\} \times (M_{2i+1} - \mathrm{Int}(M_{2i-1}))] \cup [I \times (\mathrm{Bd}(M_{2i-1}) \cup \mathrm{Bd}(M_{2i+1}))]$$

such that $\beta_0 = id$ and $\beta_t f_{2i-1} = \alpha_t f_{2i-1}$ on $I \times (M_{2i} - \mathrm{Int}(M_{2i-1}))$. Then define $f'_{2i} = \beta_1 f_{2i}$ and note that $f'_{2i-1*}f'_{2i}$ is isotopic to $f_{2i-1*}f_{2i}$ rel

$$[\{0\} \times (M_{2i+1} - \mathrm{Int}(M_{2i-1}))] \cup [I \times (\mathrm{Bd}(M_{2i-1}) \cup \mathrm{Bd}(M_{2i+1}))] .$$

Similarly we define f'_{2i+1} so that $f'_{2i+1*}f'_{2i+2}$ is isotopic to $f_{2i+1*}f_{2i+2}$ rel

$$[\{0\} \times (M_{2i+3} - \mathrm{Int}(M_{2i+1}))] \cup [I \times (\mathrm{Bd}(M_{2i+1}) \cup \mathrm{Bd}(M_{2i+3}))] .$$

This gives us f'_i defined for each i. Then the sequence $\{f'_i\}_{i=1}^{\infty}$ may be used in conjunction with $\{M_i\}_{i=1}^{\infty}$ to define $\theta([h])'$. Recall that $\theta([h])$ is represented by

$$(*) \qquad \langle \{f_{1*}f_2\}, \{(f_{2*}f_3)^{-1}h\}, \cdots \rangle$$

and $\theta([h])'$ is represented by

$$(**) \qquad \langle \{f'_1{}_*f'_2\}, \{(f'_2{}_*f'_3)^{-1}h\}, \cdots \rangle .$$

We observed above that $f'_{2i-1*}f'_{2i}$ is isotopic to $f_{2i-1*}f_{2i}$, therefore $\{f'_{2i-1*}f'_{2i}\} = \{f_{2i-1*}f_{2i}\}$. This takes care of the odd terms. We will not be able to show that $\{(f'_{2i*}f'_{2i+1})^{-1}h'\} = \{(f_{2i*}f'_{2i+1})^{-1}h\}$, but we will show that they have the same inclusion-induced image in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_{2i-1}))$. The reader can use this fact to easily show that (*) and (**) represent the same element of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$.

To establish this fact note that the inclusion-induced image of $[(f_{2i*}f_{2i+1})^{-1}h]$ in $\pi_0\mathscr{C}(M_{2i+3} - \mathrm{Int}(M_{2i-1}))$ is

$$(^+) \qquad [f_{2i-1*}f_{2i*}f_{2i+1*}f_{2i+2})^{-1}(f_{2i*}h_*f_{2i+2})]$$

and the inclusion-induced image of $[(f'_{2i*}f'_{2i+1})^{-1}h']$ in $\pi_0\mathscr{C}(M_{2i+3} - \mathrm{Int}(M_{2i-1}))$ is

$$(^{++}) \qquad [(f'_{2i-1*}f'_{2i*}f'_{2i+1*}f'_{2i+2})^{-1}(f'_{2i-1*}h'_*f'_{2i+2})] .$$

We have already seen that $[f'_{2i-1*}h'_*f'_{2i+2}] = [f_{2i-1*}h_*f_{2i+2}]$, $[f'_{2i-1*}f'_{2i}] = [f_{2i-1*}f_{2i}]$ and $[f'_{2i+1}*f_{2i+2}] = [f_{2i+1*}f_{2i+2}]$. This gives us $(+)$ equal to $(++)$.

This concludes the proof that $\theta([h])$ is well-defined. We next prove that θ is a homomorphism.

LEMMA 5.5. *θ is a homomorphism.*

Proof. Choose elements $[h]$, $[h'] \in \operatorname{Ker}(\tau_\infty)$. We must prove that $\theta([h'h]) = \theta([h'])\theta([h])$. We can write $M = \bigcup_{i=1}^{\infty} M_i$ so that for each i,

(1) M_i is clean,
(2) $M_i \subset\subset M_{i+1}$,
(3) $\{0\} \times (M_{4i-1} - \operatorname{Int}(M_{4i-2})) \hookrightarrow h(I \times M_{4i-1}) - (I \times \operatorname{Int}(M_{4i-2}))$ is a simple equivalence,
(4) $\{0\} \times (M_{4i} - \operatorname{Int}(M_{4i-1})) \hookrightarrow I \times M_{4i} - h(I \times \operatorname{Int}(M_{4i-1})$ is a simple equivalence,
(5) $\{0\} \times (M_{4i-2} - \operatorname{Int}(M_{4i-3})) \hookrightarrow h'(I \times M_{4i-2}) - (I \times \operatorname{Int}(M_{4i-3}))$ is a simple equivalence,
(6) $\{0\} \times (M_{4i+1} - \operatorname{Int}(M_{4i})) \hookrightarrow (I \times M_{4i+1}) - h'(I \times \operatorname{Int}(M_{4i})$ is a simple equivalence.

Then we can find homeomorphisms as follows:

(1) $f_{4i-2}: I \times (M_{4i-1} - \operatorname{Int}(M_{4i-2})) \rightarrow h(I \times M_{4i-1}) - (I \times \operatorname{Int}(M_{4i-2}))$ is the identity on $[I \times \operatorname{Bd}(M_{4i-2})] \cup [\{0\} \times (M_{4i-1} - \operatorname{Int}(M_{4i-2}))]$ and equals h on $I \times \operatorname{Bd}(M_{4i-1})$.
(2) $f_{4i-1}: I \times (M_{4i} - \operatorname{Int}(M_{4i-1})) \rightarrow (I \times M_{4i}) - h(I \times \operatorname{Int}(M_{4i-1}))$ is the identity on $[I \times \operatorname{Bd}(M_{4i})] \cup [\{0\} \times (M_{4i} - \operatorname{Int}(M_{4i-1}))]$ and equals h on $I \times \operatorname{Bd}(M_{4i-1})$.
(3) $f'_{4i-3}: I \times (M_{4i-2} - \operatorname{Int}(M_{4i-3})) \rightarrow h'(I \times M_{4i-2}) - (I \times \operatorname{Int}(M_{4i-3}))$ is the identity on $[I \times \operatorname{Bd}(M_{4i-3})] \cup [\{0\} \times (M_{4i-2} - \operatorname{Int}(M_{4i-3}))]$ and equals h' on $I \times \operatorname{Bd}(M_{4i-2})$.
(4) $f'_{4i}: I \times (M_{4i+1} - \operatorname{Int}(M_{4i})) \rightarrow I \times M_{4i+1} - h'(I \times \operatorname{Int}(M_{4i}))$ is the identity on $[I \times \operatorname{Bd}(M_{4i+1})] \cup [\{0\} \times (M_{4i+1} - \operatorname{Int}(M_{4i}))]$ and equals h' on $I \times \operatorname{Bd}(M_{4i})$.

Writing $M = M_1 \cup M_3 \cup \cdots$ and representing $\varprojlim^1 \pi_0\mathscr{C}(M)$ by

$$\varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i)) \mid i \text{ odd}\}$$

we calculate $\theta([h])$ to be

$$\langle \alpha_1, \alpha_3, \cdots \rangle \in \prod_{i=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_{2i-1}))/\operatorname{Im}(\Delta) ,$$

where

$$\alpha_1 = \{id_* f_{2*} f_{3*} id\} ,$$
$$\alpha_3 = \{(f_{3*} id_* id_* f_6)^{-1} h\} ,$$
$$\vdots$$

Note that $id_* f_{2*} f_{3*} id \in \mathscr{C}(M_5 - \mathrm{Int}\,(M_1))$ and $(f_{3*} id_* id_* f_6)^{-1} h \in \mathscr{C}(M_7 - \mathrm{Int}\,(M_3))$.

Now write $M = M_1 \cup M_2 \cup M_5 \cup M_6 \cup M_9 \cup M_{10} \cup \cdots$ and represent $\varprojlim^1 \pi_0 \mathscr{C}(M)$ by

$$\varprojlim{}^1 \{\varinjlim \pi_0 \mathscr{C}(M - \mathrm{Int}\,(M_i)) \,|\, i = 1, 2, 5, 6, \cdots\} .$$

Then we calculate $\theta([h'])$ to be

$$\theta([h']) = \langle \alpha'_1, \alpha'_2, \alpha'_5, \alpha'_6, \cdots \rangle \in \prod_{i=1}^{\infty} \{\varinjlim \pi_0 \mathscr{C}(M - \mathrm{Int}\,(M_i)) \,|\, i$$
$$= 1, 2, 5, 6, \cdots\}/\mathrm{Im}\,(\Delta) ,$$

where

$$\alpha'_1 = \{f'_1{}_* h'{}_* f'_4\} ,$$
$$\alpha'_2 = \{(h'{}_* f'_4{}_* f'_5)^{-1} h'\} ,$$
$$\vdots$$

Note that $f'_1{}_* h'_* f'_4 \in \mathscr{C}(M_5 - \mathrm{Int}\,(M_1))$ and $(h'{}_* f'_4{}_* f'_5)^{-1} h' \in \mathscr{C}(M_6 - \mathrm{Int}\,(M_2))$.

Again writing $M = M_1 \cup M_3 \cup \cdots$ and representing $\varprojlim^1 \pi_0 \mathscr{C}(M)$ as above we calculate $\theta([h'h])$ to be

$$\theta([h'h]) = \langle \beta_1, \beta_3, \cdots \rangle \in \prod_{i=1}^{\infty} \varinjlim \pi_0 \mathscr{C}(M - \mathrm{Int}\,(M_{2i-1}))/\mathrm{Im}\,(\Delta) ,$$

where

$$\beta_1 = \{f'_1{}_* h' f_{2*} h' f_{3*} f'_4\} ,$$
$$\beta_3 = \{(h' f_{3*} f'_4{}_* f'_5{}_* h' f_6)^{-1} h'h\} ,$$
$$\vdots$$

Note that $f'_1{}_* h' f_2 * h' f_{3*} f'_4 \in \mathscr{C}(M_5 - \mathrm{Int}\,(M_1))$ and $(h' f_{3*} f'_4{}_* f'_5{}_* h' f_6)^{-1} h'h \in \mathscr{C}(M_7 - \mathrm{Int}\,(M_3))$. We now have representatives of $\theta([h])$, $\theta([h'])$ and $\theta([h'h])$. Note that

$$f'_1{}_* h' f_{2*} h' f_{3*} f'_4 = (f'_1{}_* h'{}_* f'_4)(id_* f_{2*} f_{3*} id) .$$

This implies that $\beta_1 = \alpha'_1 \alpha_1$; similarly $\beta_5 = \alpha'_5 \alpha_5$, $\beta_9 = \alpha'_9 \alpha_9$, $\cdots$. However we cannot directly compare the remaining terms because they do not all lie in the same groups. For example α_3 and β_3 lie in $\varinjlim \pi_0 \mathscr{C}(M -$

$\mathrm{Int}(M_3))$, but α_2' lies in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_2))$. Let $(\alpha_3)_1$, $(\alpha_2')_1$ and $(\beta_3)_1$ denote the inclusion induced images of α_3, α_2' and β_3 in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_1))$. We will show that $(\beta_3)_1 = (\alpha_2')_1(\alpha_3)_1$. Similarly it will follow that $(\beta_7)_5 = (\alpha_6')_5(\alpha_7)_5$, and so forth. Here $(\beta_7)_5$, $(\alpha_6')_5$ and $(\alpha_7)_5$ are the inclusion-induced images of β_7, α_6' and α_7 in $\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_5))$. This will suffice to prove that $\theta([h'h]) = \theta([h'])\theta([h])$.

To prove that $(\beta_3)_1 = (\alpha_2')_1(\alpha_3)_1$ it will suffice to prove that the composition of

(a) $[id_*(h'_*f_4'{}_*f_5')^{-1}h'_*id_*id_*id]$

and

(b) $[id_*id_*(f_{3*}id_*id_*f_6)^{-1}h_*id_*id]$

gives

(c) $[id_*id_*(h'f_{3*}f_4'{}_*f_5'{}_*h'f_6)^{-1}h'h_*id_*id]$.

These are elements of $\pi_0\mathscr{C}(M_9 - \mathrm{Int}(M_1))$. Obviously

(a) $= [f_1'{}_*h'_*f_{4*}'f_5'{}_*h'_*f_8']^{-1}[f_1'{}_*h'_*f_8']$,

(b) $= [id_*f_{2*}f_{3*}id_*id_*f_{6*}f_{7*}id]^{-1}[id_*f_{2*}h_*f_{7*}id]$,

(c) $= [f_1'{}_*h'f_{2*}h'f_{3*}f_4'{}_*f_5'{}_*h'f_{6*}h'f_{7*}f_8']^{-1}[f_1'{}_*h'f_{2*}h'h_*h'f_{7*}f_8']$.

Using the commutativity of $\pi_0\mathscr{C}(M_9 - \mathrm{Int}(M_1))$ it is clear that (c) is the composition of (a) and (b).

Proof of Theorem 3. We now show that there is an exact sequence

$$\varprojlim \pi_0\mathscr{C}(M) \xrightarrow{j} \varinjlim \pi_0\mathscr{C}(M) \xrightarrow{k} \mathrm{Ker}(\tau_\infty) \xrightarrow{\theta} \varprojlim{}^1 \pi_0\mathscr{C}(M) \longrightarrow 0\ .$$

1. Exactness at $\varprojlim{}^1 \pi_0\mathscr{C}(M)$. We say that an element of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ is *good* provided that it can be represented by $\langle \alpha_1, \alpha_2, \cdots\rangle$, for $M = \bigcup_{i=1}^\infty M_i$, where $\alpha_i \in \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))$ is the inclusion-induced image of $[h] \in \pi_0\mathscr{C}(M_{i+1} - \mathrm{Int}(M_i))$ and $h = id$ on $\mathrm{I} \times (\mathrm{Bd}(M_i) \cup \mathrm{Bd}(M_{i+1}))$. We will first show that each element of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ is the product of two good elements. To see this write $M = \bigcup_{i=1}^\infty M_i$, where the M_i's are clean and $M_i \subset\subset M_{i+1}$, and choose any $\langle \alpha_1, \alpha_2, \cdots\rangle \in \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))\}_{i=1}^\infty$. Let α be the element of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ represented by $\langle \alpha_1, \alpha_2, \cdots\rangle$. It is easy to see that we can find a subsequence of $\{M_i\}_{i=1}^\infty$, call it $\{M_i'\}_{i=1}^\infty$, and an element $\langle \alpha_1', \alpha_2', \cdots\rangle$ of $\varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i'))\}_{i=1}^\infty$ such that

(1) $\langle \alpha_1', \alpha_2', \cdots\rangle$ represents α,

(2) for each i there is an element $h_i \in \mathscr{C}(M_{i+2}' - \mathrm{Int}(M_i'))$ such that $h_i = id$ on $I \times (\mathrm{Bd}(M_i') \cup \mathrm{Bd}(M_{i+2}'))$ and $\alpha_i' = \{h_i\}$.

Writing $M = M_1' \cup M_3' \cup \cdots$ we have an element $\beta \in \varprojlim \pi_0\mathscr{C}(M)$ represented by

$$\langle\{h_1\}, \{h_3\}, \cdots\rangle \in \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}\,(M_i')) \mid i \text{ odd}\}$$

and writing $M = M_2' \cup M_4' \cup \cdots$ we have an element ν of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$ represented by

$$\langle\{h_2\}, \{h_4\}, \cdots\rangle \in \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}\,(M_i')) \mid i \text{ even}\} .$$

Clearly β, ν are good and the reader can easily check that $\alpha = \beta\nu$.

Thus to see that θ is onto we need only consider a good element α of $\varprojlim{}^1 \pi_0\mathscr{C}(M)$. Let α be represented by $\langle\alpha_1, \alpha_2 \cdots\rangle$, where as above we have $\alpha_i = \{h_i\}$. Let $h \in \mathscr{C}(M)$ be defined by

$$h = id_* h_{1*} h_{2*} \cdots .$$

Then it is easily seen that $\theta([h]) = \alpha$.

II. Exactness at $\mathrm{Ker}\,(\tau_\infty)$. We must first define the homomorphism $k\colon \varinjlim \pi_0\mathscr{C}(M) \to \mathrm{Ker}\,(\pi_\infty)$. An element of $\varinjlim \pi_0\mathscr{C}(M)$ can be represented by $\{h_1\}$, where $h_1 \in \mathscr{C}(M_1)$ and $M_1 \subset M$ is clean such that $h_1 = id$ on $I \times \mathrm{Bd}\,(M_1)$. Let $h \in \mathscr{C}(M)$ extend h_1 by the identity and define $k([h_1]) = [h]$. It is easy to see that k gives a well-defined homomorphism and it is clear that $\theta k = 0$. The other half of the proof of exactness is more difficult.

Thus for any $[h] \in \mathrm{Ker}\,(\theta)$ we want to prove that $[h] \in \mathrm{Im}\,(k)$. For the time being we assume that h can be written as $h_{0*}h_{1*}h_{2*}\cdots$, where $M = \bigcup_{i=1}^\infty M_i$ and

(1) the M_i's are clean,

(2) $M_i \subset\subset M_{i+1}$,

(3) $h_i \in \mathscr{C}(M_{i+1} - \mathrm{Int}\,(M_i))$ and is the identity on $I \times (\mathrm{Bd}\,(M_i) \cup \mathrm{Bd}\,(M_{i+1}))$, for $i \geqq 1$,

(4) $h_0 \in \mathscr{C}(M_1)$ and is the identity on $I \times \mathrm{Bd}\,(M_1)$.

Then $\theta([h])$ can be represented by

$$\langle\{h_1\}, \{h_2\}, \cdots\rangle \in \varprojlim{}^1 \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}\,(M_i))\}_{i=1}^\infty .$$

Since $[h] \in \mathrm{Ker}\,(\theta)$ we have

$$(\{h_1\}, \{h_2\}, \cdots) = (\{g_1\}\{id_* g_2\}^{-1}, \{g_2\}\{id_* g_3\}^{-1}, \cdots) ,$$

where for each i we have $g_i \in \mathscr{C}(M_{j_i} - \mathrm{Int}\,(M_i))$, for some $j_i > i$, such that $g_i = id$ on $I \times (\mathrm{Bd}\,(M_i) \cup \mathrm{Bd}\,(M_{j_i}))$. By passing to a subsequence we may assume that $j_i = i + 1$ and that

$$[h_{i*} id] = [(g_{i*} id)(id_* g_{i+1})^{-1}] ,$$

computations being performed in $\pi_0\mathscr{C}(M_{i+2} - \mathrm{Int}\,(M_i))$. Details are left to the reader.

For each i we have $[g_i^{-1}h_{i*}id] = [id_* g_{i+1}^{-1}]$ and therefore

$$[g_1^{-1}h_{1*}id_* g_3^{-1}h_{3*}id_* \cdots] = [id_* g_2^{-1}{}_* id_* g_4^{-1}{}_* \cdots] ,$$

these computations being performed in $\pi_0\mathscr{C}(M - \operatorname{Int}(M_1))$. Similarly we get

$$[id_* g_2^{-1}h_{2*}id_* g_4^{-1}h_* \cdots] = [id_* id_* g_3^{-1}{}_* id_* g_5^{-1}{}_* \cdots] .$$

Composing we get

$$[g_1^{-1}{}_* g_2^{-1}{}_* \cdots][h_{1*}h_{2*} \cdots] = [id_* g_2^{-1}{}_* g_3^{-1}{}_* \cdots] ,$$

or

$$[h_{1*}h_{2*} \cdots] = [g_{1*}id_* id_* \cdots] .$$

This implies that $[h] \in \operatorname{Im}(k)$.

We now treat the general case. Choose any $[h] \in \operatorname{Ker}(\theta)$. We will prove that $[h] = [h']$, where h' decomposes as $h' = h'_{0*}h'_1 * h'_{2*} \cdots$, in the above sense. Thus we are going to reduce the general case to the specific case treated above. Write $M = \bigcup_{i=1}^{\infty} M_i$ and choose $\{f_i\}_{i=1}^{\infty}$ as in the definition of $\theta([h])$. We can easily choose the M_i's and f_i's so that there are even pairs (M'_4, M''_4), (M'_6, M''_6), $\cdots$ such that

(1) $M_{2i} \subset\subset M'_{2i} \subset\subset M''_{2i} \subset\subset M_{2i+1}$,
(2) $f_{2i} = id$ on $I \times (M_{2i+1} - \operatorname{Int}(M'_{2i}))$,
(3) $h(I \times \operatorname{Bd}(M''_{2i})) \subset I \times (M_{2i+1} - \operatorname{Int}(M'_{2i}))$.

By definition we have

$$\theta([h]) = (\{f_{1*}f_2\}, \{(f_{2*}f_3)^{-1}h\}, \cdots) = \varDelta\{g_1\}, \{g_2\}, \cdots) ,$$

for some $(\{g_1\}, \{g_2\}, \cdots) \in \prod_{i=1}^{\infty} \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i))$. Then we have $g_1 \in \mathscr{C}(M_{n_1} - \operatorname{Int}(M_1))$, for some $n_1 > 1$. We are going to show that for any even $m < n_1$, h restricted to a neighborhood of $I \times \operatorname{Bd}(M''_m)$ is isotopic to the identity, with the isotopy taking place in $I \times (M - \operatorname{Int}(M_1))$. The Isotopy Extension Theorem then implies that h is isotopic to a new homeomorphism which is the identity on $I \times \operatorname{Bd}(M''_m)$. Repeated applications of this observation will suffice to prove that our required h' exists.

Choose any even $m > n_1$ and consider the sequence

$$f_{1*}f_2,\ (f_{2*}f_3)^{-1}h,\ \cdots .$$

Define a sequence $u_1, u_2, \cdots$ such that $u_1 \in \mathscr{C}(M - \operatorname{Int}(M_1))$ extends $f_{1*}f_2$ by the identity, $u_2 \in \mathscr{C}(M - \operatorname{Int}(M_2))$ extends $(f_{2*}f_3)^{-1}h$ by the identity, etc. Then it is easily seen that $u_n u_{n-1} \cdots u_2 u_1 = h$ on a neighborhood of $I \times \mathrm{Bd}(M_4'')$, for $n \geqq 4$, $u_n u_{n-1} \cdots u_2 u_1 = h$ on a neighborhood of $I \times \mathrm{Bd}(M_6'')$ for $n \geqq 6$, etc. The equation

$$(\{f_{1*}f_2\}, \{(f_{2*}f_3)^{-1}h\}, \cdots) = \Delta(\{g_1\}, \{g_2\}, \cdots)$$

implies that $[u_m u_{m-1} \cdots u_2 u_1] = [(g_{1*}id)(id_* g_{m+1*}id)^{-1}]$. But

$$(g_{1*}id)(id_* g_{m+1*}id)^{-1}$$

is the identity on a neighborhood of $I \times \mathrm{Bd}(M_m'')$ and we are done.

III. Exactness at $\varinjlim \pi_0\mathscr{C}(M)$. We must first define the homomorphism $j\colon \varprojlim \pi_0\mathscr{C}(M) \to \varinjlim \pi_0\mathscr{C}(M)$. A typical element of $\varprojlim \pi_0\mathscr{C}(M)$ may be represented by $(\{g_1\}, \{g_2\}, \cdots)$, where $M = \bigcup_{i=1}^{\infty} M_i$ and $\{g_i\} \in \varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))$. Then we define $j(\{g_1\}, \{g_2\}, \cdots) \in \varinjlim \pi_0\mathscr{C}(M)$ to be the inclusion-induced image of $\{g_1\}$ in $\varinjlim \pi_0\mathscr{C}(M)$. It is clear that j is well-defined.

To see that $kj = 0$ choose $(\{g_1\}, \{g_2\}, \cdots)$ as above such that $g_i \in \mathscr{C}(M_{n_i} - \mathrm{Int}(M_i))$, for $n_i > i$, and such that $g_i = id$ on $I \times (\mathrm{Bd}(M_i) \cup \mathrm{Bd}(M_{n_i}))$. Then $kj = [h]$, where $h \in \mathscr{C}(M)$ extends g_1 by the identity. The condition

$$(\{g_1\}, \{g_2\}, \cdots) \in \varprojlim \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))\}_{i=1}^{\infty}$$

implies that

$$[h] = [id_* g_{1*}id] = [id_* g_{2*}id] = \cdots ,$$

and this provides our required isotopy of $h = id_* g_{1*}id$ to the identity.

For the other half of the proof choose $M_1 \subset M$ clean and $h_0 \in \mathscr{C}(M_1)$ such that $h_0 = id$ on $I \times \mathrm{Bd}(M_1)$. Then $\{h_0\} \in \varinjlim \pi_0\mathscr{C}(M)$ represents a typical element of $\varinjlim \pi_0\mathscr{C}(M)$. Clearly $k(\{h_0\}) = [h_{0*}id]$ and we assume that $[h_{0*}id] = [id]$. We will construct an $h_1 \in \mathscr{C}(M_n - \mathrm{Int}(M_1))$, for $n > 1$ large, so that $h_1 = id$ on $I \times (\mathrm{Bd}(M_1) \cup \mathrm{Bd}(M_n))$, $[h_{1*}id] = [id]$ in $\pi_0\mathscr{C}(M - \mathrm{Int}(M_1))$, and $\{h_1\} = \{h_0\}$ in $\varinjlim \pi_0\mathscr{C}(M)$. Repeated applications of this construction will produce an element

$$(\{h_1\}, \{h_2\}, \cdots) \in \varprojlim \{\varinjlim \pi_0\mathscr{C}(M - \mathrm{Int}(M_i))\}_{i=1}^{\infty}$$

which is sent to $\{h_0\}$ by j. Let $g_t\colon I \times M \to I \times M$ be an isotopy rel $\{0\} \times M$ such that $g_0 = h_{0*}id$ and $g_1 = id$. Choose $n > 1$ large enough so that $g_t(I \times M_1) \subset\subset I \times M_n$, for each t. Using the Isotopy Extension Theorem we can find an isotopy $G_t\colon I \times M_n \to I \times M_n$ rel $(\{0\} \times M_n) \cup (I \times \mathrm{Bd}(M_n))$ such that $G_0 = h_{0*}id$ and $G_t = g_t$ on $I \times M_1$. Let $h_1 \in \mathscr{C}(M_n - \mathrm{Int}(M_1))$ be the restriction of G_1 to $M_2 - \mathrm{Int}(M_1)$. It is clear that $[id_* h_1] = [h_{0*} id]$ in $\pi_0\mathscr{C}(M_n)$; therefore $\{h_1\} = \{h_0\}$ in $\varinjlim \pi_0\mathscr{C}(M)$. To see that $[h_{1*} id] = [id]$ in $\pi_0\mathscr{C}(M - \operatorname{Int}(M_1))$ consider the isotopy $f_t: I \times (M - \operatorname{Int}(M_1)) \to I \times (M - \operatorname{Int}(M_1)$ defined by $f_t = g_t^{-1}(G_t * id) | I \times (M - \operatorname{Int}(M_1))$. Then $f_0 = id$ and $f_1 = h_{1*} id$.

6. Proofs of Theorems 4 and 5.

Proof of Theorem 4. We are given a Q-manifold M which is movable at ∞ and we want to prove that $\varprojlim^1 \pi_0\mathscr{C}(M) = 0$. Write $M = \bigcup_{i=1}^\infty M_i$, where the M_i's are clean and $M_i \subset\subset M_{i+1}$. Recall from Theorem 4.4 that all we have to do is prove that the inverse system

$$\{\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i))\}_{i=1}^\infty$$

is Mittag-Leffler. Choose any i and use the definition of movable at ∞ to find a $j > i$ such that $(M - \operatorname{Int}(M_j))$ can be homotoped into any neighborhood of ∞, with homotopy taking place in $M - \operatorname{Int}(M_i))$. Now let $k \geqq j$ be given. We will prove that the inclusion-induced homomorphisms

$$\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_k)) \longrightarrow \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i)),$$
$$\varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_j)) \longrightarrow \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i))$$

have the same image.

Consider the inclusions

$$\alpha: M - \operatorname{Int}(M_j) \hookrightarrow M - \operatorname{Int}(M_i),$$
$$\beta: M - \operatorname{Int}(M_k) \hookrightarrow M - \operatorname{Int}(M_i)$$

and use the assumption of movability to get a map $\nu: M - \operatorname{Int}(M_j) \to M - \operatorname{Int}(M_k)$ such that $\beta\nu$ is homotopic to α. Then the induced homomorphisms

$$\alpha_*: \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_j)) \longrightarrow \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i)),$$
$$\beta_*: \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_k)) \longrightarrow \varinjlim \pi_0\mathscr{C}(M - \operatorname{Int}(M_i))$$

have the same image because $\beta_*\nu_* = \alpha_*$ and $\alpha_*\delta_* = \beta_*$, where $\delta: M - \operatorname{Int}(M_k) \hookrightarrow M - \operatorname{Int}(M_j)$.

Proof of Theorem 5. We are given a pair (M, N) of compact Q-manifolds such that N is a Z-set in M and we want to establish an exact sequence

$$\pi_0\mathscr{C}(N) \longrightarrow \pi_0\mathscr{C}(M) \longrightarrow \pi_0\mathscr{C}(M - N) \longrightarrow Wh\pi_1(N) \longrightarrow Wh\pi_1(M).$$

This follows from Theorems 2 and 3 provided that we can verify the following facts.

(1) $\varprojlim^1 \pi_0\mathscr{C}(M-N) = 0$.

(2) $\varprojlim \pi_0\mathscr{C}(M-N) \approx \pi_0\mathscr{C}(N)$.

(3) $\varinjlim \pi_0\mathscr{C}(M-N) \approx \pi_0\mathscr{C}(M)$.

(4) $\varprojlim Wh\pi_1(M-N) \approx Wh\pi_1(N)$.

(5) $Wh_1(M-N) \approx Wh\pi_1(M)$.

Since N is a Z-set in M it must be collared in M. This implies that $M-N$ is movable at ∞ and therefore $\varprojlim^1 \pi_0\mathscr{C}(M-N) = 0$. The isomorphisms $2-5$ are easy.

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