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FIXED POINTS OF AUTOMORPHISMS OF COMPACT LIE GROUPS

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Hopf's proof that the real Čech cohomology $H^*(G)$ of a compact, connected Lie group G is an exterior algebra with odd-dimensional generators was followed by a demonstration that the number of such generators is equal to the rank of the group, that is, to the dimension of a maximal torus. We show that the latter result is a special case of a relationship between an automorphism of such a group and the automorphism it induces on the cohomology.

1. Introduction. For a set X and a function $f: X \to X$, let $\Phi(f)$ denote the set of fixed points of f: those $x \in X$ for which $f(x) = x$. If X is a topological group and f is a homomorphism, we use the symbol $\Phi_0(f)$ for the component of the group $\Phi(f)$ which contains the identity element of X. By a *graded vector space* V we mean a sequence $\{V_0, V_1, V_2, \cdots\}$ of (real) vector spaces. The *dimension* of V is the sum of the dimensions of the V_i. A *subspace* of V is a graded vector space $V' = \{V'_i\}$ such that V'_i is a subspace of V_i; for all i.

Now let G be a compact, connected Lie group and let h be an automorphism of G. Denote by $PH^*(G)$ the graded vector subspace of primitives[1] in the Hopf algebra $H^*(G)$ and let Ph^* be the restriction to $PH^*(G)$ of the automorphism $h^*: H^*(G) \to H^*(G)$ induced by h.

The main result of this paper is

THEOREM 1.1. *Let G be a compact connected Lie group and let h be an automorphism of G. Then the rank of the Lie group $\Phi_0(h)$ is equal to the dimension of the graded vector space $\Phi(Ph^*)$.*

If h is the identity automorphism, then $\Phi(h) = G$ while $\Phi(Ph^*) = PH^*(G)$. Since $H^*(G)$ may be generated by primitives, the number of generators is the dimension of $PH^*(G)$. Thus Hopf's result in [4] on the rank of a compact Lie group is this case of Theorem 1.1.

The next section presents a digression concerning the kernel of an endomorphism of a compact, connected abelian topological group. The setting is more general than is necessary for the later sections because some readers may find this material of independent interest. The proof of Theorem 1.1 is accomplished in §3. The remaining

[1] A *primitive* in $H^*(G)$ is an element z for which $m^*(z) = 1 \otimes z + z \otimes 1$; where $m^*: H^*(G) \to H^*(G) \otimes H^*(G)$ is induced by the multiplication $m: G \times G \to G$ of the group.

sections discuss consequences of the main result. Section 4 demonstrates that the existence of an automorphism h on a compact, connected Lie group G such that the fixed point group of h is of low rank implies that G has a very restricted type of infinitesimal structure. In §5, we obtain necessary and sufficient conditions for the power map $p_k(x) = x^k$ on a Lie group with compact components to map a component onto another. This theorem extends the main result of [2] which established the conditions only for compact Lie groups. The results contained in this paper were announced in [1][2].

2. Endomorphisms of abelian topological groups. The kernel of a homomorphism h will be denoted by $\mathrm{Ker}(h)$. For a homomorphism h on a topological group, the symbol $\mathrm{Ker}_0(h)$ will represent the component of the kernel of h that contains the identity element of the group.

Let G be a compact, connected abelian topological group and let h be an endomorphism of G. Denote the character group of G by $G^\wedge$ and write the endomorphism of $G^\wedge$ induced by h as $h^\wedge$.

For a subgroup H of G, let $i: H \to G$ be inclusion and let $\mathrm{Ann}(H)$ denote the subgroup of $G^\wedge$ consisting of all elements which vanish on H. By [6, p. 253], there is an exact sequence

$$0 \longrightarrow \mathrm{Ann}(H) \longrightarrow G^\wedge \xrightarrow{i^\wedge} H^\wedge \longrightarrow 0 .$$

In particular, letting $\mathrm{Im}(h)$ be the image of h, we have the exact sequence

$$0 \longrightarrow \mathrm{Ann}(\mathrm{Im}(h)) \longrightarrow G^\wedge \xrightarrow{i^\wedge} \mathrm{Im}(h)^\wedge \longrightarrow 0 .$$

Since $\mathrm{Im}(h)^\wedge$ is free, there exists a homomorphism $e: \mathrm{Im}(h)^\wedge \to G^\wedge$ such that $i^\wedge e$ is the identity function.

LEMMA 2.1. *Let $\bar{h}: G/\mathrm{Ker}(h) \to G$ be the homomorphism induced by h, then $\bar{h}^\wedge$ takes $G^\wedge$ onto $(G/\mathrm{Ker}(h))^\wedge$.*

Proof. Let $\beta: G/\mathrm{Ker}(h) \to S^1$ (the circle) be any element of $(G/\mathrm{Ker}(h))^\wedge$. Consider the diagram.

$$\begin{array}{ccc} G/\mathrm{Ker}(h) & \xrightarrow{\bar{h}} & G \\ & \tilde{h} \searrow \cong \quad \nearrow i & \\ & Im(h) & \end{array}$$

which defines $\tilde{h}$. Now define $\alpha' = \beta\tilde{h}^{-1}: \mathrm{Im}(h) \to S^1$ and set $\alpha =$

[2] In [1] and [2], $PH^*(G)$ and Ph^* are denoted by $\boldsymbol{H}^*(G)$ and $\boldsymbol{h}^*$, respectively.

$e(\alpha') \in G^\wedge$, then $\bar{h}^\wedge(\alpha) = \beta$.

PROPOSITION 2.2. *Let G be a compact, connected abelian topological group, and let h be an endomorphism of G. Then the dimension of the topological group* $\mathrm{Ker}_0(h)$ *is equal to the rank of the abelian group* $\mathrm{Ker}(h^\wedge)$.

Proof. By the lemma, we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}(h^\wedge) \longrightarrow G^\wedge \xrightarrow{\bar{h}^\wedge} (G/\mathrm{Ker}(h))^\wedge \longrightarrow 0 .$$

Since G is connected, $(G/\mathrm{Ker}(h))^\wedge$ is free so

$$G^\wedge \cong \mathrm{Ker}(h^\wedge) \oplus (G/\mathrm{Ker}(h))^\wedge .$$

The sequence

$$0 \longrightarrow \mathrm{Ann}(\mathrm{Ker}_0(h)) \longrightarrow G^\wedge \longrightarrow (\mathrm{Ker}_0(h))^\wedge \longrightarrow 0$$

also splits, that is,

$$G^\wedge \cong (\mathrm{Ker}_0(h))^\wedge \oplus \mathrm{Ann}(\mathrm{Ker}_0(h)) .$$

By [6, p. 243], $\mathrm{Ann}(\mathrm{Ker}_0(h))$ is the character group of $G/\mathrm{Ker}_0(h)$. Since $\mathrm{Ker}(h)/\mathrm{Ker}_0(h)$ is finite, $G/\mathrm{Ker}(h)$ and $G/\mathrm{Ker}_0(h)$ have the same dimension, so the abelian groups $(G/\mathrm{Ker}(h))^\wedge$ and $\mathrm{Ann}(\mathrm{Ker}_0(h))$ have the same rank [6, p. 34 and p. 259]. We conclude that $(\mathrm{Ker}_0(h))^\wedge$ and $\mathrm{Ker}(h^\wedge)$ have the same rank and the result is proved.

Let $h^{*,1}$ denote the restriction of h^* to $H^1(G)$.

PROPOSITION 2.3. *Let G be a compact, connected abelian topological group, and let h be an endomorphism of G. Then the rank of* $\mathrm{Ker}(h^\wedge)$ *is equal to the dimension of the vector space* $\mathrm{Ker}(h^{*,1})$.

Proof. There is a natural isomorphism from $G^\wedge$ to $H^1(G; J)$ (integer Čech cohomology) [9; Appendix 1], so we may identify $h^\wedge$ with the endomorphism $h_J^{*,1}$ of $H^1(G; J)$ induced by h. The Universal Coefficient Theorem implies the existence of a natural isomorphism between $H^1(G; J) \otimes R$ (R = the reals) and $H^1(G)$ so that $h_J^{*,1} \otimes$ (identity) corresponds to $h^{*,1}$. Consequently, $\mathrm{Ker}(h^\wedge)$ and $\mathrm{Ker}(h_J^{*,1})$ are isomorphic free groups and their rank is equal to the dimension of the vector space $\mathrm{Ker}(h^{*,1})$.

3. Proof of the main theorem. We will use the symbol $\mathrm{Aut}(G)$ to denote the group of automorphisms of a Lie group G and $\mathrm{Inn}(G)$ for the inner automorphisms.

LEMMA 3.1. *Let G be a compact, connected Lie group and suppose $h \in \mathrm{Aut}(G)$ has the property that $h^m \in \mathrm{Inn}(G)$ for some $m \geqq 1$. Then the rank of the Lie group $\Phi_o(h)$ is equal to the dimension of the graded vector space $\Phi(Ph^*)$.*

Proof. By [8, p. 46], there is a subgroup U of $\mathrm{Aut}(G)$ which intersects each coset of $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ in a single automorphism. Thus there exists C_a, defined by $C_a(x) = axa^{-1}$ for all $x \in G$, such that $hC_a \in U$. Let J_m denote the cyclic group of order m, then since $h^m \in \mathrm{Inn}(G)$ implies that $(hC_a)^m$ is the identity, we can define $\chi: J_m \to \mathrm{Aut}(G)$ by $\chi(q) = (hC_a)^q$. Let C_r^χ denote conjugation by r in $(G \times J_m)_\chi$, the semi-direct product of G and J_m induced by χ. For $b = h(a^{-1})$, we compute that $C^\chi_{(b,1)}(x, 0) = (h(x), 0)$, so $\Phi_0(h)$ is isomorphic to the identity component of the centralizer of $(b, 1)$ in $(G \times J_m)_\chi$. By [7; 1.2] and [2; 4.3], the rank of $\Phi_0(h)$ is equal to the multiplicity of $+1$ as an eigenvalue of Ph^*. Finally, $(Ph^*)^m$ is the identity transformation E, so that multiplicity is equal to the dimension of the kernel of $Ph^* - E$, which is $\Phi(Ph^*)$.

Now we turn to the proof of Theorem 1.1, that is, we obtain the conclusion of Lemma 3.1 without the hypothesis "$h^m \in \mathrm{Inn}(G)$". Let Z be the identity component of the center of G and let S be the maximal connected semisimple normal subgroup of G, then $G = ZS$ and $Z \cap S$ is finite. Since Z and S are characteristic subgroups of G, an automorphism h of G restricts to automorphisms h_Z and h_S of Z and S respectively. Then

$$\mathrm{rank}\,(\Phi_o(h)) = \mathrm{rank}\,(\Phi_o\,(h_Z)) + \mathrm{rank}\,(\Phi_o(h_S))$$

because the equation is true of the corresponding Lie algebras. By DeRham's theorem and the Künneth theorem, there are natural isomorphisms of algebras

$$H^*(G) \cong H^*(\mathfrak{G}) \cong H^*(Z \times S) \cong H^*(Z) \otimes H^*(S)$$

where $\mathfrak{G}$ denotes the Lie algebra of G. The isomorphisms permit us to identify $PH^*(G)$ with $PH^*(Z) \oplus PH^*(S)$ (direct sum). Furthermore, naturality permits us to identify Ph^* with $Ph_Z^* \oplus Ph_S^*$, so it must be that

$$\dim\,(\Phi(Ph^*)) = \dim\,(\Phi(Ph_Z^*)) + \dim\,(\Phi(Ph_S^*))$$

where "dim" means dimension. Thus we have reduced the theorem to the corresponding statement for h_Z and for h_S. Since Z is a torus, $Ph_Z^* = h_Z^{*,1}$. Write the group operation on Z additively and define $\tilde{h}_Z(x) = h_Z(x) - x$ for $x \in Z$, then $\Phi_0(h_Z) = \mathrm{Ker}_o\,(\tilde{h}_Z)$. By 2.2 and 2.3, the dimension of the topological group $\mathrm{Ker}_o\,(\tilde{h}_Z)$ is equal to the dimension of the vector space Ker $(\tilde{h}_Z^{*,1})$ and therefore to the dimension of $\Phi(h_Z^{*,1})$. Since S is semisimple, Aut (S)/Inn (S) is a finite group so 3.1 establishes the result for h_S and completes the proof of Theorem 1.1.

The fact that the rank of $\Phi_0(h)$ is equal to the dimension of $PH^*(\Phi_0(h))$ might lead one to suspect that Theorem 1.1 is a consequence of some more elaborate relationship between $H^*(\Phi_0(h))$ and $\Phi(h^*)$. However, let $g \in G$ be such that the closure of the subgroup it generates is a maximal torus and let h be conjugation by g, then h^* is the identity isomorphism, but $\Phi_0(h)$ is just the maximal torus.

4. A bound on the rank. Let $\mathfrak{A}$ be a simple Lie algebra and let $\rho(\mathfrak{A})$ denote the number of algebra generators of $H^*(\mathfrak{A})$ which are fixed under η^* for all automorphisms η of $\mathfrak{A}$.

PROPOSITION 4.1. *Let G be a compact, connected Lie group with Lie algebra* $\mathfrak{G}$. *Write*

$$\mathfrak{G} \cong \mathfrak{Z} \oplus \mathfrak{A}_1^1 \oplus \cdots \oplus \mathfrak{A}_1^{k(1)} \oplus \cdots \oplus \mathfrak{A}_u^1 \oplus \cdots \oplus \mathfrak{A}_u^{k(u)}$$

where $\mathfrak{Z}$ *is abelian,* $\mathfrak{A}_s^i \cong \mathfrak{A}_s^j \cong \mathfrak{A}_s$ *for each* $s = 1, 2, \cdots, u$ *and all* $i, j = 1, \cdots, k(s)$, *where* $\mathfrak{A}_s$ *is simple, and* $\mathfrak{A}_s^i \not\cong \mathfrak{A}_t^j$ *if* $s \neq t$. *Then*

$$\sum_{s=1}^{u} \rho(\mathfrak{A}_s) \leqq \text{rank } \Phi_o(h)$$

for all automorphisms h of G.

Proof. By Theorem 1.1, the rank of $\Phi_0(h)$ is equal to the dimension of $\Phi(Ph^*)$. Let $PH^*(\mathfrak{G})$ denote the image of $PH^*(G)$ under the deRham isomorphism. By [3; p. 257], Ph^* can be identified with the restriction of η^* to $PH^*(\mathfrak{G})$, for an automorphism η of $\mathfrak{G}$. Consequently, it is sufficient to consider each type of simple Lie algebra in $\mathfrak{G}$ by itself. That is, it is enough to prove that if G is a Lie group such that $\mathfrak{G} \cong \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}$ (k factors) where $\mathfrak{A}$ is simple, and h is an automorphism of G, then the rank of $\Phi_o(h)$ is at least $\rho(\mathfrak{A})$. In this case, the matrix M of the restriction of η^* to $PH^*(\mathfrak{G})$ contains k rows corresponding to each generator of $PH^*(\mathfrak{A})$. If the generator is one which contributes to $\rho(\mathfrak{A})$, then those same rows of $M - E$ (E the identity matrix) are linearly dependent. Thus the multiplicity of $+1$ as an eigenvalue of M is at least $\rho(\mathfrak{A})$ and we conclude that the dimension of $\Phi(Ph^*)$ for any automorphism h of such a group G is indeed at least $\rho(\mathfrak{A})$.

If G is a simply-connected compact Lie group with its Lie algebra $\mathfrak{G}$ as in the proposition, then there is an automorphism h of G for which the rank of $\Phi_0(h)$ is precisely $\sum_{s=1}^{u} \rho(\mathfrak{A}_s)$, so Proposition 4.1

cannot be improved in general.

By [3, p. 258], which is just a restatement of pp. 81–82 of [10], we have the following table:

type of $\mathfrak{A}$	$\rho(\mathfrak{A})$
A_r r even	$r/2$
A_r $r \geqq 3$ odd	$(r+1)/2$
D_4	2
D_r $r \geqq 5$	$r-1$
E_6	4
all others	rank $(\mathfrak{A})$

Observing that $\rho(\mathfrak{A}) \geqq 1$, we have the following known result.

COROLLARY 4.2 (de Siebenthal [8]). *If G is a compact, connected Lie group and there is an automorphism of G with a finite set of fixed points, then G is abelian.*

More generally, we see that knowledge of the rank of $\Phi_0(h)$ for an automorphism h of G implies restrictions on the Lie algebra of G by means of the table above. For example, since $\rho(\mathfrak{A}) = 1$ only if $\mathfrak{A}$ is of type A_1 or type A_2, then

COROLLARY 4.3. *If there is an automorphism h of a compact, connected Lie group G such that $\Phi_0(h)$ is a sphere, then either G is abelian or its Lie algebra $\mathfrak{G}$ is of the form $\mathfrak{G} \cong \mathfrak{Z} \oplus \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}$ where $\mathfrak{Z}$ is abelian and $\mathfrak{A}$ is a simple Lie algebra, either of type A_1 or of type A_2.*

5. **The power map.** Let G be a Lie group whose components are compact. In other words, G is an extension of a compact, connected Lie group G_0 by a discrete, but not necessarily finite, group. We define the *rank* of a component K of G to be the rank of the identity component of the centralizer of g in G, for some element g of K. To see that the definition is independent of the choice of the element, let $C_g\colon G_0 \to G_0$ be conjugation by g and notice that the identity component of the centralizer of g is $\Phi_0(C_g)$. A path in K from g to any other point g' induces a homotopy from C_g to $C_{g'}$, so $C_g^* = C_{g'}^*$. Since Theorem 1.1 states that the rank of $\Phi_0(C_g)$ can be computed from C_g^*, the theorem implies that the rank of a component is independent of the choice of element used to define it.

When G is compact, the definition of the rank of a component which we have just given agrees with the definition we used in [2] because, by [7; 1.2], a maximal torus of the centralizer of $g \in K$ is the identity component of a Cartan subgroup generated from K.

The "power map" $p_k: G \to G$, $k \geqq 2$, is defined by $p_k(g) = g^k$. The component of G containing an element g is gG_0, so $p_k(gG_0) \subseteq g^kG_0$. We will establish necessary and sufficient conditions for $p_k(gG_0) = g^kG_0$.

PROPOSITION 5.1. *The degree of the map* $p_k: gG_0 \to g^kG_0$ *is nonzero if and only if* $\operatorname{rank}(gG_0) = \operatorname{rank}(g^kG_0)$.

Proof. Let $A = PC_g^*$, then by Theorem 1.1 the rank of gG_0 is the dimension of $\Phi(A)$. Define $A^{(k)} = A^{k-1} + A^{k-2} + \cdots + A + E$, then $A^{(k)}$ is nonsingular if and only if $\Phi(A) = \Phi(A^k)$. Thus $\operatorname{rank}(gG_0) = \operatorname{rank}(g^kG_0)$ if and only if $A^{(k)}$ is nonsingular. By [2; 2.3] (note the remark following that theorem), the determinant of $A^{(k)}$ is the degree of $p_k: gG_0 \to g^kG_0$.

Following a suggestion of K. H. Hofmann, we define, for $g \in G$ and $k \geqq 2$, a map $\varphi_k^g: G_0 \to G_0$ by $\varphi_k^g(x) = g^{-k}(gx)^k$. Observe that the degree of φ_k^g is equal to the degree of $p_k: gG_0 \to g^kG_0$. We next wish to prove that if the degree of φ_k^g is zero, then the dimension of $\varphi_k^g(G_0)$ is less than the dimension of G_0. We establish the usual special cases first.

LEMMA 5.2. *Let G be a Lie group in which the identity component G_0 is a torus. If the degree of φ_k^g is zero, then* $\dim(\varphi_k^g(G_0)) < \dim(G_0)$.

Proof. Abbreviate φ_k^g as φ and let $G_0^\wedge$ be the character group of G_0. Just as we did in the proof of Proposition 2.3, we may identify $\varphi^\wedge$ with $\varphi_J^*: H^1(G_0; J) \to H^1(G_0; J)$. Again let C_g denote conjugation by g and set $A = PC_g^*$. Noting that

$$\varphi(x) = C_g^{k-1}(x) \cdot C_g^{k-2}(x) \cdots C_g(x) \cdot x ,$$

an induction argument shows that φ_J^* may be identified with $A^{(k)} = A^{k-1} + A^{k-2} + \cdots + A + E$ so the hypothesis implies that $\varphi^\wedge$ has a nontrivial kernel. We have an exact sequence

$$0 \longrightarrow \operatorname{Ann}(\varphi(G_0)) \longrightarrow G_0^\wedge \xrightarrow{i^\wedge} \varphi(G_0)^\wedge \longrightarrow 0$$

where $\operatorname{Ann}(\varphi(G_0)) \neq 0$ since it contains the kernel of $\varphi^\wedge$. Thus $\varphi(G_0)$ is a proper subtorus of G_0 and consequently it is of lower dimension.

Let G be a Lie group with compact components and suppose $g \in G$ has the property that g^m is in the centralizer of G_0; for some $m \geqq 1$. Let $\Gamma = \bigcup_{r=0}^{m-1} g^rG_0$ and define operation "$\circ$" on Γ by

$$(g^sx) \circ (g^ty) = g^{\psi(s+t)}z$$

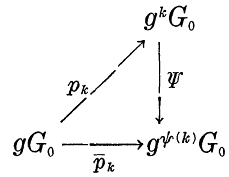
where $(g^s x)(g^t y) = g^{s+t} z$ in G ($x, y, z \in G_0$) and ψ is reduction modulo m. Consider the subgroup $G^g = \bigcup_r g^r G_0$ of G and define $\Psi: G^g \to \Gamma$ by $\Psi(g^s x) = g^{\psi(s)} x$. Since we can compute that

$$\Psi((g^s x)(g^t y)) = \Psi(g^s x) \circ \Psi(g^t y) ,$$

we conclude that Γ is a group and Ψ is a homomorphism.

LEMMA 5.3. *Let G be a Lie group in which G_0 is compact and semisimple. If the degree of φ_k^q is zero, then* $\dim(\varphi_k^q(G_0)) < \dim(G_0)$.

Proof. Since G_0 is semisimple, we may assume that g^m is in the centralizer of G_0, for some $m \geqq 1$. The computation above proves the commutativity of the diagram



where $\bar{p}_k$ denotes the power map in Γ. The hypothesis thus implies that the degree of $\bar{p}_k$ is zero so, since Γ is a compact Lie group, the proof of Theorem 5.2 of [2] establishes that $\dim(\bar{p}_k(gG_0)) < \dim(G_0)$. Observing that $p_k(gG_0) = g^k \varphi_k^q(G_0)$ completes the argument.

PROPOSITION 5.4. *Let G be a Lie group with compact components. If the degree of φ_k^q is zero, then* $\dim(\varphi_k^q(G_0)) < \dim(G_0)$.

Proof. Returning to the notation of §3, we write $G_0 = ZS$ where Z is abelian and S is semisimple, and for $\varphi = \varphi_k^q$, we let φ_Z and φ_S be the restrictions (note that φ is a product of automorphisms). Since $P\varphi^* = P\varphi_Z^* \oplus P\varphi_S^*$, the degree of φ is the product of the degrees of φ_Z and φ_S; so at least one must vanish by the hypothesis. Thus by 5.2 applied to the subgroup of G generated by g and Z, if the degree of φ_Z is zero then $\dim(\varphi_Z(Z)) < \dim(Z)$. Similarly, if the degree of φ_S is zero, then $\dim(\varphi_S(S)) < \dim(S)$ by 5.3. Let $\mu: Z \times S \to G_0$ be defined by $\mu(z, s) = zs$, then since Z is central, $\varphi\mu = \mu(\varphi_Z \times \varphi_S)$ and we have

$$\dim(\varphi(G_0)) = \dim(\varphi(Z)) + \dim(\varphi(S)) < \dim(G_0) .$$

Proposition 5.1 and 5.4 together imply the following extension of the main result, Theorem 5.2, of [2].

THEOREM 5.5. *Let G be a Lie group with compact components.*

The statements below are equivalent:

(i) $p_k(gG_0) = g^kG_0$.

(ii) *The degree of* $p_k: gG_0 \to g^kG_0$ *is not zero*.

(iii) $\operatorname{rank}(gG_0) = \operatorname{rank}(g^kG_0)$.

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