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ON EXTENDING HIGHER DERIVATIONS GENERATED BY CUP PRODUCTS TO THE INTEGRAL CLOSURE

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Let $A=k[x_1,\cdots,x_g]$ be a finitely generated integral domain over a field k of characteristic zero. Let $\bar{A}$ denote the integral closure of A in its quotient field. A well known result due to A. Seidenberg says that any first order k-derivation of A can be extended to $\bar{A}$. This result is known to be false for higher order derivations. In this paper, the authors investigate what types of higher derivations on A can be extended to $\bar{A}$. The main results are for higher derivations which are cup products. Set $\mathrm{Der}_k^1(A) = \mathrm{Der}_k^1(A)_0$ and inductively define $\mathrm{Der}_k^n(A)_0$ as follows:

$$\mathrm{Der}_k^n(A)_0 = \{\varphi \in \mathrm{Der}_k^n(A) \mid \Delta\varphi \in \sum_{i=1}^{n-1} \mathrm{Der}_k^i(A)_0 \cup \mathrm{Der}_k^{n-i}(A)_0\} .$$

The authors show that if $\varphi \in \mathrm{Der}_k^n(A)_0$, then $\varphi(\bar{A}) \subseteq \bar{A}$. Various examples are given which indicate that the above mentioned result is about as good as possible.

Introduction. Throughout this paper, $A = k[x_1, \cdots, x_g]$ will denote a finitely generated integral domain over a field k of characteristic zero. We shall let Q denote the quotient field of A and $\bar{A}$ the integral closure of A in Q. For each $n = 1, 2, \cdots$, we shall let $\mathrm{Der}_k^n(A)$ denote the A-module of all nth order k-derivations of A to A. Thus, $\varphi \in \mathrm{Der}_k^n(A)$ if and only if $\varphi \in \mathrm{Hom}_k(A, A)$, and for all $a_0, \cdots, a_n \in A$ we have

$$(1)\quad \varphi(a_0a_1 \cdots a_n) = \sum_{s=1}^{n} (-1)^{s-1} \sum_{i_1<\cdots<i_s} a_{i_1} \cdots a_{i_s}\varphi(a_0 \cdots \check{a}_{i_1} \cdots \check{a}_{i_s} \cdots a_n) .$$

The authors refer the reader to [3] for the various facts about $\mathrm{Der}_k^n(A)$ used in this paper. Of particular importance is the fact that any nth order derivation $\varphi \in \mathrm{Der}_k^n(A)$ can naturally be extended to an nth order derivation of any localization of A [Thm 15; 3].

We shall need the Hochschild coboundary operator Δ which is defined as follows: If $\varphi \in \mathrm{Hom}_k(A, A)$, then $\Delta\varphi\colon A \times A \to A$ is the k-bilinear mapping defined by $\Delta\varphi(a_1, a_2) = \varphi(a_1a_2) - a_1\varphi(a_2) - a_2\varphi(a_1)$. We shall also need the cup product $\varphi \cup \psi$ of two k-linear mappings φ and ψ of A. $\varphi \cup \psi\colon A \times A \to A$ is the k-bilinear mapping defined by $\varphi \cup \psi(a_1, a_2) = \varphi(a_1)\psi(a_2)$ If P and P are two A-submodules of $\mathrm{Hom}_k(A, A)$, then $P \cup P$ will denote the set of all k-bilinear mappings of $A \times A$ into A which are finite A-linear combinations of mappings of the form $\varphi \cup \psi$ for $\varphi \in P$, $\psi \in P'$. Thus, if φ is an nth order k-derivation of A such that $\Delta\varphi \in \sum_{i=1}^{n-1} \mathrm{Der}_k^i(A) \cup \mathrm{Der}_k^{n-i}(A)$, then there exist constants $e_{lj} \in A$ and k-derivations $\psi_l^{(j)}$, $\lambda_l^{(j)} \in \mathrm{Der}_k^j(A)$ such that for all a and b in A, we have

$$\varphi(ab) = a\varphi(b) + b\varphi(a) + \sum e_{l1}\psi_l^{(1)}(a)\lambda_l^{(n-1)}(b) + \cdots + \sum e_{ln-1}\psi_l^{(n-1)}(a)\lambda_l^{(1)}(b) \,. \tag{2}$$

Now the purpose of this paper is to study which nth order k-derivations $\varphi: A \to A$ can be extended to $\bar{A}$. In [4], A. Seidenberg showed that any 1st order derivation of A must map $\bar{A}$ to $\bar{A}$. In [1], an example was given which shows that 2nd order derivations $\varphi \in \mathrm{Der}_k^2(A)$ need not have the property that $\varphi(\bar{A}) \subset \bar{A}$. Since we shall have use of this example latter, we present it here

EXAMPLE 1. Consider the curve $X^2 = Y^3$ over the rational numbers $\boldsymbol{Q}$. Let A be the coordinate ring of this curve i.e. $A = \boldsymbol{Q}[x, y] = \boldsymbol{Q}[X, Y]/(X^2 - Y^3)$. One can easily check that A is a domain whose integral closure is given by $\bar{A} = A[x/y]$. Since the quotient field of A is a finite separable extension of $\boldsymbol{Q}(y)$, it follows that any 2nd order derivation $\varphi \in \mathrm{Der}_Q^2(A)$ is determined by its values on y and y^2. A simple calculation shows that if $\varphi(y) = a$, and $\varphi(y^2) = b$ (where a and b lie in the quotient field of A), then

$$\varphi(x) = \frac{3y}{8}\left(\frac{2ya + b}{x}\right),\ \varphi(x^2) = 3yb - 3y^2a$$

and

$$\varphi(xy) = \frac{5y^2}{8}\left(\frac{3b - 2ya}{x}\right) .$$

If we set $a = 1$ and $b = -2y$, then $\varphi \in \mathrm{Der}_Q^2(A)$, and one easily checks that $\varphi(x/y) = x/y^2 \notin \bar{A}$.

Thus, higher derivations on A need not extend to $\bar{A}$. At the end of [1], the author conjectured that any $\varphi \in \mathrm{Der}_k^2(A)$ such that $\Delta\varphi \in \mathrm{Der}_k^1(A) \cup \mathrm{Der}_k^1(A)$ must map $\bar{A}$ to $\bar{A}$. In this paper, we shall show that this conjecture is correct. We shall also formulate sufficient conditions on $\varphi \in \mathrm{Der}_k^n(A)$ in order that $\varphi(\bar{A}) \subset \bar{A}$. We assume the reader is familiar with [1].

Main results.

THEOREM 1. *Let $A = k[x_1, \cdots, x_g]$ be a finitely generated integral domain over a field k of characteristic zero. Let $\bar{A}$ denote the integral closure of A in its quotient field Q. Let $\varphi \in \mathrm{Der}_k^2(A)$ and* *assume* $\Delta\varphi \in \mathrm{Der}_K^1(A) \cup \mathrm{Der}_k^1(A)$. *Then* $\varphi(\bar{A}) \subset \bar{A}$.

Proof. Let $\mathrm{Min}(\bar{A})$ denote the collection of height one primes in $\bar{A}$. Since $\bar{A}$ is a Krull domain, we have $\bar{A} = \bigcap \{\bar{A}_q \mid q \in \mathrm{Min}(\bar{A})\}$. Here as usual $\bar{A}_q$ means $\bar{A}$ localized at the prime q. Let $q \in \mathrm{Min}(\bar{A})$. Then $p = q \cap A \in \mathrm{Min}(A)$. Let us set $R = A_p$ and $\bar{R} = (\bar{A})_p = \bar{A}_p$ the integral closure of R in Q. Let $\bar{q}$ denote the extended prime ideal $q\bar{R}$ in $\bar{R}$. Then $\bar{R}_{\bar{q}} = \bar{A}_q$. Now since R is a localization of A, we see that $\varphi \in \mathrm{Der}_k^2(R)$. Suppose we could show that $\varphi(\bar{R}) \subseteqq \bar{R}$. Then $\varphi(\bar{R}_{\bar{q}}) \subseteqq \bar{R}_{\bar{q}}$ or equivalently $\varphi(\bar{A}_q) \subseteqq \bar{A}_q$. Since $\bar{A}$ is the intersection of the $\bar{A}_q$, the theorem would be proven. Thus to prove Theorem 1, it suffices to prove the following assertion:

"Under the same hypotheses as Theorem 1, let $p \in \mathrm{Min}(A)$, $R = A_p$ and $\bar{R} = \bar{A}_p$. Then $\varphi(\bar{R}) \subseteqq \bar{R}$."

So fix a minimal prime $p \in \mathrm{Min}(A)$, and set $R = A_p$, $\bar{R} = \bar{A}_p$. We have already noted that $\varphi \in \mathrm{Der}_k^2(R)$, and one easily sees that $\Delta\varphi \in \mathrm{Der}_k^1(R) \cup \mathrm{Der}_k^1(R)$. Now if $A = \bar{A}$, there is nothing to prove. Hence, we may assume $\bar{A} \neq A$. Then the conductor C of A in $\bar{A}$ is a proper ideal in A. If $C \not\subset p$, then $R = \bar{R}$ and again there is nothing to prove. Hence we may assume $C \subset p$. In this case, CR is the conductor of R in $\bar{R}$.

We now follow the proof of Theorem 3 in [1]. Let the transcendence degree of A over k be r, and let m denote the maximal ideal in R. Then R/m is the quotient field of A/p and hence has transcendence degree $r-1$ over k. Let $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_{r-1}\}$ be a transcendence basis of R/m over k. Pull these $\bar{\alpha}_i$ back to elements α_i in $R - m$. Then $F = k(\alpha_1, \cdots, \alpha_{r-1})$ is a field of transcendence degree $r-1$ over k, and $F \subset R$.

We know that $\bar{R}$ is a semilocal ring with maximal ideals $m_1, \cdots, m_t$ lying over m in R. Set $J = \bigcap_{i=1}^t m_i$, the Jacobson radical of $\bar{R}$. Each local ring $V_i = \bar{R}_{m_i}$, $i = 1, \cdots, t$, is a discrete rank one valuation ring dominating R. By [Thm 18, p. 45; 6], we can find an element $\beta \in J$ such that β generates the maximal ideal in each V_i. Since the Krull dimension of $\bar{R}$ is one, we see that J is the radical of the ideal CR in $\bar{R}$. Thus, some power of β, say β^n, lies in CR. We shall have use of this remark later.

It was shown in [1], that $\mathrm{Der}_k^1(\bar{R})$ is a free $\bar{R}$-module with basis $\{\delta_0, \delta_1, \cdots, \delta_{r-1}\}$. The derivations δ_i satisfy the following relations:

$$(3) \qquad \delta_0(\beta) = 1,\ \delta_0(\alpha_i) = 0 = \delta_i(\beta) \quad \text{for} \quad i = 1, \cdots, r-1$$

and

$$\delta_i(\alpha_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad 1 \leqq i \leqq j \leqq r-1 \,.$$

We observe that the derivations δ_i commute on the field $F(\beta)$. Since β is a uniformizing parameter for V_i, β is transcendental over F. Hence Q is a separable algebraic extension of $F(\beta)$. Therefore the derivations on $F(\beta)$ have a unique extension to Q. It follows that the δ_i commute on Q. It follows from [2; Thm 16, 11. 2] that the union $\bigcup_{n=1}^{\infty} \mathrm{Der}_k^n(Q)$ is a free Q-algebra generated by $\delta_0, \cdots, \delta_{r-1}$. In particular, φ can be written as a unique polynomial of degree two in $\delta_0, \cdots, \delta_{r-1}$. The coefficients of this polynomial lie in Q. Let us write φ as follows:

$$(4) \qquad \varphi = \sum_{i=0}^{r-1} a_i \delta_i + \sum_{0 \leqq i < j \leqq r-1} a_{ij} \delta_i \delta_j + \sum_{i=0}^{r-1} a_{ii} \delta_i^2 \,.$$

Since $\Delta\varphi \in \mathrm{Der}_k^1(R) \cup \mathrm{Der}_k^1(R)$, we can write for all a and b in R:

$$(5) \qquad \varphi(ab) = a\varphi(b) + b\varphi(a) + \sum_l e_l \psi_l(a) \lambda_l(b)$$

where $e_l \in R$ and $\psi_l, \lambda_l \in \mathrm{Der}_k^1(R)$. One easily checks that equation (5) continues to hold for all a and b in Q. Now by [Thm 1; 4], each ψ_l and λ_l extends to $\bar{R}$. It then easily follows that CR is differential under ψ_l and λ_l, i.e. $\psi_l(CR) \subset CR$ and $\lambda_l(CR) \subset CR$. Thus, CR remains differential under ψ_l and λ_l when considered as an ideal in $\bar{R}$. Hence, [Thm 1; 5] implies that each m_i in $\bar{R}$ is differential under ψ_l and λ_l. Write each ψ_l and λ_l as a linear combination of $\delta_0, \delta_1, \cdots, \delta_{r-1}$:

$$(6) \qquad \psi_l = \sum_{i=0}^{r-1} \mu_{li} \delta_i \quad \lambda_l = \sum_{i=0}^{r-1} \gamma_{li} \delta_i \,.$$

Here the coefficients μ_{li} and γ_{li} lie in $\bar{R}$. Then $\psi_l(J) \subset J$ and $\lambda_l(J) \subset J$ imply that μ_{l0} and γ_{l0} lie in J. If we now substitute the expressions in equations (6) and (4) into equation (5) and then make various substitutions of the form $a, b = \alpha_1, \cdots, \alpha_{r-1}, \beta$, we see that all the coefficients, except possibly a_0, appearing in (4) lie in $\bar{R}$. We further get that $a_{0i} \in J$ for $i = 1, \cdots, r-1$, and $a_{00} \in J^2$.

Thus, to complete the proof of the assertion $\varphi(\bar{R}) \subseteqq \bar{R}$, we must show that a_0 in (4) lies in $\bar{R}$. We shall show this by arguing that $a_0 \in V_i$ for every $i = 1, \cdots, t$.

So fix an $i = 1, \cdots, t$, and let $v_i : V_i \to \boldsymbol{Z}$ be the valuation of V_i given by $v_i(\beta) = 1$. We wish to show that $v_i(a_0) \geqq 0$. Let us assume $v_i(a_0) < 0$. We need the following lemma:

LEMMA 1. *There exist two elements x and y in R such that*
(a) *The value $N = v_i(x)$ of x is the smallest positive value of*

any element in R.

(b) *The value* $v_i(y)$ *of* y *is not a multiple of* N.

Proof. Since $R \subset V_i$, we have $v_i(z) \geqq 0$ for every element z in R. So we can certainly find an element x in R which satisfies (a). As pointed out earlier, $\beta^n \in CR \subset R$. Thus, $\beta^{n+l} \in R$ for any nonnegative integer l.

Now suppose no $y \in R$ can be found satisfying (b). Then for every nonnegative integer l, we must have $n + l = v_i(\beta^{n+l})$ is a multiple of N. This can only happen if $N = 1$. We shall show this is impossible.

If $N = 1$, then $x = \gamma\beta$ for some unit γ in V_i. We want to consider

$$\varphi(x) = \sum_{i=0}^{r-1} a_i\delta_i(x) + \sum_{0 \leqq i < j \leqq r-1} a_{ij}\delta_i\delta_j(x) + \sum_{i=0}^{r-1} a_{ii}\delta_i^2(x)$$

which is an element of R. Now we have

$$\begin{aligned} &\delta_0(x) = \beta\delta_0(\gamma) + \gamma \\ &\delta_i(x) = \beta\delta_i(\gamma) \qquad i = 1, \cdots, r-1 \\ &\delta_0\delta_i(x) = \beta\delta_0\delta_i(\gamma) + \delta_i(\gamma) \qquad i = 1, \cdots, r-1 \\ &\delta_i\delta_j(x) = \beta\delta_i\delta_j(\gamma) \qquad 0 < i \leqq j \leqq r-1 \end{aligned} \tag{7}$$

and

$$\delta_0^2(x) = \beta\delta_0^2(\gamma) + 2\delta_0(\gamma) .$$

Since the δ_j are derivations on $\bar{R}$, they naturally extend to V_i. Thus, the elements in equation (7) are all elements of V_i, and clearly $\delta_0(x)$ is a unit in V_i. If we now use the facts that $a_1, \cdots, a_{r-1}, a_{ij} \in \bar{R}$, $a_{0i} \in J$ and $a_{00} \in J^2$, we see that

$$v_i\left[\sum_{i=1}^{r-1} a_i\delta_i(x) + \sum_{0 \leqq i < j \leqq r-1} a_{ij}\delta_i\delta_j(x) + \sum_{i=0}^{r-1} a_{ii}\delta_i^2(x)\right] \geqq 1 \tag{8}$$

Thus, $v_i(\varphi(x)) = v_i(a_0) + v_i(\delta_0(x)) = v_i(a_0) < 0$. But, $\varphi(x) \in R$ means the value of $\varphi(x)$ must be nonnegative. Thus, we have reached a contradiction and the proof of Lemma 1 is complete.

Now among all the elements z of R such that $v_i(z)$ is not a multiple of N pick one, say y, of smallest value M. Lemma 1 guarantees that such an element $y \in R$ exists. Then $M - N > 0$, and $M - N$ is not the value of any element of R. Since $v_i(x) = N$, $x = \gamma\beta^N$ for some unit $\gamma \in V_i$. An argument similar to that in Lemma 1 shows that $v_i(\varphi(x)) = v_i(a_0) + N - 1$. Now there are two cases to consider. Either $\varphi(x)$ is a unit in R or it is not. If $\varphi(x)$ is a nonunit, then $v_i(\varphi(x)) \geqq N$. But this implies $v_i(a_0) \geqq 1$ which is contrary to our assumption. Thus, $\varphi(x)$ is a unit. So $v_i(a_0) = 1 - N$. But now a similar computation applied to y gives us that $v_i(\varphi(y)) = v_i(a_0) + M - 1 = M - N$. Since $\varphi(y) \in R$, and $M - N$ is not the value of anything in R, we have reached a contradiction.

Thus, $v_i(a_0) \geqq 0$ and the proof of Theorem 1 is complete.

In our proof of Theorem 2 below, we shall need the fact that the coefficient a_0 in equation (4) actually lies in J. The proof of Theorem 1 shows that $a_0 \in \bar{R}$. To see that $a_0 \in J$, we proceed as follows: Since $\varphi(\bar{R}) \subseteqq \bar{R}$, equation (5) immediately implies that $\varphi(CR) \subseteqq CR$. In the notation of Theorem 1, we wish to argue that $v_i(a_0) \geqq 1$. Suppose $v_i(a_0) = 0$. Let N be the minimum positive value of any element in CR, and let $x \in CR$ have value N. Then as in Lemma 1, $v_i(\varphi(x)) = v_i(a_0) + N - 1 = N - 1$. Since $\varphi(x) \in CR$ this is impossible. Thus $v_i(a_0) \geqq 1$.

For Theorem 2, we shall need the following definition:

DEFINITION. Set $\mathrm{Der}_k^1(A)_0 = \mathrm{Der}_k^1(A)$ and inductively define $\mathrm{Der}_k^n(A)_0$ as follows:

$$\mathrm{Der}_k^n(A)_0 = \left\{\varphi \in \mathrm{Der}_k^n(A) \mid \Delta\varphi \in \sum_{i=1}^{n-1} \mathrm{Der}_k^i(A)_0 \cup \mathrm{Der}_k^{n-i}(A)_0\right\} .$$

Thus, Theorem 1 states that if $\varphi \in \mathrm{Der}_k^2(A)_0$, then $\varphi(\bar{A}) \subset \bar{A}$. We can now prove the general result.

THEOREM 2. *Let $A = k[x_1, \cdots, x_g]$ be a finitely generated integral domain over a field k of characteristic zero. Let $\bar{A}$ denote the integral closure of A in its quotient field Q. Let $\varphi \in \mathrm{Der}_k^n(A)_0$. Then $\varphi(\bar{A}) \subset \bar{A}$.*

Proof. The proof proceeds along the same lines as in Theorem 1. It suffices to show that for every prime p of height one in A, $\varphi(\bar{R}) \subset \bar{R}$. Here, as in Theorem 1, $\bar{R}$ denotes the integral closure of $R = A_p$ in Q. One easily checks that $\varphi \in \mathrm{Der}_k^n(R)_0$. We shall adopt all the notation used in Theorem 1. Thus, CR is the conductor of R in $\bar{R}$.

For the purposes of this proof, let us define $\mathrm{Der}_k^n(R)_{\bar{R}}$ inductively as follows:

$$\mathrm{Der}_k^1(R)_{\bar{R}} = \mathrm{Der}_k^1(R) \tag{9}$$

$$\mathrm{Der}_k^n(R)_{\bar{R}} = \left\{\varphi \in \mathrm{Der}_k^n(R) \mid \Delta\varphi \in \sum_{i=1}^{n-1} \mathrm{Der}_k^i(R)_{\bar{R}} \cup \mathrm{Der}_k^{n-i}(R)_{\bar{R}} \text{ and } \varphi(\bar{R}) \subset \bar{R}\right\} .$$

Then we have already proven that $\mathrm{Der}_k^2(R)_0 = \mathrm{Der}_k^2(R)_{\bar{R}}$ in Theorem 1, and we shall show that $\mathrm{Der}_k^n(R)_0 = \mathrm{Der}_k^n(R)_{\bar{R}}$ for all n.

Now we know that $\bigcup_n \mathrm{Der}_k^n(Q)$ is a free Q-algebra generated by $\delta_0, \cdots, \delta_{r-1}$. Thus if $\varphi \in \mathrm{Der}_k^n(R)$, then $\varphi = g(\delta_0, \cdots, \delta_{r-1})$ for some polynomial $g(X_0, \cdots, X_{r-1}) \in Q[X_0, \cdots, X_{r-1}]$ of degree less than or equal to n. We further know this polynomial is unique. We now need the following lemma:

LEMMA 2. *Let $\varphi \in \mathrm{Der}_k^n(R)_{\bar{R}}$, and write $\varphi = g(\delta_0, \cdots, \delta_{r-1})$. Then the coefficients of any monomials of g which contain $\delta_0^j (1 \leqq j \leqq n)$ lie in J^j.*

Proof. We proceed by induction on n. The case $n = 1$ was proven in Theorem 1. The case $n = 2$ was proven in Theorem 1 and the remarks following Theorem 1. Thus, we may assume Lemma 2 has been proven for all elements of $\mathrm{Der}_k^m(R)_{\bar{R}}$ with $m < n$.

Let $\varphi \in \mathrm{Der}_k^n(R)_{\bar{R}}$. Then there exist constants $e_{lj} \in R$ and derivations $\psi_l^{(j)}, \lambda_l^{(j)} \in \mathrm{Der}_k^j(R)_{\bar{R}}$, $j = 1, \cdots, n-1$, such that for all a and b in Q equation (2) is satisfied. Our induction hypothesis applies to the derivations $\psi_l^{(j)}$ and $\lambda_l^{(j)}$. So we can write:

$$\begin{aligned} \psi_l^{(j)} &= \sum c_t^{l,j}\delta_t + \sum c_{t_1t_2}^{l,j}\delta_{t_1}\delta_{t_2} + \cdots + \sum c_{t_1\cdots t_j}^{l,j}\cdot\delta_{t_1}\cdots\delta_{t_j} \\ \lambda_l^{(j)} &= \sum d_t^{l,j}\delta_t + \sum d_{t_1t_2}^{l,j}\delta_{t_1}\delta_{t_2} + \cdots + \sum d_{t_1\cdots t_j}^{l,j}\delta_{t_1}\cdots\delta_{t_j}\,. \end{aligned} \tag{10}$$

In (10), the coefficient of any monomial in either expression which contains δ_0^j will lie in J^j. We note that since $\psi_l^{(j)}, \lambda_l^{(j)}: \bar{R} \to \bar{R}$, all the coefficients of (10) lie in $\bar{R}$.

Now write out the polynomial $g(\delta_0, \cdots, \delta_{r-1})$ which gives us φ as follows:

$$\varphi = \sum a_t\delta_t + \sum a_{t_1t_2}\delta_{t_1}\delta_{t_2} + \cdots + \sum a_{t_1\cdots t_n}\delta_{t_1}\cdots\delta_{t_n}\,. \tag{11}$$

Since $\varphi(\bar{R}) \subset \bar{R}$, one easily checks that all the coefficients $a_t, a_{t_1t_2}, \cdots, a_{t_1\cdots t_n}$ of (11) lie in $\bar{R}$. We now substitute equations (10) and (11) into (2) and get:

$$\begin{aligned} &\sum a_t\delta_t(ab) + \sum a_{t_1t_2}\delta_{t_1}\delta_{t_2}(ab) + \cdots + \sum a_{t_1\cdots t_n}\delta_{t_1}\cdots {}_{t_n}(ab) \\ &\quad = a\{\sum a_t\delta_t(b) + \cdots + \sum a_{t_1\cdots t_n}\delta_{t_1}\cdots\delta_{t_n}(b)\} \\ &\qquad + b\{\sum a_t\delta_t(a) + \cdots + \sum a_{t_1\cdots t_n}\delta_{t_1}\cdots\delta_{t_n}(a)\} \\ &\qquad + \sum_l e_{l,1}\Big\{\sum_t c_t^{l,1}\delta_t(a)\Big\}\Big\{\sum_t d_t^{l,n-1}\,\delta_t(b) + \cdots \\ &\qquad + \sum d_{t_1\cdots t_{n-1}}^{l,n-1}\,\delta_{t_n}\cdots\delta_{t_{n-1}}(b)\Big\} + \cdots \\ &\qquad + \sum e_{l,n-1}\{\sum c_t^{l,n-1}\delta_t(a) + \cdots + \sum c_{t_1\cdots t_{n-1}}^{l,n-1}\,\delta_{t_1}\cdots\delta_{t_{n-1}}(a)\} \\ &\qquad \times \{\sum d_t^{l,1}\delta_t(b)\}\,. \end{aligned} \tag{12}$$

After simplifying (12) and comparing coefficients, we see that any coefficient of (11) (except possibly for a_0) in a monomial containing δ_0^j lies in J^j. Thus, the lemma will be complete if we show $a_0 \in J$.

Since $\varphi(\bar{R}) \subset \bar{R}$, one easily sees using (2) that $\varphi(CR) \subset CR$. Thus, to argue $a_0 \in J$, one can proceed exactly as in the remarks following Theorem 1. Pick an element $x \in CR$ of minimum value $N = v_i(x)$. If $v_i(a_0) = 0$, then $v_i(\varphi(x)) = N - 1$ which is a contradiction. This completes the proof of Lemma 2.

We now proceed to prove Theorem 2 by induction on n. A. Seidenberg's original result [Thm; 4], and Theorem 1 give us the case $n = 1$ and $n = 2$. Thus, assume Theorem 2 is correct for all $m < n$, and let $\varphi \in \mathrm{Der}_k^n(R)_0$. We can expand φ as in equation (2) for some choice of constants $e_{lj} \in R$ and derivations $\psi_l^{(j)}, \lambda^{(j)} \in \mathrm{Der}_k^j(R)_0$. By our induction hypothesis, $\mathrm{Der}_k^j(R)_0 = \mathrm{Der}_k^j(R)_{\bar{R}}$. So by Lemma 2, each $\psi_l^{(j)}$ and $\lambda_l^{(j)}$ can be written as in equation (10) with the coefficients of any monomials containing δ_0^j lying in J^j. Now write φ as in equation (11). Following the same substitutions as in Lemma 2, we see that all the coefficients $a_1, \cdots, a_{r-1}, a_{t_1t_2}, \cdots, a_{t_1\cdots t_n}$ lie in $\bar{R}$. Further, the coefficients appearing in terms containing δ_0^j lie in J^j, except possibly for a_0. Thus, as in Theorem 1, we have to argue that $v_i(a_0) \geqq 0$ for all $i = 1, \cdots, t$. But this argument is exactly the same as in Theorem 1. Assume $v_i(a_0) < 0$. The coefficients of (11) lying in the right powers of J exactly mean that $v_i(\varphi(z)) = v_i(a_0) + v_i(z) - 1$ for any nonunit z of R. Thus we proceed exactly as before to argue that $v_i(a_0) < 0$ is impossible. This completes the proof of Theorem 2.

The reader may be wondering if a slightly weaker hypothesis on $\varphi \in \mathrm{Der}_k^n(A)$ will imply $\varphi(\bar{A}) \subset \bar{A}$. In particular, it is natural to ask the following question: Suppose $\varphi \in \mathrm{Der}_k^n(A)$ such that

$$\Delta\varphi \in \sum_{i=1}^{n-1} \mathrm{Der}_k^i(A) \cup \mathrm{Der}_k^{n-i}(A) .$$

Then is $\varphi(\bar{A}) \subseteqq \bar{A}$? Theorem 1 implies this is true if $n = 2$. We shall give an example which shows that for $n > 2$ the answer to the above question is in general negative.

EXAMPLE 2. We return to Example 1 at the beginning of this paper. We may equally well describe the ring A as $A = \boldsymbol{Q}[t^3, t^2]$. Set $\delta = \partial/\partial_t$, a first order derivation on the quotient field of A. One can easily check that $t\delta$, $t^2\delta$, $\delta^2 - (2/t)\delta$, $t\delta^2 - \delta$ and $\delta^3 - (3/t)\delta^2 + (3/t^2)\delta$ are all derivations on A. Set

$$\varphi = t^2\delta\left(\delta^3 - \frac{3}{t}\delta^2 + \frac{3}{t^2}\delta\right) - \frac{9t}{2}\delta\left(\delta^2 - \frac{2}{t}\delta\right) + \frac{3}{2}\left(\delta^2 - \frac{2}{t}\delta\right)(t\delta) . \tag{13}$$

Then $\varphi \in \mathrm{Der}^4_Q(A)$. If we expand φ out, we get $\varphi = t^2\delta^4 - 6t\delta^3 + 15\delta^2 - (18/t)\delta$. Now the integral closure $\bar{A}$ of A is just $Q[t]$, and thus $\varphi(\bar{A}) \not\subset \bar{A}$. However one can easily check that

$$\Delta\varphi = 4\left(\delta^3 - \frac{3}{t}\delta^2 + \frac{3}{t^2}\delta\right) \cup (t^2\delta) + 6(t\delta^2 - \delta) \cup (t\delta^2 - \delta)$$
$$+ (t^2\delta) \cup \left(\delta^3 - \frac{3}{t}\delta^2 + \frac{3}{t^2}\delta\right) .$$

Thus

$$\Delta\varphi \in \mathrm{Der}^1_Q(A) \cup \mathrm{Der}^3_Q(A) + \mathrm{Der}^2_Q(A) \cup \mathrm{Der}^2_Q(A) + \mathrm{Der}^3_Q(A) \cup \mathrm{Der}^1_Q(A) ,$$

but $\varphi(\bar{A}) \not\subset \bar{A}$.

This example shows that we really need the stronger statement $\varphi \in \mathrm{Der}^n_k(A)_0$ in order to conclude the $\varphi(\bar{A}) \subset \bar{A}$.

Finally, we note that the methods used in Theorems 1 and 2 give a new proof of A. Seidenberg's original theorem for finitely generated domains:

THEOREM (A. Seidenberg). *Let $A = k[x_1, \cdots, x_g]$ be a finitely generated integal domain over a field k of characteristic zero. Let $\bar{A}$ denote the integral closure of A in its quotient field Q. Let $\delta \in \mathrm{Der}^1_k(A)$. Then $\delta(\bar{A}) \subset \bar{A}$.*

Proof. Using the same notation as in Theorem 1, we see that it suffices to prove $\delta(\bar{R}) \subset \bar{R}$. Write $\delta = a_0\delta_0 + \cdots + a_{r-1}\delta_{r-1}$ with the $a_i \in Q$. Since $\delta(\alpha_i) \in R$, we see $a_1, \cdots, a_{r-1} \in R$. As before, it remains to argue that $v_i(a_0) \geqq 0$ for all $i = 1, \cdots, t$. So fix an $i = 1, \cdots, t$ and assume $v_i(a_0) < 0$. Pick $x \in R$ such that $N = v_i(x)$ is the minimum positive value of any element of R. Then $v_i(\delta(x)) = v_i(a_0) + N - 1$. Since $\delta(x) \in R$, we conclude that $v_i(a_0) = 1 - N$. By an argument similar to that in Lemma 1, we can find an element $y \in R$ such that $M = v_i(y)$ is the minimum positive value of anything in R which is not a multiple of N. Then $v_i(\delta(y)) = M - N$ which is impossible.

REFERENCES

1. W. C. Brown, *Higher derivations on finitely generated integral domains II*, Proc. Amer. Math. Soc., **51** (1975), 8-14.
2. A. Grothendieck, *Elements de Geometrie Algebreque IV*, pt. 4, Pub. Math. de L'IHES #32 Paris, 1967.
3. Y. Nakai, *High order derivations I*, Osaka J. Math., **7** (1970), 1-27.

4. A. Seidenberg, *Derivations and integral closure*, Pacific J. Math., **16** (1966), 167-173.
5. ———, *Differential ideals in rings of finitely generated type*, Amer. J. Math., **89** (1967), 22-42.
6. O. Zariski and P. Samuel, *Commutative Algebra II*, University Series in Higher Math. Van Nostrand, Princeton, N. J. 1958, MR19 #833.
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