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THE C^*-ALGEBRAS OF SOME REAL AND p-ADIC SOLVABLE GROUPS

JONATHAN ROSENBERG

When G is a locally compact group, the unitary representation theory of G is the "same" as the *-representation theory of the group C^*-algebra $C^*(G)$. Hence it is of interest to determine the isomorphism class of $C^*(G)$ for a wide variety of groups G. Using methods suggested by papers of Z'ep and Delaroche, we determine explicitly the C^*-algebras of the "$ax+b$" groups over all nondiscrete locally compact fields and of a number of two-step solvable Lie groups. Only finitely many C^*-algebras arise as the group C^*-algebras of 3-dimensional simply connected Lie groups, and we characterize many of them. We also discuss the C^*-algebras of unipotent $\mathfrak{p}$-adic groups.

1. Introduction. The C^*-algebra of a locally compact group is easily defined as the enveloping C^*-algebra of the group L^1-algebra [11, 13.9.1], but until recently the only groups the structure of whose C^*-algebra was explicitly "known" have been abelian and compact groups and a few semi-simple Lie groups: $SL(2, \boldsymbol{C})$ [13], [10]; $SL(2, \boldsymbol{R})$ [19]; the other groups with the same universal covering group as $SL(2, \boldsymbol{R})$ [18]; and $Spin\ (4, 1)$ [5]. A fair amount is known about the C^*-algebras of nilpotent Lie groups (see, for instance, [21]), but the problem of characterizing the C^*-algebra of the Heisenberg group up to isomorphism among the family of all C^*-algebras having the same spectrum has proved difficult and remains unsolved. It is therefore interesting that Z'ep [23] has now noticed that the study of certain C^*-algebra extensions by Brown, Douglas and Fillmore [8] can be applied to characterize the C^*-algebra of the "improper $ax+b$ group," the affine group of the real line.

We extend Z'ep's method to study the C^*-algebras of other groups with "relatively few" infinite-dimensional irreducible representations. The affine groups of the affine lines over all nondiscrete locally compact fields K (the remaining important cases being $K = \boldsymbol{C}$ and K a $\mathfrak{p}$-adic field) are treated in §§ 2 and 3; the calculations are routine and the results are quite similar to Z'ep's. More interesting is the fact that similar methods can be applied to some groups with perhaps infinitely many inequivalent infinite-dimensional irreducible representations. A class of such groups, all of which are two-step solvable connected Lie groups, is studied in § 4. The simplest group in this class is the "proper $ax+b$ group," the connected component of the identity element in the group considered by Z'ep.

In § 5, we examine the problem of classifying the group C^*-algebras of all simply connected solvable Lie groups of dimension 3. Although we cannot give a complete solution, we at least show that such groups yield only finitely many C^*-algebras (up to isomorphism). It appears that the C^*-algebra of a solvable Lie group depends on the root structure of the Lie algebra but not on the exact values of the roots. Our analysis points out two curious but elementary facts: there exist one-parameter families of groups with isomorphic group C^*-algebras, and it is impossible to tell whether or not a group is unimodular merely by looking at its C^*-algebra. Section 6 contains a few remarks about the C^*-algebras of unipotent $\mathfrak{p}$-adic groups.

It should be mentioned that every group considered in this paper is a semidirect product of abelian groups, and so has for its C^*-algebra the C^*-algebra of some topological transformation group. (See [12] for the theory of these algebras.) Philip Green, in studying the classification of certain transformation group algebras, has been able to obtain some of the results of this paper by different methods; his work will appear shortly [15]. The author is indebted to Professor Marc A. Rieffel for many useful comments and suggestions.

The notations $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Z}$, and $\boldsymbol{T}$ for the real and complex numbers, the rational integers, and the circle group are used throughout this paper. If K is a field, K^* denotes its multiplicative group. If E is a subset of some (understood) larger set, χ_E denotes its characteristic function. A raised dot sometimes denotes the usual inner product in Euclidean space and sometimes denotes a group law or action. Which is intended should be clear from the context.

2. The C^*-algebra of the complex "$ax+b$" group. The most obvious analogue of the group considered by Z'ep is the complex "$ax+b$" group, the complex Lie group of complex affine motions of the complex line. We realize this as the group G of matrices of the form

$$\begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix}$$

with $z \in \boldsymbol{C}^*$, $w \in \boldsymbol{C}$. An easy application of the "Mackey machine" (e.g., [3, Ch. I, § 10]) shows that G has (up to equivalence) the following irreducible unitary representations: one infinite-dimensional representation σ, and a family of one-dimensional representations parametrized by the dual of the group $\boldsymbol{C}^*$. Since $\boldsymbol{C}^* \simeq \boldsymbol{R} \times \boldsymbol{T}$, $(\boldsymbol{C}^*)^\wedge \simeq \boldsymbol{R} \times \boldsymbol{Z}$, and one sees that the group C^*-algebra of G satisfies an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow C^*(G) \longrightarrow C_\infty(\boldsymbol{R} \times \boldsymbol{Z}) \longrightarrow 0 ,$$

where $\mathscr{K}$ denotes the C^*-algebra of compact operators on an infinite-dimentional separable Hilbert space and, for a locally compact space Y, $C_\infty(Y)$ denotes the C^*-algebra of continuous complex-valued functions on Y vanishing at infinity. As in the case of the group considered by Z'ep, it is convenient to find not $C^*(G)$ but $C^*(G)^\sim$, the corresponding algebra with identity adjoined. Since the one-point compactification of $\boldsymbol{R} \times \boldsymbol{Z}$ is homeomorphic to the "Hawaiian necklace" space

$$X = \{z \in \boldsymbol{C}: |z - 2^{-n}| = 2^{-n} \quad \text{for some} \quad n = 1, 2, \cdots\} ,$$

we have an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow C^*(G)^\sim \longrightarrow C(X) \longrightarrow 0 . \tag{1}$$

An explicit calculation of functions of positive type associated with σ easily shows that σ weakly contains every one-dimensional representation of G. (Alternatively, one can note that σ, being quasi-equivalent to the regular representation of G, is faithful on $C^*(G)$ since G is amenable [11, Proposition 18.3.9].) Hence $C^*(G)^\sim$ can be realized as a subalgebra, containing 1 and the compact operators, of $\mathscr{L}(\mathscr{H})$, for some separable Hilbert space $\mathscr{H}$, and so (1) is an extension in the sense of Brown, Douglas, and Fillmore [8]. By [8] or [2], the equivalence class of this extension is characterized by an element of $\mathrm{Hom}\,(\pi^1(X), \boldsymbol{Z})$ given by the Fredholm index. As $\mathrm{Hom}\,(\pi^1(X), \boldsymbol{Z})$ is evidently the product of countably many copies of $\boldsymbol{Z}$, one for each generator of $\pi^1(X)$, the equivalence class of the extension (1) may be given by a sequence of integers, namely, the Fredholm indices of the images under σ of a sequence of elements of $C^*(G)^\sim$ mapping to generators of $\pi^1(X)$ in $C(X, \boldsymbol{T}) \subset C(X)$. (So far, everything is as for the group considered by Z'ep except for the number of generators of $\pi^1(X)$.) These integers thus determine the C^*-algebra $C^*(G)^\sim$ up to isomorphism. In fact, since the choice of orientation of the cohomotopy generators is arbitrary, only the absolute values of these integers are needed to determine this isomorphism class.

To perform the necessary calculations, we view G (as a manifold) as $\boldsymbol{R}_+^* \times \boldsymbol{T} \times \boldsymbol{C}$, where $\boldsymbol{R}_+^*$ is the multiplicative group of positive real numbers. The group operation is then $(r, t, w)\cdot(r', t', w') = (rr', tt', w + rtw')$, and left Haar measure on G is the same as the product of the Haar measures on $\boldsymbol{R}_+^*$, $\boldsymbol{T}$, and $\boldsymbol{C}$. We normalize these measures to be dx/x on $\boldsymbol{R}_+^*$, $d\theta/2\pi$ on $\boldsymbol{T}$, and usual Lebesgue measure on $\boldsymbol{C}$. The one-dimensional representations of G are $U_{n,\lambda}$, $n \in \boldsymbol{Z}$, $\lambda \in \boldsymbol{R}$, where $U_{n,\lambda}(r, t, w) = r^{i\lambda}t^n$. The infinite-dimensional representation σ may be realized on $L^2(\boldsymbol{C}^*)$, the action being given by $(\sigma(r, t, w)f)(z) = \exp(i \operatorname{Re} wz)f(rtz)$.

For $m \in \boldsymbol{Z}$, let $\varphi_m(r, t, w) = -(2/\pi)\chi_{[1,\infty)}(r)\chi_D(w)r^{-1}t^{-m}$, where D is the closed unit disk. Then $\varphi_m \in L^1(G)$, which we view as a subalgebra of $C^*(G)$. Let $\tilde{\varphi}_m = \varphi_m + 1 \in C^*(G)^\sim$. Then

$$U_{n,\lambda}(\tilde{\varphi}_m) = 1 - 2\int_T t^{n-m}dt \int_1^\infty r^{i\lambda}r^{-1}(dr/r)$$
$$= \begin{cases} 1 & \text{if} \quad n \neq m \,, \\ (i\lambda + 1)/(i\lambda - 1) & \text{if} \quad n = m \,. \end{cases}$$

Since $\lambda \mapsto (i\lambda + 1)/(i\lambda - 1)$ is a mapping of $\boldsymbol{R} \cup \{\infty\}$ onto the unit circle with winding number 1, the $(n, \lambda) \mapsto U_{n,\lambda}(\tilde{\varphi}_m)$ may be viewed as generators of $\pi^1(X)$ and it suffices to compute the Fredholm indices of the $\sigma(\tilde{\varphi}_m)$.

PROPOSITION 1. *With the above notation,* $\sigma(\tilde{\varphi}_m)$ *has Fredholm index* -1 *for all* m. *Thus the equivalence class of the extension* (1) *is given by the invariants* $\cdots, -1, -1, \cdots$. (Of course, we would get $+1$'s if we reversed the orientations of the generators of $\pi^1(X)$.) *These invariants also determine the isomorphism class of* $C^*(G)$.

Proof. Let $f \in L^2(\boldsymbol{C}^*)$, and write $f(rt) = g(r, t)$ $(r > 0, |t| = 1)$. Then

$$(\sigma(\varphi_m)g)(r', t')$$
$$= -(2/\pi)\iint_{w \in D}\int_T \int_1^\infty \exp(i \operatorname{Re} wr't')g(rr', tt')r^{-2}t^{-m}drdtdxdy \,,$$

where $w = x + iy$. Changing variables in the integrand, this becomes

$$-(2/\pi)(t')^m(r')^{-1}\int_T\int_{r'}^\infty g(r, t)r^{-2}drt^{-m}dt \iint_{|w| \leq r'} \exp(i \operatorname{Re} w)dxdy \,.$$

The integral over w can be computed in terms of Bessel functions by changing to polar coordinates and using standard formulas; thus

$$(\sigma(\varphi_m)g)(r', t') = -4J_1(r')(t')^m \int_T\int_{r'}^\infty g(r, t)r^{-2}drt^{-m}dt \,.$$

First, we calculate the kernel of $\sigma(\tilde{\varphi}_m)$. If $\sigma(\tilde{\varphi}_m)f = 0$, then

$$g(r', t') = 4(t')^m J_1(r')\int_T\int_{r'}^\infty g(r, t)r^{-2}drt^{-m}dt \,,$$

so that g is (equal a.e. to a function) of the form $g(r, t) = t^m h(r)$, where h satisfies the integral equation

$$h(r) = 4J_1(r)\int_r^\infty h(\rho)\rho^{-2}d\rho\ .$$

Hence $h(r)/J_1(r)$ is differentiable with derivative $-4h(r)r^{-2}$. If h is not identically zero, then near $r = 0$ (so that $J_1(r) \approx r/2$), we have

$$\left(\frac{d}{dr}\right)\left(\frac{h(r)}{r}\right) \approx -2h(r)r^{-2}\ ,$$

and $h(r)$ behaves like a multiple of r^{-1}, which is impossible since we must have $h \in L^2(\boldsymbol{R}_+^*, dr/r)$. So $\sigma(\widetilde{\varphi}_m)$ has trivial kernel.

One easily computes, using the Fubini theorem, that the adjoint of $\sigma(\varphi_m)$ is given by the integral operator on $L^2(\boldsymbol{C}^*)$ with

$$(\sigma(\varphi_m)^*g)(r, t) = -4t^m r^{-1}\int_T\int_0^r g(r', t')J_1(r')\left(\frac{dr'}{r'}\right)(t')^{-m}dt'\ .$$

As before, $\sigma(\widetilde{\varphi}_m)^*f = 0$ if and only if g is (equal a.e. to a function) of the form $g(r, t) = t^m k(r)$, where k satisfies the integral equation

$$rk(r) = 4\int_0^r k(r')J_1(r')\frac{dr'}{r'}\ .$$

This equation has a one-dimensional family of solutions in $L^2(\boldsymbol{R}_+, dr/r)$, as may be seen by solving the associated differential equation

$$\left(\frac{d}{dr}\right)(rk(r)) = 4k(r)\frac{J_1(r)}{r}\ .$$

Near $r = 0$, $J_1(r)/r \approx 1/2$, so that if k is a solution, $(d/dr)(rk(r)) \approx 2k(r)$ and $k(r) = O(r)$. However, $J_1(r)$ tends to 0 as $r \to \infty$, so that for any solution k, $k(r) = O(1/r)$ for r large. Thus the solutions of the differential equation are in L^2, and since they tend to 0 as $r \to 0$, they also satisfy the integral equation. This proves that $\sigma(\widetilde{\varphi}_m)$ has index -1, as asserted.

This completes the calculation of the isomorphism class of $C^*(G)^\sim$. Now for a general C^*-algebra A, knowing $A^\sim$ does not determine A up to isomorphism—for instance, there exist nonhomeomorphic locally compact spaces having homeomorphic one-point compactifications. But in our specific situation, $C^*(G)$ is clearly the inverse image in $C^*(G)^\sim$, under the map of (1), of the functions in $C(X)$ vanishing at the point at infinity adjoined to $\boldsymbol{R} \times \boldsymbol{Z}$, and thus $C^*(G)$ is determined when the equivalence class of (1) is known.

3. The C^*-algebras of "$ax + b$" groups over nonarchimedean fields. The method of the last section may also be applied to the group G of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

with $a \in K^*$, $b \in K$, K a nondiscrete totally disconnected locally compact field. For basic information about such fields, we refer the reader to [14, pp. 123–130] — the p-adic fields $\boldsymbol{Q}_p$, p a prime, are prototypical examples. We may write $K^* \simeq \boldsymbol{Z} \times H$, where H is the multiplicative group of elements of K of absolute value 1, a compact abelian group. Thus $(K^*)^\wedge \simeq \boldsymbol{T} \times \hat{H}$, where $\hat{H}$ is countably infinite and discrete. As before, we have an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow C^*(G) \longrightarrow C_\infty(\boldsymbol{T} \times \hat{H}) \longrightarrow 0$$

and an extension

$$0 \longrightarrow \mathscr{K} \longrightarrow C^*(G)^\sim \longrightarrow C(X) \longrightarrow 0\,, \tag{1A}$$

except that now X, the one-point compactification of $\boldsymbol{T} \times \hat{H}$, is homeomorphic to $\{z \in \boldsymbol{C}: z = 0 \text{ or } |z| = 2^{-n} \text{ for some } n = 1, 2, \cdots\}$.

Let $\mathscr{O}$ be the ring of integers in K (a discrete valuation ring) and choose $\pi \in \mathscr{O}$ such that (π) is the valuation ideal in $\mathscr{O}$. We fix an additive character ψ of K such that ψ is trivial on $\mathscr{O}$ but not on $(\pi)^{-1}$; then the map $a \mapsto \psi_a$, where $\psi_a(b) = \psi(ab)$, is a topological isomorphism of K onto $\hat{K}$. Identifying G as a set with $\boldsymbol{Z} \times H \times K$, Haar measure on G is the product of counting measure on $\boldsymbol{Z}$ and Haar measures on H and K. We normalize these measures so that $\int_H dh = \int_{\mathscr{O}} dk = 1$. As in § 1, G has one infinite-dimensional irreducible representation σ on $L^2(K^*)$, given by $(\sigma(n, h, b)f)(c) = \psi(bc)f(\pi^n hc)$, where $c \in K^*$, $b \in K$, $n \in \boldsymbol{Z}$, $h \in H$. The one-dimensional representations of G are of the form $U_{t,\lambda}(n, h, b) = t^n\lambda(h)$, where $n \in \boldsymbol{Z}$, $h \in H$, $b \in K$, $t \in \boldsymbol{T}$, and $\lambda \in \hat{H}$.

For $\gamma \in \hat{H}$, let $\varphi_\gamma(n, h, b) = \bar{\gamma}(h)\chi_{\mathscr{O}}(b)(\delta_1(n) - \delta_0(n))$, where $\delta_m = \chi_{\{m\}}$. Then $\varphi_\gamma \in L^1(G)$, which we view as a subalgebra of $C^*(G)$, and as before we define $\tilde{\varphi}_\gamma = 1 + \varphi_\gamma \in C^*(G)^\sim$. For $t \in \boldsymbol{T}$, $\lambda \in \hat{H}$, we have

$$\begin{aligned} U_{t,\lambda}(\varphi_\gamma) &= \sum_n \int_H \bar{\gamma}(h)\lambda(h)dh \int_K \chi_{\mathscr{O}}(b)db(\delta_1(n) - \delta_0(n))t^n \\ &= \begin{cases} 0 & \text{if} \quad \gamma \neq \lambda \\ t - 1 & \text{if} \quad \gamma = \lambda \end{cases} \end{aligned}$$

by orthogonality of characters of H and normalization of Haar measures. Hence $U_{t,\lambda}(\tilde{\varphi}_\gamma) = 1$ if $\gamma \neq \lambda$, t if $\gamma = \lambda$, and the $(t, \lambda) \mapsto U_{t,\lambda}(\tilde{\varphi}_\gamma)$ represent generators of $\pi^1(X)$. As in the last section, it is enough to compute the Fredholm indices of the $\sigma(\tilde{\varphi}_\gamma)$.

PROPOSITION 2. *$\sigma(\tilde{\varphi}_\gamma)$ has Fredholm index 1 for all γ. Thus* *the equivalence class of the extension* (1A) *is given by the invariants* $\cdots, 1, 1, \cdots$, *and these also determine the isomorphism class of* $C^*(G)$.

Proof. First, note that by choice of ψ and our normalization of Haar measures, for $h \in H$, $\int_{\mathcal{O}} \psi(\pi^n hb)db = 1$ if $n \geqq 0$, 0 otherwise. Identify K^* with $\boldsymbol{Z} \times H$ and let $f \in L^2(K^*)$. Then,

$$(\sigma(\varphi_\gamma)f)(n, x) = \sum_m \int_H \int_K \bar{\gamma}(h)\chi_{\mathcal{O}}(b)(\delta_1(m) - \delta_0(m))\psi(b\pi^n x)f(n + m, hx)dbdh$$
$$= \chi_{[0,\infty)}(n)\gamma(x)\left[\int_H \bar{\gamma}(h)f(n + 1, h)dh - \int_H \bar{\gamma}(h)f(n, h)dh\right].$$

Suppose $f \in \ker(\sigma(\tilde{\varphi}_\gamma))$. Then $f(n, x) = -(\sigma(\varphi_\gamma)f)(n, x)$ a.e., and f is (equal a.e. to a function) of the form $f(n, x) = \gamma(x)g(n)$, where g satisfies the difference equation

$$g(n) = -\chi_{[0,\infty)}(n)(g(n + 1) - g(n)) .$$

This implies that $g(n) = 0$ for $n < 0$, and that for $n \geqq 0$, $g(n + 1) = 0$. However, $g(0)$ is arbitrary, so $\ker(\sigma(\varphi_\gamma))$ is one-dimensional, spanned by the function $f(n, x) = \gamma(x)\delta_0(n)$.

The adjoint of $\sigma(\varphi_\gamma)$ is computed by the Fubini theorem to be given by the operator on $L^2(K^*)$ with

$$(\sigma(\varphi_\gamma)^*f)(n, h) = \chi_{[1,\infty)}(n)\gamma(h)\int_H f(n - 1, x)\bar{\gamma}(x)dx$$
$$- \chi_{[0,\infty)}(n)\gamma(h)\int_H f(n, x)\bar{\gamma}(x)dx .$$

If we suppose that $f \in \ker(\sigma(\tilde{\varphi}_\gamma)^*)$, then $f(n, h) = -(\sigma(\varphi_\gamma)^*f)(n, h)$ a.e., and f is (equal a.e. to a function) of the form $f(n, h) = \gamma(h)p(n)$, where p satisfies the difference equation

$$p(n) = \chi_{[0,\infty)}(n)p(n) - \chi_{[1,\infty]}(n)p(n - 1) .$$

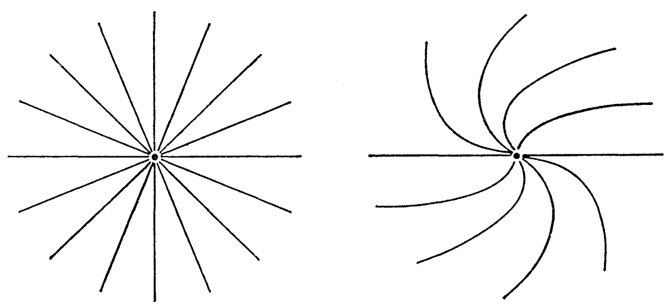
This implies that $p(n) = 0$ for $n < 0$ and that $\chi_{[1,\infty)}(n)p(n - 1) = 0$ for $n \geqq 0$, which forces $p = 0$. So $\sigma(\tilde{\varphi}_\gamma)^*$ has trivial kernel and $\sigma(\tilde{\varphi}_\gamma)$ has index 1. Exactly as in § 2, this information determines the isomorphism class of $C^*(G)$.

4. The C^*-algebras of a family of solvable Lie groups. In this section we consider the C^*-algebra of a simply connected solvable group G which is a semidirect product of $\boldsymbol{R}$ and $\boldsymbol{R}^m$ (m any positive integer), where $\boldsymbol{R}$ acts on $\boldsymbol{R}^m$ with m roots $\alpha_1, \cdots, \alpha_m$, all of which have nonzero real parts of the same sign. When $m = 1$, G is the "proper $ax + b$ group." In general, we may identify G as a manifold with $\boldsymbol{R}^{m+1}$, and left Haar measure on G is Lebesgue measure. When the action of $\boldsymbol{R}$ on $\boldsymbol{R}^m$ is diagonal, the group operation is given by the formula

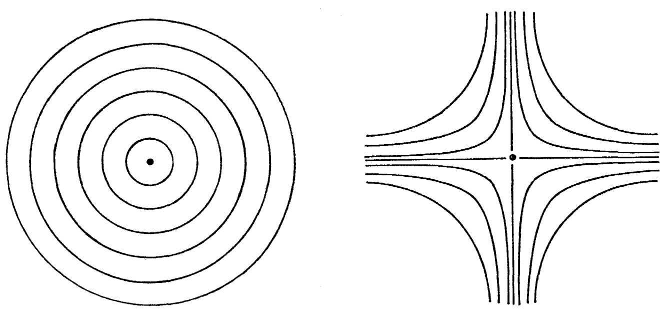
$$(x_1, x_2, \cdots, x_{m+1})(y_1, y_2, \cdots, y_{m+1}) \\ = (x_1 + y_1 \exp(\alpha_1 x_{m+1}), \cdots, x_m + y_m \exp(\alpha_m x_{m+1}), x_{m+1} + y_{m+1}) ,$$

where (with slight abuse of notation) $\alpha_1, \cdots, \alpha_m$ are now positive real numbers.

Since G is a semidirect product group with abelian normal subgroup $\boldsymbol{R}^m$, the C^*-algebra of G is the same as that of the topological transformation group $(\boldsymbol{R}, \boldsymbol{R}^m)$ [12, §3], where the group $\boldsymbol{R}$ acts on the space $\boldsymbol{R}^m \simeq (\boldsymbol{R}^m)^\wedge$ by the action dual to that given by the roots α_i. Since the α's all have real parts of the same sign, the orbits of $\boldsymbol{R}$ on $\boldsymbol{R}^m$ are just $\{0\}$ (a one-point orbit) and curves (on which $\boldsymbol{R}$ acts freely) which are topologically equivalent to rays emanating from 0. (See Figs. 1a and 1b.) It is clear that the topological conjugacy class of the transformation group does not depend on the α's, hence all groups of the same dimension in the class we are considering have isomorphic group C^*-algebras. We therefore may (and do) assume that the α's are all real-valued and equal, so that multiplica-



1a 1b



1c 1d

FIGURE 1. Orbit structures of $\boldsymbol{R}$ acting on $\boldsymbol{R}^2$ for various semidirect products.

tion in G is given by the formula in the last paragraph with $\alpha_1 = \cdots = \alpha_m = 1$. G has one-dimensional representations U_λ of the form $(x_1, x_2, \cdots, x_{m+1}) \mapsto \exp(i\lambda x_{m+1})$, $\lambda \in \boldsymbol{R}$, and infinite-dimensional irreducible representations parameterized by $(\boldsymbol{R}^m - \{0\})/\boldsymbol{R} \simeq S^{m-1}$. (In the case $m = 1$, S^{m-1} is a two-point set; as is well known, the proper $ax + b$ group has exactly two inequivalent infinite-dimensional irreducible representations.)

Next, we claim that the group C^*-algebra of G satisfies an exact sequence

$$(2) \qquad 0 \longrightarrow C(S^{m-1}, \mathscr{K}) \longrightarrow C^*(G) \longrightarrow C_\infty(\boldsymbol{R}) \longrightarrow 0 .$$

Here $C(S^{m-1}, \mathscr{K})$ denotes the C^*-algebra of norm-continuous functions from S^{m-1} into $\mathscr{K}$. To check this, we note that $C_\infty(\boldsymbol{R})$ is certainly the quotient of $C^*(G)$ by some ideal I. I is liminary, since it is clear that no infinite-dimensional irreducible representation of G is weakly contained in another such representation associated with a different orbit in $\boldsymbol{R}^m$. We need only confirm that $I \cong C(S^{m-1}, \mathscr{K})$.

For this we need formulas for the infinite-dimensional irreducible representations of G. Let $\zeta \in S^{m-1}$, and identify ζ as usual with a unit vector $(\zeta_1, \cdots, \zeta_m)$ in $\boldsymbol{R}^m$. Then ζ defines a character $x \mapsto \exp(i\zeta \cdot x)$ of $\boldsymbol{R}^m$ which when induced up to G is an irreducible representation π_ζ of G. This representation may be realized on $\mathscr{H} = L^2(\boldsymbol{R})$, the action being

$$(\pi_\zeta(x_1, x_2, \cdots, x_{m+1})f)(s) = \exp(i\zeta \cdot (e^{-s}x_1, \cdots, e^{-s}x_m))f(s - x_{m+1}) ,$$

for $f \in \mathscr{H}$. For $g \in L^1(G)$, we view g as an L^1 function on the vector space $\boldsymbol{R}^{m+1}$ and compute its Fourier transform $\hat{g}$ (in all $m + 1$ variables) as usual. We also let $\tilde{g}$ denote the partial Fourier transform of g in the first m variables. Then we have (with $f \in \mathscr{H}$)

$$U_\lambda(g) = \hat{g}(0, \cdots, 0, \lambda)$$

and

$$(3) \qquad (\pi_\zeta(g)f)(s) = \int f(s - x)\tilde{g}(e^{-s}\zeta_1, \cdots, e^{-s}\zeta_m, x)dx .$$

PROPOSITION 3. *With notation as above, the map $a \mapsto \mu(a)$, where $\mu(a)(\zeta) = \pi_\zeta(a)$, is an embedding of $C^*(G)$ into $C(S^{m-1}, \mathscr{L}(\mathscr{H}))$.*

Proof. The first step is to show that for $a \in C^*(G)$, $\mu(a)$ is a norm-countinuous function from S^{m-1} to $\mathscr{L}(\mathscr{H})$, i.e., that $\mu(C^*(G)) \subseteq C(S^{m-1}, \mathscr{L}(\mathscr{H}))$. Let $\mathscr{D}(G)$ be the convolution algebra of C^∞ functions on G with compact support. Since we may view $\mathscr{D}(G)$ as a dense subalgebra of $C^*(G)$, it is enough to prove continuity of

$\zeta \mapsto \pi_\zeta(g)$ for $g \in \mathscr{D}(G)$. Let $f \in \mathscr{H}$, $g \in \mathscr{D}(G)$, and $\zeta, \xi \in S^{m-1}$. We have, by (3),

$$\begin{aligned}
\|(\pi_\zeta(g) - \pi_\xi(g))f\|_2^2 &= \int |(\pi_\zeta(g)f)(s) - (\pi_\xi(g)f)(s)|^2 \, ds \\
&= \int \Big| \int f(x)[\tilde{g}(e^{-s}\zeta_1, \cdots, e^{-s}\zeta_m, s - x) \\
&\qquad - \tilde{g}(e^{-s}\xi_1, \cdots, e^{-s}\xi_m, s - x)]dx \Big|^2 ds \\
&\leqq \int \|f\|_2^2 \int |\tilde{g}(e^{-s}\zeta, s - x) - \tilde{g}(e^{-s}\xi, s - x)|^2 \, dxds \\
&\qquad\qquad\qquad \text{(by Hölder)} \\
&= \|f\|_2^2 \iint |\tilde{g}(e^{-s}\zeta, x) - \tilde{g}(e^{-s}\xi, x)|^2 \, dxds \\
&= \|f\|_2^2 \int_0^\infty \int_{-\infty}^\infty |\tilde{g}(y\zeta, x) - \tilde{g}(y\xi, x)|^2 \, dxdy/y \; ,
\end{aligned}$$

where for convenience we write $\tilde{g}$ as a function on $\boldsymbol{R}^m \times \boldsymbol{R}$. Now $\tilde{g}$ is of compact support in its second argument, so we may take the integral in x over an interval $[-R, R]$ $(R > 0)$. Also $\tilde{g}$ is real analytic in its first argument, so we can choose $C > 0$ such that

$$|\tilde{g}(\eta, x) - \tilde{g}(\theta, x)| \leqq C\|\eta - \theta\| \quad \text{for all} \quad \eta, \theta \in \boldsymbol{R}^m, x \in \boldsymbol{R} \; .$$

Fixing $\varepsilon > 0$, we can take R large enough so that

$$\int_R^\infty \int_{-R}^R |\tilde{g}(y\eta, x)|^2 \, dxdy/y < \varepsilon/4 \quad \text{for all} \quad \eta \in S^{m-1} \; ,$$

and then

$$\|\pi_\zeta(g) - \pi_\xi(g)\|^2 \leqq \varepsilon + \int_0^R \int_{-R}^R C^2 y \|\zeta - \xi\|^2 \, dxdy = \varepsilon + C^2R^3\|\zeta - \xi\|^2 \; .$$

Thus $\|\pi_\zeta(g) - \pi_\xi(g)\| \to 0$ as $\|\zeta - \xi\| \to 0$, and $\zeta \mapsto \pi_\zeta(g)$ is norm-continuous.

It remains only to show that μ is injective. But this can be proved in the same way we showed in § 1 that (1) is an extension in the sense of [8], for it is easy to compute from (3) that U_λ is weakly contained in π_ζ, for all $\lambda \in \boldsymbol{R}$ and $\zeta \in S^{m-1}$.

COROLLARY. *$I \simeq C(S^{m-1}, \mathscr{K})$, and* (as claimed earlier) (2) *is an exact sequence.*

Proof. From the proposition, $\mu(I) \subseteqq C(S^{m-1}, \mathscr{L}(\mathscr{H}))$, and since I is liminary, it follows that $\mu(I) \subseteqq C(S^{m-1}, \mathscr{K})$, where $\mathscr{K}$ is identified with the algebra of compact operators on $\mathscr{H}$. By Proposition 4.2.5 and Lemma 10.5.3 of [11], which together amount to a sort of Stone-Weierstrass theorem for certain liminary C^*-algebras,

$\mu(I) = C(S^{m-1}, \mathscr{K})$. And again by the proposition, μ is injective. Hence, μ implements an isomorphism of I onto $C(S^{m-1}, \mathscr{K})$.

To determine the structure of $C^*(G)$, it is again convenient to adjoin an identity element and consider in place of (2) the exact sequence

$$(4) \qquad 0 \longrightarrow C(S^{m-1}, \mathscr{K}) \longrightarrow C^*(G)^{\sim} \longrightarrow C(S^1) \longrightarrow 0 ,$$

valid since S^1 is the one-point compactification of $\boldsymbol{R}$. This time the theory of Brown, Douglas and Fillmore does not apply directly. However, we may still view (4) as an extension of C^*-algebras and consider its equivalence class in the sense of [9]. By [9, Theorem 4.3], this is uniquely determined by the associated map $\gamma: C(S^1) \rightarrow O(C(S^{m-1}, \mathscr{K}))$, where $O(A) = M(A)/A$ denotes the outer multiplier algebra of a C^*-algebra A. ($M(A)$ is the multiplier or double centralizer algebra of A.) By [1, Corollaries 3.4 and 3.5], $M(C(S^{m-1}, \mathscr{K}))$ is just the algebra $C_{s-*}(S^{m-1}, \mathscr{L}(\mathscr{H}))$ of functions $S^{m-1} \rightarrow \mathscr{L}(\mathscr{H})$ ($\mathscr{H}$ a separable Hilbert space, which we take equal to $L^2(\boldsymbol{R})$) continuous for the strong-*topology.

PROPOSITION 4. (a) *The natural map* μ: $C^*(G)^{\sim} \rightarrow M(C(S^{m-1}, \mathscr{K})) = C_{s-*}(S^{m-1}, \mathscr{L}(\mathscr{H}))$ *actually takes its values in the subalgebra* $C(S^{m-1}, \mathscr{L}(\mathscr{H}))$ *of norm-continuous functions, so that* γ *may be viewed as a map*

$$C(S^1) \rightarrow C(S^{m-1}, \mathscr{L}(\mathscr{K}))/C(S^{m-1}, \mathscr{K}) \simeq C(S^{m-1}, \mathscr{L}(\mathscr{H})/\mathscr{K}) .$$

(b) *Moreover, the image of* γ *consists of* constant *functions* $S^{m-1} \rightarrow \mathscr{L}(\mathscr{H})/\mathscr{K}$, *so that* γ *may be viewed as a map* $C(S^1) \rightarrow \mathscr{L}(\mathscr{H})/\mathscr{K}$.

Proof. Part (a) is essentially a restatement of Proposition 3, since it is clear that the restriction of the map μ to $C^*(G)$ is just the map of that proposition. To prove (b), we note first that $C(S^1)$ is generated as a C^*-algebra by the function z (identifying S^1 with the unit circle in the complex plane). So it is enough to find an element $a \in C^*(G)^{\sim}$ such that the image of a modulo $I = C(S^{m-1}, \mathscr{K})$ is the function in z, and such that $\mu(a)$ is a constant function. As in §§ 2 and 3 above, we construct a as $1 + g$, where $g \in L^1(G)$.

Let h be a function in $L^1(\boldsymbol{R}^m)$ whose Fourier transform $\hat{h}$ has the properties that $\hat{h}(x)$ depends only on $\|x\|$ and $\hat{h}(0) = 1$ (for instance, a Gaussian function will do), and let

$$g(x_1, \cdots, x_{m+1}) = -2h(x_1, \cdots, x_m)\chi_{[0,\infty)}(x_{m+1})e^{-x_{m+1}} .$$

Then $g \in L^1(G)$ and

$$\begin{aligned} U_\lambda(g) &= -2\int_0^\infty \exp\left[(i\lambda - 1)x\right]dx\hat{h}(0, \cdots, 0) \\ &= \frac{2}{(i\lambda - 1)}, \end{aligned}$$

so that if $a = 1 + g \in C^*(G)^\sim$, $U_\lambda(a) = (i\lambda + 1)/(i\lambda - 1)$, which as a function of λ maps $\boldsymbol{R} \cup \{\infty\}$ onto $\boldsymbol{T}$ with winding number 1. Hence (using suitable coordinates on S^1) a is as desired, and we need only calculate $\pi_\zeta(a)$ for varying $\zeta \in S^{m-1}$. For $f \in L^2(\boldsymbol{R})$, we have, by (3),

$$\begin{aligned} (\pi_\zeta(g)f)(s) &= \int f(s - x)\tilde{g}(e^{-s}\zeta_1, \cdots, e^{-s}\zeta_m, x)dx \\ &= -2\hat{h}(e^{-s}\zeta)\int_0^\infty f(s - x)e^{-x}dx, \end{aligned} \tag{5}$$

which, by assumption on h, is independent of the unit vector ζ. This proves (*b*).

Now we can determine the isomorphism class of $C^*(G)^\sim$ exactly. This is a consequence of the fact that equivalence classes of extensions of C^*-algebras

$$0 \longrightarrow C(S^{m-1}, \mathscr{K}) \longrightarrow A \longrightarrow C(S^1) \longrightarrow 0 \tag{6}$$

for which the associated maps γ take values in the constant functions $S^{m-1} \rightarrow \mathscr{L}(\mathscr{H})/\mathscr{K}$ are in natural bijective correspondence with $\mathrm{Hom}\,(C(S^1), \mathscr{L}(\mathscr{H})/\mathscr{K}) = \mathrm{Hom}\,(C(S^1), O(\mathscr{K}))$, and hence with equivalence classes (in the sense of [9]) of extensions

$$0 \longrightarrow \mathscr{K} \longrightarrow B \longrightarrow C(S^1) \longrightarrow 0\,. \tag{7}$$

When γ is injective, the short exact sequence (7) is also an extension in the sense of [8], and the equivalence class of (7) in the sense of Brown, Douglas and Fillmore (which is essentially its weak equivalence class in the sense of Busby) is given by the Fredholm index as described previously. From this it is easy to see that the index invariant fixes the weak equivalence class of the extension (6), and in particular determines the isomorphism class of the C^*-algebra A. Since $\pi^1(S^1) \simeq \boldsymbol{Z}$, this invariant may be given by a single integer. As before, the sign of this integer, which depends on the choice of orientation of a generator of $\pi^1(S^1)$, does not affect the isomorphism class of A.

PROPOSITION 5. *With notation as above, $\pi_\zeta(a)$ has Fredholm index -1 for all $\zeta \in S^{m-1}$, and this index invariant determines the isomorphism classes of $C^*(G)^\sim$ and $C^*(G)$.*

Proof. By Proposition 3, we know that γ is injective and so the index theory applies. Choose the function h of the proof of Proposition 4 so that $\hat{h} \in \mathscr{D}(\boldsymbol{R}^m)$ and $\hat{h}(x)$ is equal to $\chi_{[0,1]}(\|x\|)$ except on $\{x \in \boldsymbol{R}^m : 1/2 \leqq \|x\| \leqq 1\}$. Then $\pi_\zeta(a)$ is given by (5), and it is easy to see that $\pi_\zeta(a)$ is a compact perturbation of the operator $T \in \mathscr{L}(\mathscr{H})$ with

$$(Tf)(s) = f(s) - 2\chi_{[0,\infty)}(s) \int_0^\infty f(s-x)e^{-x}dx \ .$$

(The kernel defining the operator $T - \pi_\zeta(a)$ is square-integrable; hence $T - \pi_\zeta(a)$ is Hilbert-Schmidt.) Therefore, it is enough to compute the Fredholm index of T.

First, suppose $f \in \ker T$. Then

$$f(s) = 2\chi_{[0,\infty)}(s) \int_0^\infty f(s-x)e^{-x}dx \quad \text{a.e.},$$

so f is essentially supported on $[0, \infty)$ and is equal a.e. there to a solution of the integral equation

$$f(s) = 2\int_0^\infty f(s-x)e^{-x}dx = 2e^{-s}\int_0^s f(x)e^x dx \ .$$

Hence, $(d/ds)(e^s f(s)) = 2(e^s f(s))$, $e^s f(s) = Ce^{2s}$ (C a constant), and $f(s) = Ce^s$, $s > 0$. Since $f \in L^2(\boldsymbol{R})$, this is impossible unless $C = 0$. So T has trivial kernel.

An easy computation with the Fubini theorem shows that T^* is given by the formula

$$(T^*f)(s) = f(s) - 2\int_{\max(0,-s)}^\infty f(s+x)e^{-x}dx \ .$$

So $f \in \ker T^*$ if and only if f agrees a.e. with a solution of the integral equations

$$f(s) = \begin{cases} 2\displaystyle\int_{-s}^\infty f(s+x)e^{-x}dx \ , & s < 0 \ , \\ 2\displaystyle\int_0^\infty f(s+x)e^{-x}dx \ , & s > 0 \ . \end{cases}$$

It is easy to see that these equations have a one-dimensional family of solutions, spanned by the function $s \mapsto e^{-|s|}$, which is clearly in $L^2(\boldsymbol{R})$. So $\ker T^*$ is one-dimensional and T (hence also $\pi_\zeta(a)$) has Fredholm index -1.

This completes the calculation of the isomorphism class of $C^*(G)^\sim$. $C^*(G)$ is then the inverse image in $C^*(G)^\sim$ of the functions in $C(S^1)$ vanishing at the point at infinity ∞ in S^1. Alternatively, $C^*(G)$ is determined up to isomorphism by the weak equivalence class of the extension (2) and hence by a certain conjugacy class of the associated map $C_\infty(\boldsymbol{R}) \to O(C(S^{m-1}, \mathscr{K}))$. This map is just the restriction to $C_\infty(\boldsymbol{R}) = \{f \in C(S^1): f(\infty) = 0\}$ of the map γ whose conjugacy class was just determined. So the isomorphism class of $C^*(G)$ is uniquely determined.

5. The C^*-algebras of the 3-dimensional solvable Lie groups.

In this section, we apply some of the results and methods of § 4 to the problem of determining the C^*-algebras of all connected 3-dimensional solvable Lie groups. Although we do not completely answer this problem, we obtain extensive partial results.

The 3-dimensional solvable Lie algebras over $\boldsymbol{R}$ have been tabulated (see, for instance, [4, p. 182]), and they are all isomorphic to one of the following (no two of which are isomorphic): $(\mathfrak{g}_1)^3$, $\mathfrak{g}_{3,1}$, $\mathfrak{g}_1 \times \mathfrak{g}_2$, $\mathfrak{g}_{3,2}(\alpha)$ with $|\alpha| \geqq 1$, $\mathfrak{g}_{3,3}$, and $\mathfrak{g}_{3,4}(\alpha)$, $\alpha \geqq 0$. Here $\mathfrak{g}_1$ is the one-dimensional abelian Lie algebra, $\mathfrak{g}_2$ is the Lie algebra of the "$ax + b$" group, $\mathfrak{g}_{3,1}$ is the Heisenberg Lie algebra, and the remaining algebras have bases e_1, e_2, e_3 satisfying the relations

$$\mathfrak{g}_{3,3}: [e_1, e_2] = e_2 + e_3\,, \quad [e_1, e_3] = e_3$$
$$\mathfrak{g}_{3,2}(\alpha): [e_1, e_2] = e_2\,, \quad [e_1, e_3] = \alpha e_3$$
$$\mathfrak{g}_{3,4}(\alpha): [e_1, e_2] = \alpha e_2 - e_3\,, \quad [e_1, e_3] = e_2 + \alpha e_3\,.$$

Let us denote the corresponding simply connected groups by $G_1 \times G_1 \times G_1 = \boldsymbol{R}^3$, $G_{3,1}$, $G_1 \times G_2$, etc. The C^*-algebras of all abelian Lie groups are known, and by § 4 above, the C^*-algebra of G_2 is known. Since $C^*(\boldsymbol{R} \times G_2) \simeq C^*(\boldsymbol{R}) \otimes C^*(G_2)$ and $C^*(\boldsymbol{T} \times G_2) \cong C^*(\boldsymbol{T}) \otimes C^*(G_2)$ by Proposition 7 of [16], the only cases left to consider are groups covered by $G_{3,1}$, $G_{3,2}(\alpha)$, $G_{3,3}$, and $G_{3,4}(\alpha)$, for varying α. $G_{3,1}$ is the Heisenberg group — as mentioned previously, the "determination" of $C^*(G_{3,1})$ remains an open problem. However, we should mention that if G is a nonsimply connected Lie group with $G_{3,1}$ as its universal covering group, then $C^*(G)$ is easily characterized. In this case, the center of G is isomorphic to $\boldsymbol{T}$ and, as a topological space, $G^\wedge$ is the disjoint union of $\boldsymbol{R}^2$ and $\boldsymbol{Z} - \{0\}$ (a Hausdorff space). It is easily checked that $C^*(G)$ is a C^*-algebra with continuous trace, and since $H^3(G^\wedge, \boldsymbol{Z}) = 0$, $C^*(G)$ is the C^*-algebra defined by a continuous field of Hilbert spaces (one-dimensional ones over $\boldsymbol{R}^2$, infinite-dimensional separable ones over $\boldsymbol{Z} - \{0\}$) over $\hat{G}$ [11, Théorème 10.9.3]. This field is obviously trivial over $\boldsymbol{R}^2$ and over the discrete space $\boldsymbol{Z} - \{0\}$, so that $C^*(G) \simeq C_\infty(\boldsymbol{R}^2) \times C_\infty(\boldsymbol{Z} - \{0\}, \mathscr{K})$.

Of the remaining simply connected groups, only one has nontrivial center (and so covers nonsimply connected groups), namely, $G_{3,4}(0)$. This group, better recognized as $E_2^\sim$, the universal covering group of the group E_2 of Euclidean motions of the plane, is also the only nonnilpotent 3-dimensional simply connected group of type R (and thus with a liminary C^*-algebra—see [3, Chapter V]). We can characterize $C^*(E_2^\sim)$ exactly, using results of Delaroche [10].

First, we determine the irreducible unitary representations of $G = E_2^\sim$ by the Mackey method. G contains the vector group $\boldsymbol{C}$ as a normal subgroup; it is the semidirect product of $\boldsymbol{C}$ and $\boldsymbol{R}$, where $\boldsymbol{R}$ acts on $\boldsymbol{C}$ by the formula $s \cdot z = e(s)z$, where $e(s) = \exp(2\pi i s)$. The integers $\boldsymbol{Z}$ in $\boldsymbol{R}$ are central in G. The orbits of $\boldsymbol{R}$ on $\boldsymbol{C}^\wedge \simeq \boldsymbol{C}$ are $\{0\}$ (a one-point orbit) and a family of concentric circles. (See Fig. 1c.) Over the one-point orbit, G has a family of one-dimensional representations parameterized by $\boldsymbol{R}^\wedge \simeq \boldsymbol{R}$; over each circular orbit, G has infinite-dimensional representations parameterized by $\boldsymbol{Z}^\wedge \simeq \boldsymbol{T}$, since $\boldsymbol{Z}$ is the stability group in $\boldsymbol{R}$ of the orbit. Thus $G^\wedge$ is the disjoint union of $X = (0, \infty) \times \boldsymbol{T}$ and $\boldsymbol{R}$. One may check that the topology of $G^\wedge$ is given as follows: let X_1 be the compactification $[0, \infty] \times \boldsymbol{T}$ of X and define a map f from $X_1 - X$ to $\mathscr{F}(\boldsymbol{R})$, the space of closed subsets of $\boldsymbol{R}$, by $f(\infty, t) = \varnothing$, $f(0, t) = e^{-1}(t)$ for $t \in \boldsymbol{T}$. Then the topology of $G^\wedge$ is defined by f as in [10, Proposition II. 8]. The essential feature of this topology is that X and $\boldsymbol{R}$ have their usual topologies and that if ψ is a character of $\boldsymbol{Z}$ and $\{\pi_n\}$ is a sequence of infinite-dimensional irreducible representations of G restricting to multiples of ψ on $\boldsymbol{Z}$ and corresponding to orbits whose diameters tend to 0, then $\{\pi_n\}$ converges to every one-dimensional representation of G which restricts to ψ on $\boldsymbol{Z}$.

Next, we observe that $C^*(G)$ satisfies an exact sequence

$$(8) \qquad 0 \longrightarrow C_\infty(X, \mathscr{K}) \longrightarrow C^*(G) \longrightarrow C_\infty(\boldsymbol{R}) \longrightarrow 0 .$$

Again, we check this by noting that $C_\infty(\boldsymbol{R})$ is certainly the quotient of $C^*(G)$ by some ideal I, where I is liminary with $I^\wedge = X$. By an explicit calculation very similar to that of § 4 above, one can either show directly that $I \simeq C_\infty(X, \mathscr{K})$ or else show that I has continuous trace and apply [11, Corollaire 10.9.6].

Now, by [10, VI. 1.5], the extension (8) is "encadré," and applying [10, VI. 3.6 and VI. 3.7], the semi-equivalence class of the extension, and hence the isomorphism class of $C^*(G)$, is determined uniquely by the behavior of the trace function. The only invariants needed, other than the function f defined above, are "multiplicities" $m_x(s)$, $x \in X_1 - X$, $s \in f(x)$, such that if $\{x_n\}$ is a sequence in X tending to x in X_1, and if $c \in C^*(G)^+$, then

$$\operatorname{Tr} x_n(c) \longrightarrow \sum_{s \in f(x)} m_x(s) \operatorname{Tr} s(c) .$$

(Here we identify elements of X and $\boldsymbol{R}$ with the corresponding representations of $C^*(G)$.) But in this case, it is easy to see that the multiplicities (which must be positive integers) are all ones. (This is a consequence of the fact that if ψ is a character of $\boldsymbol{Z}$, then each character of $\boldsymbol{R}$ restricting to ψ appears with multiplicity 1 in the decomposition of the representation of $\boldsymbol{R}$ induced by ψ.) One might note that the extension (8) is "θ-split" in a sense generalizing [10, VII. 2.1], but cannot be split in the strict sense (of [9]). To see this, first define a map $g: \{0, \infty\} \to \mathscr{F}(\boldsymbol{Z})$ by $g(\infty) = \emptyset$, $g(0) = \boldsymbol{Z}$. If (8) were a split extension, then so would be the corresponding extension for $C^*(E_2)$. By [10, IV. 1.6], this would imply that g could be extended to a continuous map $[0, \infty] \to \mathscr{F}(\boldsymbol{Z})$. But this is impossible, since $[0, \infty]$ is connected and $\boldsymbol{Z}$ is discrete.

Once $C^*(E_2^\sim)$ is known, it is easy to determine $C^*(H)$ for any nonsimply connected group with universal covering group $E_2^\sim$, E_2 for instance. Such a group H can have any finite cyclic group as its center, say, $Z_n = \boldsymbol{Z}/n\boldsymbol{Z}$. Then $H^\wedge$ consists of $e^{-1}(Z_n) \cup ((0, \infty) \times Z_n)$ with the relative topology from $G^\wedge$, where Z_n is identified with a subgroup of $\boldsymbol{T}$. $C^*(H)$ is described as an "encadré" extension as before.

This analysis leaves only the groups $G_{3,2}(\alpha)$ ($|\alpha| > 1$), $G_{3,3}$, and $G_{3,4}(\alpha)$ ($\alpha > 0$). The cases of the $G_{3,2}(\alpha)$ ($\alpha > 0$), the $G_{3,4}(\alpha)$, and $G_{3,3}$ were dealt with in § 4 above, and all these groups were shown to have the same C^*-algebra (up to isomorphism), which we determined explicitly. (Recall that this C^*-algebra arises from various transformation group actions of $\boldsymbol{R}$ on $\boldsymbol{R}^2 \simeq (\boldsymbol{R}^2)^\wedge$. Figures 1a and 1b illustrate the orbits of the actions corresponding to the groups $G_{3,2}(1)$ and $G_{3,3}$.) To conclude this section, we examine the C^*-algebras of the $G_{3,2}(\alpha)$ with $\alpha < 0$ and note that they are all mutually isomorphic. This has one interesting consequence. A connected Lie group is unimodular if and only if for all x in its Lie algebra, $\mathrm{Tr}\,(ad\,x) = 0$. Thus the group $G_{3,2}(\alpha)$ is unimodular if and only if $\alpha = -1$. So we conclude that it is impossible to tell whether or not a group is unimodular merely by looking at its group C^*-algebra. For solvable Lie groups, $C^*(G)$ is a coarse invariant, reflecting the general root structure but not the specific values of the roots.

Of the 3-dimensional solvable groups, the $G_\alpha = G_{3,2}(-\alpha)$, $\alpha > 0$, appear to have the most complicated C^*-algebras. The duals of these groups are computed as before: G_α is the semidirect product of $\boldsymbol{R}$ and $\boldsymbol{R}^2$, and the orbits of $\boldsymbol{R}$ on $(\boldsymbol{R}^2)^\wedge \simeq \boldsymbol{R}^2$ are illustrated in Fig. 1d. By the Mackey method, $G_\alpha^\wedge$ is as a set the same as the orbit space, except that a copy of $\boldsymbol{R}$ replaces the one-point orbit $\{0\}$. One can check that the topology of $G_\alpha^\wedge$ is obtained from the quotient space topology of $\boldsymbol{R}^2/\boldsymbol{R}$ in the usual fashion. An explicit description of $C^*(G_\alpha)$ seems almost hopeless; however, $C^*(G_\alpha)$ is the C^*-algebra of the transformation group $(\boldsymbol{R}, \boldsymbol{R}^2)$ with orbit structure as just described, and the topological conjugacy class of this transformation group is independent of α. Hence the isomorphism class of $C^*(G_\alpha)$ is independent of α.

6. A word about C^*-algebras of $\mathfrak{p}$-adic unipotent groups. We conclude with some elementary remarks about C^*-algebras of some $\mathfrak{p}$-adic groups. Let K be a $\mathfrak{p}$-adic field (a nondiscrete totally disconnected locally compact field of characteristic zero — actually some of this discussion applies when K is of "large enough" prime characteristic), and let G be the group of K-rational points of some unipotent algebraic group defined over K. Then, as first noticed by Moore [20], G is a locally compact group with most of the nice properties of nilpotent Lie groups; in particular, $C^*(G)$ is liminary and $G^\wedge$ is given by the Kirillov orbit method. We wish to point out that, in principle, $C^*(G)$ can be determined as an inductive limit algebra.

Recall that an AF algebra ("approximately finite-dimensional") is an inductive limit of a sequence of finite-dimensional C^*-algebras [6]. (We do not require existence of a unit; as noted in [7], this does not affect most of the important properties.) Then any separable C^*-algebra which is a restricted product of matrix algebras, such as the C^*-algebra of a separable compact group, is AF, and an inductive limit of a sequence of AF algebras is AF. Most of the properties of an AF algebra can be determined from its "diagram" [6].

Now with G as above, G is easily seen to be the union of a sequence $\{H_n\}$ of compact-open subgroups H_n. Then $\cup L^1(H_n)$ is dense in $L^1(G)$, and the $C^*(H_n)$-norm on $L^1(H_n)$ coincides with the restriction of the $C^*(G)$-norm. So $C^*(G)$ is the inductive limit of the $C^*(H_n)$, and is therefore AF. A method for computing $C^*(H_n)$ was given in [17], and it is not hard (at least when G is a Heisenberg group) to determine the embeddings of $C^*(H_n)$ into $C^*(H_{n+1})$. Thus the diagram of $C^*(G)$ is computable. We omit the details, which are not very illuminating, and confine ourselves to one remark. Let G be the 3-dimensional Heisenberg group over K—then by an analysis almost identical to that for the real Heisenberg group, $C^*(G)$ contains $C_\infty(K^*, \mathscr{K})$ as an ideal. In looking at the diagram for $C^*(G)$, one immediately notices that $C_\infty(K^*, \mathscr{K})$ is a restricted product of algebras which are crossed products of abelian profinite groups and UHF algebras. This fact is far from obvious otherwise, but was essentially observed by Takesaki [22] in a slightly different form.

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UNIVERSITY OF CALIFORNIA-BERKELEY

Current address: University of Pennsylvania