PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 1, 1976

AMALGAMATED SUMS OF ABELIAN l-GROUPS

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A class $\mathscr{K}$ of algebraic structures is said to have the amalgamation property if, whenever G, H_1, and H_2 are in $\mathscr{K}$ and $\sigma_1: G \to H_1$ and $\sigma_2: G \to H_2$ are embeddings, then for some L in $\mathscr{K}$ there are embeddings $\tau_1: H_1 \to L$ and $\tau_2: H_2 \to L$ such that $\sigma_1\tau_1 = \sigma_2\tau_2$. Since this property has important universal-algebraic implications, this author has attempted to determine which well-known classes of abelian lattice-ordered groups (l-groups) have the amalgamation property. Theorem 1 lists those that do, and Theorem 2 lists those that do not. Finally, we focus our attention on one important class — Archimedian l-groups — in which the amalgamation property fails, and derive some sufficient conditions on G, H_1, and H_2 for amalgamation to occur.

Unless otherwise stated, all l-groups are abelian. For the basic theory of l-groups, see [3]. We write $A \oplus^* B$ for the sum, lexicographically ordered from the right, of an l-group A and an o-group B, while we write $A \oplus B$, $\prod_i A_i$, $\Sigma_i A_i$ for the cardinal sum or product of l-groups, ordered componentwise. For the o-groups of reals and integers we reserve the letters R and Z. $\mathscr{C}(G)$ and $\mathscr{P}(G)$ denote respectively the poset of convex l-subgroups and the complete Boolean algebra of polar subgroups of G. If $S \subseteq G$ then $G(S)$ denotes the convex l-subgroup of G generated by S.

Referring to the definition in the first paragraph, we call $(G, H_1, H_2, \sigma_1, \sigma_2)$ an *amalgam* and say that τ_1 and τ_2 *embed* the amalgam in L. We shall occasionally simplify the notation by assuming that σ_1 and σ_2 are inclusion maps.

THEOREM 1. *The following classes have the amalgamation property*:

(a) *all* (abelian) *l-groups*

(b) *o-groups*

(c) *l-groups with a finite basis*

(d) *l-groups with ACC on $\mathscr{C}(G)$*

(e) *l-groups with DCC on $\mathscr{C}(G)$*

(f) *l-groups with ACC and DCC on $\mathscr{C}(G)$.*

(g) *direct sums of subgroups of R, that is, Archimedian l-groups with property* (F).

A universal-algebraic proof of (a) and (b) can be found in [6], and a constructive proof of (b), via Hahn embeddings, is found in

[7]. We gain a little more by examining the following constructive proof of (a). First we need some preliminary results about prime subgroups, which have the flavor of the theory of prime ideals in rings. A proof of Lemma 1, for the abelian case, is contained in [5]. We include a proof here in which this hypothesis is eliminated.

LEMMA 1. *Let G be a* (not necessarily abelian) *l-group and let $S \subseteqq G^+$ be closed under finite meets. If $C \in \mathscr{C}(G)$ is maximal with respect to being disjoint from S, then C is prime.*

Proof. Let $a \wedge b = 0$. If neither a nor b is in C then $G(C, a)$ and $G(C, b)$, properly containing C, contain elements s and t respectively of S. But then $s \wedge t \in S \cap G(C, a) \cap G(C, b) = S \cap G(C, a \wedge b) = S \cap C$, a contradiction. Thus one of a and b is in C, and therefore C is prime.

LEMMA 2. *If G is an l-subgroup of the* (not necessarily abelian) *l-group H, then for every prime subgroup P of G there is a prime subgroup Q of H such that $Q \cap G = P$* (l-groups have "going up"). *Furthemore, if P is a minimal prime subgroup then Q can be chosen as a minimal prime subgroup of H.*

Proof. Since $G \cap H(P) = P$, we can extend $H(P)$ to an l-subgroup Q of maximal with respect to missing $G^+ \backslash P$. By Lemma 1, Q is prime, and Q must intersect G exactly in P. If P is a minimal prime then let Q' be a minimal prime subgroup contained in Q. Since $Q' \cap G$ is a prime of G inside P, then $Q' \cap G = P$.

We turn now to the proof of Theorem 1, part (a): Let $\{P_\alpha : \alpha \in A\}$ and $\{Q_\beta : \beta \in B\}$ be collections of prime subgroups of H_1 and H_2 respectively which intersect trivially. By Lemma 2, for each α in A there is a prime subgroup Q of H_2 such that $Q_\alpha \cap G = P_\alpha \cap G$, and for each β in there is a prime subgroup P_β of H_1 such that $P_\beta \cap G = Q_\beta \cap G$. For each γ in $A \cup B$, $G/P_\gamma \cap G$ is canonically an o-subgroup of the o-groups H_1/P_γ and H_2/Q_γ, whence by part (b) there exists an o-group L_γ and embeddings $\tau_{1\gamma}: H_1/P_\gamma \rightarrow L_\gamma$ and $\tau_{2\gamma}: H_2/Q_\gamma \rightarrow L_\gamma$ which agree on $G/P_\gamma \cap G$. Let $L = \prod_\gamma L_\gamma$ and define l-embeddings $\tau_1: H_1 \rightarrow L$ and $\tau_2: H_2 \rightarrow L$ by setting $h\tau_1(\gamma) = (h + P_\gamma)\tau_{1\gamma}$ and $k\tau_2(\gamma) = (k + Q_\gamma)\tau_{2\gamma}$. τ_1 and τ_2 evidently agree on G, and L is therefore the desired amalgamation.

Parts (c) through (f) involve classes of l-groups which can be represented as subdirect products of finitely many o-groups, each of which has the corresponding chain conditions on its convex subgroups. If one inspects the proof of (b) found in [7], one finds that these properties of o-groups can be preserved under amalgamation. Therefore the above construction will not lead out of these classes.

For (g), suppose $G \leqq H_1 = \sum_{\alpha \in A} H_{1\alpha}$ and $G \leqq H_2 = \sum_{\beta \in B} H_{2\beta}$, where $H_{1\alpha}, H_{2\beta} \leqq R$. Let $\{g_i : i \in I\}$ be a basis for G. Since each element of G is a real linear combination of basis elements, then every l-homomorphism on G is uniquely determined by its action on the basis, and therefore it suffices to find embeddings of H_1 and H_2 into a direct sum of reals which agree on the basis for G. For each $i \in I$ let $A_i = \{\alpha \in A : g_i(\alpha) > 0\}$ and let $B_i = \{\beta \in B : g_i(\beta) > 0\}$. Let $\Gamma = [\bigcup_{i \in I} A_i \times B_i] \cup [A \backslash \bigcup_{i \in I} B_i] \cup [B \backslash \bigcup_{i \in I} B_i]$ and let $L' = \prod_{\gamma \in \Gamma} R_\gamma$, Where $R_\gamma = R$. Define the embeddings $\tau_1 : H_1 \to L'$ and $\tau_2 : H_2 \to L'$ componentwise as follows: For $\gamma = (a, \beta) \in A_i \times B_i$ define $h\tau_1(\gamma) = h(\alpha)/g_i(\alpha)$ and $k\tau_2(\gamma) = k(\beta)/g_i(\beta)$; for $\gamma = \alpha \in A \backslash \bigcup_i A_i$ define $h\tau_1(\gamma) = h(\alpha)$ and $k\tau_2(\gamma) = 0$; and for $\gamma = \beta \in B \backslash \bigcup_i B_i$ define $h\tau_1(\gamma) = 0$ and $k\tau_2(\gamma) = k(\beta)$. Evidently $H_1\tau_1 + H_2\tau_2 \leqq L = \sum_\gamma R_\gamma$ and the embeddings agree on the basis for G. Thus the amalgam has been embedded in L.

REMARK. There are classes $\mathscr{K}$ for which the amalgamation property is a trivial consequence of (a), for reason that any abelian l-group is embeddable in a member of $\mathscr{K}$. Two rather trivial examples are

(a) l-groups with basis (take for L the direct product of o-groups), and

(b) compactly generated l-groups (see [2] for a proof that every abelian l-group is embeddable in such a group).

THEOREM 2. *The following classes do not have the amalgamation property*:

(a) *l-groups with property* (F)

(b) *direct sums of o-groups*

(c) *Archimedian l-groups*

(d) *Archimedian l-groups with basis*

(e) *subdirect products of subgroups of R*

(f) *hyper-archimedian l-groups.*

Proof. For (a) and (b) let G be the o-group $\langle a_1 \rangle \oplus^* \langle a_2 \rangle \oplus^* \langle a_3 \rangle \oplus^* \cdots$, let $H_1 = G \oplus G \oplus G \oplus G \oplus \cdots$, and let $H_2 = G \oplus^* \langle c \rangle$. Let $\sigma_2 : G \to H_2$ be the natural inclusion map, and embed G in H_1 by defining

$$a_1\sigma_1 = (a_1, 0, 0, \cdots),$$
$$a_2\sigma_1 = (a_2, a_2, 0, 0, \cdots),$$
$$a_3\sigma_1(a_3, a_3, a_3, 0, \cdots),$$

and so on. Suppose that this amalgam is embedded in L via the maps τ_1 and τ_2. For each natural number i let $h_i \in H_1$ have i^{th} component a_i and zeros elsewhere. Then $h_i\tau_i \leqq a_i\sigma_1\tau_1 = a_i\sigma_2\tau_2 \leqq c\tau_2$, which implies that $c\tau_2$ bounds an infinite set of mutually orthogonal elements. Thus L could neither have property (F) nor be a sum of o-groups.

For (c), (d) and (e), let $H_1 = \prod_{i\in\omega} Z_i$, let G be the l-subgroup of H_1 consisting of all sequences which are eventually constant, and let $H_2 = G \oplus Z$. Embed G in H_1 and H_2 be letting σ_1 be inclusion and setting $g\sigma_2 = (g, g_\infty)$, where $g_\infty = \lim_{i\to\infty} g(i)$. Suppose that this amalgam is embedded in L. Let $x = (1, 2, 3, 4, \cdots) \in H_1$, $z = (1, 1, 1, \cdots) \in G$, and $y = ((0, 0, \cdots), 1) \in H_2$. We will show that, in L, $x\tau_1$ exceeds every multiple of $y\tau_2$, and thus L cannot be Archimedian. By Lemma 2.17 of [3] it suffices to show that $n(y\tau_2) \leqq x\tau_1(\text{mod } P)$ for every prime P of L. Now if $y\tau_2 \in P$ this is obvious. If $y\tau_2 \notin P$ then $M\sigma_2\tau_2 \subseteqq P$, where $M = \sum Z_i \leqq G$, since every element of $M\sigma_2$ is orthogonal to y. Since $n(z\sigma_1) \leqq x(\text{mod } M\sigma_1)$, and since $y \leqq z\sigma_2$, then $n(y\tau_2) \leqq n(z\sigma_2\tau_2) = n(z\sigma_1\tau_1) \leqq x\tau_1$ modulo $M\sigma_1\tau_1 = M\sigma_2\tau_2$ and hence also modulo P.

For (f), let $G = \sum_{i\in\omega} Z_i$, and embed G in H_1 and H_2, the hyper-Archimedian l-subgroups of $\prod_{i\in\omega} Z_i$ generated respectively by G and $h = (1, 1, \cdots)$ and G and $K = (1, 2, 3, \cdots)$. Assume that this amalgam were embedded in L. For each natural number m let $P_{m+1} = \{x \in H_1: x(m+1) = 0\}$, and by Lemma 2 let Q_{m+1} be a prime of L such that $Q_{m+1} \cap H_1\tau_1 = P_{m+1}\tau_1$. Now $(Q_{m+1} \cap H_2\tau_2)\tau_2^{-1}$ is a prime subgroup of H_2, in fact, an elementary argument shows that it is the prime $R_{m+1} = \{x \in H_2: x(m+1) = 0\}$. Let g be the element of G whose $(m+1)$-coordinate is 1 and whose other coordinates are 0. Since $mh = mg(\text{mod } P_{m+1})$ and $mg < k(\text{mod } R_{m+1})$ then $0 < m(h\tau_1) < k\tau_2(\text{mod } Q_{m+1})$. Thus there is no natural number m for which $[k\tau_2 - (mh\tau_1 \wedge k\tau_2)] \wedge h\tau_1 = 0$, and hence L cannot be hyper-Archimedian.

ARCHIMEDIAN AMALGAMATIONS. Our first construction makes use of Bernau's representation of Archimedian l-groups, found in [1], which we summarize here; if B is a maximal set of mutually orthogonal positive elements of G and X is the Stone space (compact Hausdorff and extremally disconnected space) associated with $\mathscr{P}(G)$, then there is an l-embedding η of G into $D(X)$, the l-group of almost finite continuous extended-real-valued functions on X, with the properties

(a) $G\eta$ is a large subgroup of $D(X)$ (i.e., if $0 < f \in D(x)$ then $0 < g\eta < nf$ for some $g \in G$ and some natural number n),

(b) $B\eta$ is a set of characteristic functions of a family of mutually disjoint clopen subsets of X whose union is dense in X, and

(c) for each $g \in G$, $S(g\eta)$, the support of $g\eta$, is the clopen subset of X corresponding to the polar subgroup $\{g\}''$.

Moreover, there is the following uniqueness: If Y is a Stone space and if η' is an l-embedding of G into $D(Y)$ having properties (a) and (b), then there is a homeomorphism θ from X to Y such that $(g\eta)(x) = (g\eta')(x\theta)$ for all $g \in G$ and all $x \in X$.

One consequence of (c) which we shall use is the following: If $M \in \mathscr{P}(G)$ and if Y is the clopen subset of X corresponding to M, then, under the natural association, Y is the Stone space of $\mathscr{P}(M)$, every element of M is zero outside Y, and $\eta' = \eta \mid M$ is an l-embedding of M into $D(Y)$ satisfying (a) and (b). Furthermore, if $B \cap M$ is a maximal orthogonal subset of M, then (c) will also be satisfied.

THEOREM 3. *If $(G, H_1, H_2, \sigma_1, \sigma_2)$ is an amalgam of Archimedian l-groups in which $G\sigma_i$ is a large subgroup of a polar M_i of $H_i (i = 1, 2)$. then the amalgam is embeddable in an Archimedian l-group.*

Proof. First pick a maximal orthogonal subset A of G, and then extend $A\sigma_i$ to a maximal orthogonal subset B_i of H_i. Let X_i be the Stone space of $\mathscr{P}(H_i)$ and, via B_i, let η_i be the l-embedding of H_i into $D(X_i)$ satisfying (a)-(c) above. Because "largeness" is a transitive relation (cf [4]), it follows from the above remarks that $\sigma_i\eta_i$ is an l-embedding of G into $D(Y_i)$ satisfying (a) and (b), where Y_i is the clopen subset of X_i corresponding to M_i. Thus by uniqueness, Y_1 and Y_2 are homeomorphic, and, if we actually identify Y_1 and Y_2, $\sigma_1\eta_1$ and $\sigma_2\eta_2$ are identical on $Y_1 = Y_2$ and zero elsewhere. We now form the disjoint union $Z = Y_1 \cup (X_1 \backslash Y_1) \cup (X_2 \backslash Y_2)$ and let τ_1 and τ_2 be the natural embedding of H_1 and H_2 respectively into $D(Z)$. Since $\sigma_1\tau_1 = \sigma_2\tau_2$, this finishes the proof of the theorem.

Our second construction requires a modification of Bernau's embedding. Let X be a topological space and let $F^*(X)$ be the set of all real-valued functions which are defined and continuous on a dense open subset of x. If lattice and group operations are defined pointwise, with domain of the resultant being the intersection of the domains of the operands, then $F^*(X)$ is an abelian lattice-ordered semigroup with zero. Saying that f and g are equivalent if they agree on a dense open subset of X defines a congruence relation on $F^*(X)$, and the quotient structure, which we denote by $F(X)$, is an Archimedian l-group.

An Archimedian l-group G is said to be *amalgamable* (in Archimedian l-groups) if every amalgam (G, H_1, H_2) of Archimedian l-groups is embeddable in an Archimedian l-group.

THEOREM 4. *Direct sums of subgroups of the reals are amalgamable in Archimedian l-groups.*

Proof. We divide the proof into two parts, the sum of which proves something stronger than the theorem.

(A) *Subgroups of the reals are amalgamable.* Let g be a positive element of G, let B_i be a maximal orthogonal subset of H_i containing g, and, via B_i, embed H_i in $D(X_i)$ in the manner described above. Let Y_i denote the support of g in X_i, and let Z be the disjoint union of the sets $Y_1 \times Y_2$, $X_1 \backslash X_1$, and $X_2 \backslash X_2$, topologized in the obvious manner. Define an embedding $\tau_1: H_1 \to D(Z)$ by

$$h\tau_1(z) = \begin{cases} h(x) & \text{if either } z = (x, w) \text{ or } z = x \in X_1 \backslash Y_1 \\ 0 & \text{otherwise} \end{cases}$$

and define $\tau_2: H_2 \to D(Z)$ in an analogous fashion. Clearly τ_1 and τ_2 agree on G since $g\tau_1$ and $g\tau_2$ are both the characteristic function of $Y_1 \times Y_2$. But Z may not be a Stone space, since products of Stone spaces are not necessarily Stone spaces, and thus $H_1\tau_1 \cup H_2\tau_2$ may not generate a group in $D(Z)$. However, the natural map from $D(Z)$ to $F(Z)$ embeds the amalgam in an Archimedian l-group.

(B) *Direct sums of amalgamable l-groups are amalgamable.* Let $G = \sum_{\alpha \in A} G_\alpha$, each G_α being amalgamable. If $M_{i\alpha} = (G_\alpha \sigma_i)'$, then $H_i/M_{i\alpha}$ is Archimedian, since quotients of Archimedian l-groups by polar subgroups are Archimedian, and G_α is naturally l-embedded in $H_i/M_{i\alpha}$. Suppose that $\tau_{1\alpha}$ and $\tau_{2\alpha}$ embed the amalgam $(G_\alpha, H_1/M_{1\alpha}, H_2/M_{2\alpha})$ in the Archimedian l-group L_α. Let $M_i = (\bigcup_\alpha G_\alpha \sigma_i)''$, let $L = H_1/M_1 \oplus H_2/M_2 \oplus \prod_\alpha L_\alpha$, and define maps from H_1 and H_2 into L as

$$h\tau_1 = (h + M_1, 0, \cdots, (h + M_{1\alpha})\tau_{1\alpha}, \cdots)\,,$$
$$h\tau_2 = (0, h + M_2, \cdots, (h + M_{2\alpha})\tau_{2\alpha}, \cdots)\,.$$

Since $M_i \cap (\bigcap_\alpha M_{i\alpha}) = 0$, then τ_1 and τ_2 are l-embeddings. Furthermore, $G_\alpha \sigma_1 \tau_1 = G_\alpha \sigma_2 \tau_2$ for each α, and so they must agree on the direct sum.

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Received March 5, 1975.

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