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LINKS WHICH ARE UNKNOTTABLE BY MAPS

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Let L be a piecewise linear (PL) link of two components in the Euclidean 3-sphere S^3 (i.e., $L = L_1 \cup L_2$ where L_1, L_2 are disjoint polygonal simple closed curves in S^3. In Theorem 1 of this paper we give a geometric condition on L which implies it is unknottable. In Theorem 2, we show that there is an infinite class of links of two components which are unknottable.

We call a continuous (PL) map $f: S^3 \to S^3$ strongly $1-1$ on L if $f|L$ is a homeomorphism onto $f(L)$, $f(S^3 - L) \cap f(L) = \emptyset$ and f is locally $1-1$ at each point of L. In Theorem 1 of [3], the link $L_0 = L_{01} \cup L_{02}$ where L_{01} is unknotted and L_{02} is the square knot is shown to have the property that there is no strongly $1-1$ map f on L_0 such that $f(L_{01})$ and $f(L_{02})$ are unknotted. Call L "unknottable" if there does not exist a strongly $1-1$ map f on L such that $f(L_1)$ and $f(L_2)$ are unknotted. This paper and [3] resulted from an attempt to generalize Hempel's result [2] that given any knot K in S^3 there exists a strongly $1-1$ map f on K such that $f(K)$ is unknotted.

Let S_1 be a (PL) orientable surface such that $\mathrm{Bd}\, S_1 = L_1$ and L_2 intersects and pierces S_1 in a finite number of points. Let $N(L) = N(L_1) \cup N(L_2)$ be a regular neighborhood of L such that $S_1 \cap N(L_1)$ is an annulus and $S_1 \cap N(L_2)$ consists of transverse disks. Call S_1 essential if $S_1 - \mathrm{Int}\, N(L)$ is incompressible [7] and boundary incompressible [7] in $S^3 - \mathrm{Int}\, N(L)$.

DEFINITION 1. L is boundary incompressibly unlinked with respect to L_1 (B.I.U.) if, whenever S_1 is essential, we have $S_1 \cap L_2 = \emptyset$. L is said to be 1-linked [5] if L_1, L_2 do not bound disjoint orientable surfaces in S^3.

THEOREM 1. *If L is 1-linked,* B.I.U. *and L_1 is knotted, then L is unknottable.*

Proof. Suppose there exists a $f: S^3 \to S^3$ which is strongly $1-1$ on L and $f(L_1)$, $f(L_2)$ are unknotted. Let D_1 be a disk in S^3 such that $\mathrm{Bd}\, D_1 = f(L_1)$ and $f(L_2)$ intersects and pierces D_1 in a finite number t of points. Suppose also that t is chosen to be smallest possible. Now, following the techniques used in [7], we adjust f so that it is transverse to D_1, in particular $D_1' = f^{-1}(D_1) \cap (S^3 - \mathrm{Int}\, N(L))$ is an orientable surface with one boundary component in $\operatorname{Bd} N(L_1)$ which is a longitude of $N(L_1)$ and t boundary components in $\operatorname{Bd} N(L_2)$, each of which is a meridian of $N(L_2)$. Now suppose D_1' is compressible in $S^3 - \operatorname{Int} N(L)$, i.e. there exists a disk Q in $S^3 - \operatorname{Int} N(L)$ such that $Q \cap D_1' = \operatorname{Bd} Q \cap D_1' = \operatorname{Bd} Q$ and $\operatorname{Bd} Q$ does not bound a disk in D_1'. Now if the loop $f(\operatorname{Bd} Q)$ separates a point of $D_1 \cap f(L_2)$ from $\operatorname{Bd} D_1$, we may apply Dehn's lemma [4] to conclude that t was not minimal. If $f(\operatorname{Bd} Q)$ separates no point of $D_1 \cap f(L_2)$ from $\operatorname{Bd} D_1$, then we may cut out a small regular neighborhood of $\operatorname{Bd} Q$ in D_1' and add two parallel copies of Q to form a new orientable surface D_1'' with less genus than D_1'. We may then redefine the map f so that $D_1'' = f^{-1}(D_1) \cap (S^3 - \operatorname{Int} N(L))$. If D_1' is boundary compressible, then there exists a disk Q such that $\operatorname{Int} Q \cap D_1' = \emptyset$ and $\operatorname{Bd} Q$ consists of two arcs, one in $\operatorname{Bd} N(L_2)$, the other in D_1' and the arc in D_1' together with any arc in $\operatorname{Bd} D_1'$ do not bound a disk in D_1'. In this case we may use a modified version of the loop theorem (see [6]) on the loop $f(\operatorname{Bd} Q)$ in $S^3 - \operatorname{Int} f(N(L))$ to conclude that t was not minimal. Hence we may assume that D_1' is incompressible and boundary incompressible. Since L is B.I.U. we have $t = 0$. Then $f(L_2)$ bounds a disk D_2 which is disjoint from D_1. We may adjust f so that $f^{-1}(D_1)$, $f^{-1}(D_2)$ are disjoint orientable surfaces, contradicting the assumption that L is 1-linked, and the proof is complete.

We now define the class of links $L_{1j} \cup L_{2j}$ illustrated in Figures 1 and 2. Each L_{1j} is a curve with j full twists (j is any positive or negative integer and one of the full twists is shown in the figure). If $j \neq 0$, then in [1] it is shown that L_{1j} is knotted.

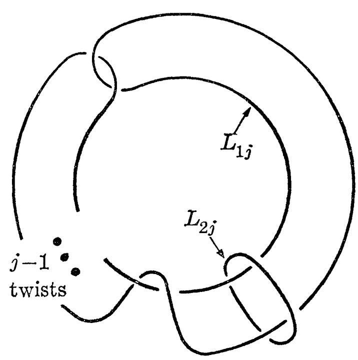

FIGURE 1

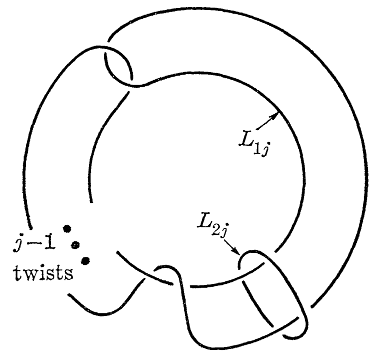

FIGURE 2

LEMMA 1. *$L_{1j} \cup L_{2j}$ is 1-linked for all j.*

Proof. Suppose L_{1j}, L_{2j} bound disjoint orientable surfaces S_{1j}, S_{2j}, respectively. Let D' be a disk bounded by L_{2j} such that L_{1j} intersects and pierces D' in two points and the two components of

$L_{1j} - D'$ self link each other. By cut and paste techniques (see [7] or we used some of these methods in Theorem 1) we may assume that $(\text{Int } D') \cap S_{2j} = \emptyset$ and $D' \cap S_{1j}$ consists of one arc connecting the two points of $D' \cap L_{1j}$. Let D'' be a disk whose boundary consists of the arc $D' \cap S_{1j}$ and one of the two arcs of $L_{1j} - D'$. Assume $D'' \cap D' = D' \cap S_{1j}$ and the other arc of $L_{1j} - D'$ intersects and pierces D'' in one point. But it now follows that there is a curve in $S_{2j} \cap D''$ which is not homologous to zero in $S^3 - L_{1j}$, contradicting that $S_{1j} \cap S_{2j} = \emptyset$.

In Figure 3 we view L_{1j} as being contained in a cube with two handles C where $N(L_{1j}) \subset \text{Int } C \subset S^3 - L_{2j}$. Let H_1, H_2 be the two annuli illustrated in Figure 3, where $H_1 \cap H_2$ is an arc.

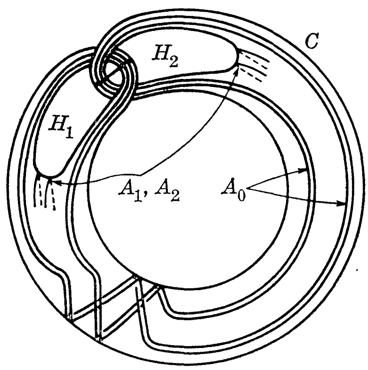

FIGURE 3

LEMMA 2. *Each link $L_{1j} \cup L_{2j}$, $j \neq 0$, is boundary incompressibly unlinked* (B.I.U.).

Proof. Suppose S' is an orientable surface in the solid torus $T = S^3 - \text{Int } N(L_{2j})$ with one boundary component L_{1j} and each of the remaining t boundary components is a meridian of $N(L_{2j})$ in $\text{Bd } N(L_{2j})$. Suppose also that $S = S' - \text{Int } N(L_{1j})$ is incompressible and boundary incompressible in $T - \text{Int } N(L_{1j})$. We may choose the cube with two handles C so that $S' \cap C$ consists of an annulus A_0 and s disks $A_1, \cdots, A_s$ (see Figure 3). Now, by following the techniques used in Lemma 1 of [3], we may adjust S' so that $S' \cap H_1$ is one arc parallel to $H_1 \cap H_2$ in H_1. (To see this, put S' in general position relative to H_1 and push arcs of $S' \cap H_1$ with both endpoints in the same component of $\text{Bd } H_1$ off H_1 and then off C, i.e. we reduce s by 1 or 2 and hence we may suppose $s = 0$.) By the same reasoning we may suppose further that $S' \cap H_2$ consists of one arc parallel to $H_1 \cap H_2$ in H_2. Let $N(H_1)$, $N(H_2)$ be regular neighborhoods of H_1, H_2, resp., taken in $T - \text{Int } C$. Let T' be the solid torus

$C \cup N(H_1) \cup N(H_2)$. Then $T - \operatorname{Int} T'$ is homeomorphic to the product space $(S^1 \times S^1) \times I$. None of the three simple closed curves of $S' \cap \operatorname{Bd} T'$ is homotopic to the t curves of $S' \cap \operatorname{Bd} T$. (Note that one component of $S' \cap \operatorname{Bd} T'$ bounds a disk in $\operatorname{Bd} T'$ and the other two go once around the longitude of T' and j times, $j \neq 0$, around the meridian of T'.) Since S is incompressible and boundary incompressible, it follows that $\pi_1(S \cap (T - \operatorname{Int} T'))$ injects into the abelian group $\pi_1(T - \operatorname{Int} T')$. Hence $S \cap (T - \operatorname{Int} T')$ consists of one disk and one annulus, so $t = 0$ and the proof of Lemma 2 is finished.

Theorem 1, Lemma 1 and Lemma 2 now imply the following:

THEOREM 2. *Each of the links* $L_{1j} \cup L_{2j}$, $j \neq 0$, *is unknottable*, i.e. *there does not exist a strongly* $1-1$ *map* f *on* $L_{1j} \cup L_{2j}$ *such that* $f(L_{1j})$ *and* $f(L_{2j})$ *are unknotted.*

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