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COMMUTATORS AND NUMERICAL RANGES OF POWERS OF OPERATORS

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If 0 does not lie in the closure of the numerical range of any positive integral power of a Hilbert space operator T, then an odd power of T is normal. If, in addition, T is convexoid, then T itself is normal; in fact, T is the direct sum of at most three rotated positive operators. A version of these results is given in terms of commutators.

1. Introduction. In [8] C. R. Johnson proved: For an $m \times m$ complex matrix A, if A^n is not normal for any positive integer n, then there exist a positive integer n_0 and a nonzero vector $x \in C^m$ such that $(A^{n_0}x, x) = 0$. Later he and M. Neuman [9] obtained a number theoretic result which strengthens the above theorem. We generalize these theorems to the Hilbert space operator case in this paper.

Let $\mathscr{B}(\mathscr{H})$ denote the set of bounded operators on a Hilbert space $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, $\bar{W}(T)$ denotes the closure of the numerical range of T. Our main results are: If $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \cdots$, then an odd power of T is normal; in fact, T is similiar to the direct sum of at most three rotated positive operators. Moreover, under the above hypothesis, T is normal if and only if T is convexoid.

These results can be applied to the theory of commutators: Let $\mathfrak{H}$ denote a separable infinite dimensional Hilbert space. For $T \in \mathscr{B}(\mathfrak{H})$, if $T^n \notin \{SX - XS\colon S, X \in \mathscr{B}(\mathfrak{H}),\ S \text{ positive}\}$, $n = 1, 2, 3, \cdots$, then there are an odd integer k and a compact operator K such that $T^k + K$ is normal; furthermore, T is a compact perturbation of a normal operator if and only if the essential numerical range of T is a polygon (possibly degenerate).

2. Preliminaries. Let C denote the set of complex numbers and R^+ the set of strictly positive numbers. For $\Omega \subset C$, $\mathrm{Co}(\Omega)$ denotes its convex hull; $\Omega^n = \{z^n\colon z \in \Omega\}$, n a positive integer. We write $\Omega > r$, r a real number, if Ω is a real subset and each number in Ω is greater than r. Let $\alpha, \beta \in C$ and $\varepsilon \in (0, 1]$, $\Theta(\alpha, \beta; \varepsilon)$ denotes the closed elliptical disc with eccentricity ε and foci at α and β,

$$\Theta(\alpha, \beta; \varepsilon) = \{z \in C\colon |z - \alpha| + |z - \beta| \leqq |\alpha - \beta|/\varepsilon\} .$$

Note that $\Theta(\alpha, \beta; 1)$ is the line segment joining α and β.

LEMMA 1. *Let α, β be two distinct nonzero complex numbers. For $\varepsilon \in (0, 1]$, if $|\mathrm{Arg}(\alpha/\beta)| \geqq \arccos(-\varepsilon^2)$, then $0 \in \Theta(\alpha, \beta; \varepsilon)$.*

For $T \in \mathscr{B}(\mathscr{H})$, $\sigma(T)$ denotes the spectrum and $W(T)$ the numerical range of T, $W(T) = \{(Tx, x): \|x\| = 1\}$. We say T is positive and write $T > 0$ if $\bar{W}(T) > 0$. T is called convexoid if $\mathrm{Co}(\sigma(T)) = \bar{W}(T)$ [6, p. 114].

The following result describes the numerical range of a 2×2 matrix with distinct eigenvalues ([12], [10]).

LEMMA 2. *If $\alpha \neq \beta$, then $W\left(\begin{pmatrix}\alpha & \gamma \\ 0 & \beta\end{pmatrix}\right) = \Theta(\alpha, \beta; (1 + |\gamma/(\alpha - \beta)|^2)^{-1/2})$.*

Let $\mathscr{H} \oplus \mathscr{K}$ denote the direct sum of two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$; an operator on $\mathscr{H} \oplus \mathscr{K}$ may be expressed as a 2×2 matrix whose entries are operators. See [6, Chapter 7].

LEMMA 3. *Let $T \in \mathscr{B}(\mathscr{H} \oplus \mathscr{K})$, $T = \begin{pmatrix}A & B \\ C & D\end{pmatrix}$. Then*

$$W(T) = \cup \left\{ W\left(\begin{pmatrix}(Ax, x) & (By, x) \\ (Cx, y) & (Dy, y)\end{pmatrix}\right): x \in \mathscr{H}, y \in \mathscr{K}, \|x\| = \|y\| = 1 \right\}.$$

Let $T \in \mathscr{B}(\mathscr{H})$ with $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint, nonempty and closed. Let E be the spectral projection associated with σ_1 [18, §5.7]; then $E^2 = E$, $ET = TE$, $\sigma(T|_{E\mathscr{H}}) = \sigma_1$ and $\sigma(T|_{(I-E)\mathscr{H}}) = \sigma_2$. We note that E may not be Hermitian.

LEMMA 4 (cf. [13, §0.4]). *Let T and E be as above and let P be the orthogonal projection on $E\mathscr{H}$. Then, with respect to the decomposition $E\mathscr{H} \oplus (E\mathscr{H})^\perp$, the operator matrix corresponding to T has the form $\begin{pmatrix}T_1 & T_1A - AT_2 \\ 0 & T_2\end{pmatrix}$, where $\begin{pmatrix}0 & A \\ 0 & 0\end{pmatrix} = E - P$ and*

$$\sigma(T_i) = \sigma_i,\ i = 1, 2.$$

Furthermore, $T_1A - AT_2 = 0$ if and only of $A = 0$.

The following result is proved in ([14], [15]).

LEMMA 5. *For $T \in \mathscr{B}(\mathscr{H})$ and $\sigma(T) > \gamma > 0$, if $\{z \in \boldsymbol{C}: |z| \leqq \gamma^n\} \not\subset W(T^n)$ for infinitely many positive integers n, then $T > 0$.*

3. Main results.

The following generalizes [8, Theorem 1].

THEOREM 1. *Let $T \in \mathscr{B}(\mathscr{H})$ with $\sigma(T) \cap \boldsymbol{R}^+ \neq \varnothing$. Suppose*

$0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \cdots$, *then either* (i) *there is a positive odd integer* m *such that* $T^m > 0$ *or* (ii) *there exist a proper closed subspace* $\mathscr{H}_1$ *of* $\mathscr{H}$ *and positive operators* T_1 *and* T_2 *on* $\mathscr{H}_1$ *and* $\mathscr{H}_1^{\perp}$ *respectively such that* $T = T_1 \oplus e^{i\theta} T_2$, θ *being irrational modulo* 2π.

Proof. Since $0 \notin \bar{W}(T^n) \supset \mathrm{Co}(\sigma(T^n)) = \mathrm{Co}(\sigma(T)^n)$, $n = 1, 2, 3, \cdots$, either (i) there is an odd integer m such that $\sigma(T)^m \subset \boldsymbol{R}^+$ or (ii) $\sigma(T) \subset \boldsymbol{R}^+ \cup e^{i\theta} \cdot \boldsymbol{R}^+$, θ being irrational modulo 2π.

In case (i), $\sigma(T^m) > 0$. Thus we have $T^m > 0$ by Lemma 5.

In case (ii) we apply Lemma 4 with $\sigma_1 = \sigma(T) \cap \boldsymbol{R}^+$. Then

$$T = \begin{pmatrix} T_1 & T_1 A - e^{i\theta} A T_2 \\ 0 & e^{i\theta} T_2 \end{pmatrix},$$

where $\sigma(T_1) > 0$ and $\sigma(T_2) > 0$. Since $T^n = \begin{pmatrix} T_1^n & T_1^n A - e^{in\theta} A T_2^n \\ 0 & e^{in\theta} T_2^n \end{pmatrix}$, $W(T^n) \supset W(T_1^n)$ and $W(T^n) \supset W(e^{in\theta} T_1^n)$, we have $T_1 > 0$ and $T_2 > 0$ by Lemma 5.

To show that $T = T_1 \oplus e^{i\theta} T_2$, we have to show $A = 0$. Assume $A \neq 0$. For a positive integer n and $y \in (E\mathscr{H})^{\perp}$, with $\|y\| = 1$ and $Ay \neq 0$, let $\Theta[n, y]$ denote the numerical range of the 2×2 matrix

$$\begin{pmatrix} (T_1^n Ay, Ay)/\|Ay\|^2 & ((T_1^n Ay, Ay) - e^{in\theta}(AT_2^n y, Ay))/\|Ay\| \\ 0 & e^{in\theta}(T_2^n y, y) \end{pmatrix}.$$

By Lemma 3, $\Theta[n, y] \subset W(T^n)$. By Lemma 2, $\Theta[n, y] = \Theta(\alpha, \beta; \varepsilon[n, y])$, where $\alpha \in \boldsymbol{R}^+$, $\beta \in e^{in\theta}\boldsymbol{R}^+$ and

$$\varepsilon[n, y] = \left(1 + \left|\frac{((T_1^n Ay, Ay) - e^{in\theta}(AT_2^n y, Ay))/\|Ay\|}{(T_1^n Ay, Ay)/\|Ay\|^2 - e^{in\theta}(T_2^n y, y)}\right|^2\right)^{-1/2}.$$

Let y_m, $m = 1, 2, 3, \cdots$ be a sequence in $(E\mathscr{H})^{\perp}$ such that $\|y_m\| = 1$ and $\lim_{m\to\infty} \|Ay_m\| = \|A\|$. For each n,

$$\begin{aligned} &\frac{((T_1^n Ay_m, Ay_m) - e^{in\theta}(T_2^n y_m, A^*Ay_m))/\|Ay_m\|^2}{(T_1^n Ay_m, Ay_m)/\|Ay_m\|^2 - e^{in\theta}(T_2^n y_m, y_m)} \\ &\quad = 1 + \frac{e^{in\theta}(T_2^n y_m, (\|Ay_m\|^2 - A^*A)y_m)}{(T_1^n Ay_m A, y_m)/\|Ay_m\|^2 - e^{in\theta}(T_2^n y_m, y_m)} \longrightarrow 1 \text{ as } m \longrightarrow \infty . \end{aligned}$$

Hence $\lim_{m\to\infty} \varepsilon[n, y_m] = (1 + \|A\|^2)^{-1/2}$. Thus for each integer n, there is an integer $m(n)$ such that

$$\varepsilon[n, y_{m(n)}] \leqq (1 + \|A\|^2/2)^{-1/2} < 1 .$$

Since θ is irrational modulo 2π, we can pick a positive integer N for which $|\mathrm{Arg}\, e^{iN\theta}| \geqq \arccos\,(-/(1 + \|A\|^2/2))$. Then $0 \in \Theta[N, y_{m(N)}]$ by Lemma 1. However, $0 \notin W(T^N)$ by hypothesis; $A = 0$ and $T = T_1 \oplus e^{i\theta} T_2$.

We note that if $\mathscr{H}$ is finite dimensional, the proof of case (ii) can be greatly simplified: Let $\alpha, \beta \in \boldsymbol{C}$ and $\alpha^n \neq \beta^n$, $n = 1, 2, 3, \cdots$, then $W\left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}^n\right) = \Theta(\alpha^n, \beta^n; (1 + |\gamma/(\alpha - \beta)|^2)^{-1/2})$ by Lemma 2.

For $\mathscr{C} \subset \boldsymbol{C}\backslash\{0\}$, let $\sharp\mathrm{Arg}\ \mathscr{C}$ denote the cardinality of the set $\{\lambda/|\lambda| : \lambda \in \mathscr{C}\}$. The result in [9] may be stated as follows: Let $\mathscr{C}$ be a compact set of nonzero complex numbers such that $\mathscr{C} \cap \boldsymbol{R}^+ \neq \varnothing$. If $0 \notin \mathrm{Co}\,(\mathscr{C}^n)$, $n = 1, 2, 3 \cdots$, and if $\sharp\mathrm{Arg}\ \mathscr{C} \geqq 3$, then $\sharp\mathrm{Arg}\ \mathscr{C} = 3$ and $\mathscr{C}^7 \subset \boldsymbol{R}^+$.

THEOREM 1'. *Let $T \in \mathscr{B}(\mathscr{H})$ with $\sigma(T) \cap \boldsymbol{R}^+ \neq \varnothing$. Suppose $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, , \cdots$. We have the following cases:*

(i) *$\sharp\mathrm{Arg}\,\sigma(T) = 1$ then $T > 0$.*

(ii) *$\sharp\mathrm{Arg}\,\sigma(T) \geqq 3$, then $\sharp\mathrm{Arg}\,\sigma(T) = 3$ and $T^7 > 0$.*

(iii) *$\sharp\mathrm{Arg}\,\sigma(T) = 2$, then either there is a positive odd integer m such that $T^m > 0$ or there exist a closed subspace $\mathscr{H}_1$ of $\mathscr{H}$ and positive operators T_1 and T_2 on $\mathscr{H}_1$ and $\mathscr{H}_1^\perp$ respectively such that $T = T_1 \oplus e^{i\theta}T_2$, θ being irrational modulo 2π.*

THEOREM 2. *Let $T \notin \mathscr{B}(\mathscr{H})$. Suppose $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \cdots$. Then T is normal if T is convexoid.*

Proof. By Theorem 1', $\sharp\mathrm{Arg}\,\sigma(T) \leqq 3$. First, we consider the case $\sharp\mathrm{Arg}\,\sigma(T) = 2$, i.e., there are two real numbers θ_1 and θ_2 such that $\sigma(T) \subset e^{i\theta_1}\cdot\boldsymbol{R}^+ \cup e^{i\theta_2}\cdot\boldsymbol{R}^+$. Let E be the spectral projection associated with $\sigma(T) \cap e^{i\theta_1}\cdot\boldsymbol{R}^+$. With respect to $E\mathscr{H} \oplus (E\mathscr{H})^\perp$, put $E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$, then $T = \begin{pmatrix} e^{i\theta_1}T_1 & e^{i\theta_1}T_1A - Ae^{i\theta_2}T_1 \\ 0 & e^{i\theta_2}T_2 \end{pmatrix}$, where $T_1 > 0$ and $T_2 > 0$. Assume $A \neq 0$; thus there is a two-dimensional compression of T whose numerical range consists of an elliptical disc with foci on each of the two half-rays $e^{i\theta_j}\cdot\boldsymbol{R}^+$, $j = 1, 2$, and eccentricity strictly less than unity. However, T is a convexoid by hypothesis and $\mathrm{Co}\,(\sigma(T))$ is a quadrilateral, a triangle or a line segment with all of its vertices lying on the two half-rays $e^{i\theta_j}\cdot\boldsymbol{R}^+$, $j = 1, 2$. Therefore, $A = 0$ and $T = e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2$.

The case that $\sharp\mathrm{Arg}\,\sigma(T) = 3$ is treated in a similar fashion. Nevertheless, we note that the above geometric argument fails if $\sharp\mathrm{Arg}\,\sigma(T) \geqq 4$. Fortunately this case cannot arise.

By the term polygon, we mean the rectilinear figure together with its interior domain; moreover, we do not exclude the degenerate cases of singletons and line segments. For $T \in \mathscr{B}(\mathscr{H})$, if $\bar{W}(T)$ is a polygon, then T is convexoid [7, Satz 1]. Thus we have

COROLLARY 1. *Let* $T \in \mathscr{B}(\mathscr{H})$. *Suppose* $0 \notin \bar{W}(T^n)$, $n = 1, 2, 3, \cdots$. *Then* T *is normal if and only if* $\bar{W}(T)$ *is a polygon.*

We note that the polygon mentioned in Corollary 1 may have at most six sides.

4. Commutators. There are interesting applications of the above results to the theory of commutators. Let $\mathfrak{H}$ be a separable infinite dimensional Hilbert space, $\mathscr{K}(\mathfrak{H})$ the set of all compact operators on $\mathfrak{H}$ and Π the canonical homomorphism from $\mathscr{B}(\mathfrak{H})$ onto the Calkin algebra, $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$. There exists an isometric *-isomorphism τ of the Calkin algebra onto a closed self-adjoint subalgebra of $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a suitably chosen Hilbert space [16, Theorem 12.41]. For $T \in \mathscr{B}(\mathfrak{H})$, the Weyl spectrum $\sigma_W(T)$ is the largest subset of $\sigma(T)$ which is invariant under compact perturbations, $\sigma_W(T) = \cap \{\sigma(T + K) \colon K \in \mathscr{K}(\mathfrak{H})\}$. In [5] it is shown that $\sigma_W(T)$ consists of $\sigma(\tau(\Pi(T)))$ together with some of the bounded components of the complement of $\sigma(\tau(\Pi(T)))$. Consequently if $\sigma_W(T)$ lies on a simple arc, $\sigma_W(T) = \sigma(\tau(\Pi(T)))$.

LEMMA 6 ([11], [4, p. 62]). *Let* $T \in \mathscr{B}(\mathfrak{H})$. *Suppose* $\tau(\Pi(T))$ *is normal and* $\sigma(\tau(\Pi(T)))$ *lies on a simple arc. Then, there exists a compact operator* K *such that* $T + K$ *is normal and* $\sigma(T + K) = \sigma(\tau(\Pi(T)))$.

The essential numerical range of $T \in \mathscr{B}(\mathfrak{H})$ is the set $W_e(T) = \cap \{\bar{W}(T + K) \colon K \in \mathscr{K}(\mathfrak{H})\}$. By [17, Theorem 9] and [2, Theorem 3], $W_e(T) = \bar{W}(\tau(\Pi(T)))$. Let $\mathscr{R}$ denote $\{SX - XS \colon S, X \in \mathscr{B}(\mathfrak{H}),\ S > 0\}$. In [1], J. H. Anderson proved the following deep result: $\mathscr{R} = \{T \in \mathscr{B}(\mathfrak{H}) \colon 0 \in W_e(T)\}$; also see [3, §34]. Corresponding to Theorem 1′, we have

THEOREM 3. *Let* $T \in \mathscr{B}(\mathfrak{H})$. *Suppose* $T^n \notin \mathscr{R}$, $n = 1, 2, 3, \cdots$. *Then we have the following cases*:

(i) $\sharp \mathrm{Arg}\, \sigma_W(T) = 1$, *then there exist* $\theta \in [0, 2\pi)$ *and a compact operator* K *such that* $(e^{i\theta}T + K) > 0$.

(ii) $\sharp \mathrm{Arg}\, \sigma_W(T) \geqq 3$, *then* $\sharp \mathrm{Arg}\, \sigma_W(T) = 3$ *and there exist* $\theta \in [0, 2\pi)$ *and a compact operator* K *such that* $(e^{i\theta}T^7 + K) > 0$.

(iii) $\sharp \mathrm{Arg}\, \sigma_W(T) = 2$, *then either there exist a positive odd integer* m, $\theta \in [0, 2\pi)$ *and a compact operator* K *such that* $(e^{i\theta}T^m + K) > 0$, *or there exist a closed subspace* $\mathfrak{H}_1$ *of* $\mathfrak{H}$ *and positive operators* T_1 *and* T_2 *on* $\mathfrak{H}_1$ *and* $\mathfrak{H}_1^\perp$ *respectively such that* $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ *is compact, where* $(\theta_1 - \theta_2)$ *is a number irrational modulo* 2π.

Proof. We only need to prove the second half of case (iii).

We know $\tau(\Pi(T)) = e^{i\theta_1}V_1 \oplus e^{i\theta_2}V_2$ on $\mathscr{H}_1 \oplus \mathscr{H}_1^{\perp} = \mathscr{H}$, where $V_1 > 0$ and $V_2 > 0$. Thus $\tau(\Pi(T))$ is normal and $\sigma(\tau \circ \Pi(T))$ lies on a simple arc. By Lemma 6, there is a compact operator K such that $T + K$ is normal and $\sigma(T + K) = \sigma(\tau(\Pi(T)))$. Consequently, there exist a closed subspace $\mathfrak{H}_1$ of $\mathfrak{H}$ and positive operators T_1 and T_2 on $\mathfrak{H}_1$ and $\mathfrak{H}_1^{\perp}$ respectively such that $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ is compact.

THEOREM 4. *Let* $T \in \mathscr{B}(\mathfrak{H})$. *Suppose* $T^n \notin \mathscr{R}$, $n = 1, 2, 3, \cdots$. *Then T is a compact perturbation of a normal operator if and only if* $W_e(T)$ *is a polygon.*

Proof. Apply Corollary 1.

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