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NEGATIVE THEOREMS ON GENERALIZED CONVEX APPROXIMATION

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In this paper we show that there exist functions $f \in C[-1, +1]$ with all $(r+1)$-st order divided differences uniformly bounded away from zero for r fixed ($f[x_0, x_1, \cdots, x_{r+1}] \geqq \delta > 0$ for fixed δ and all sets $x_0 < \cdots < x_{r+1}$ in $[-1, +1]$), for which infinitely many of the polynomials of best approximation to f do not have nonnnegative $(r+1)$-st derivatives on $[-1, +1]$.

1. **Introduction.** In [6]-[10] there appear many examples of functions f in $C[a, b]$ with nonnegative $(r+1)$-st divided differences there for which infinitely many of the polynomials of best approximation to f fail to have nonnegative $(r+1)$st derivatives. None of these examples has the $(r+1)$st divided differences uniformly bounded away from zero. In [11] Roulier shows that if $f \in C^{2r+2}[-1, +1]$ and if $f^{(r+1)}(x) \geqq \delta > 0$ on $[-1, 1]$ then for n sufficiently large the polynomial of best approximation of degree less than or equal to n has a positive $(r+1)$st derivative on $[-1, +1]$.

On the other hand for the case $r = 0$ Roulier in [12] shows that first divided differences of f uniformly bounded away from zero is not sufficient to insure that for n sufficiently large the polynomial of best approximation to f is increasing on $[-1, 1]$.

In this paper we extend the results of [12] to the case when $r \geqq 0$. The proofs are similar to those in [12] but make use of higher order divided differences and their properties.

2. **Notation and preliminary concepts.** For $n = 0, 1, 2, \cdots$ define H_n to be the set of all algebraic polynomials of degree less than or equal to n. For $f \in C[a, b]$, let

$$||f|| = \sup \{|f(x)|: a \leqq x \leqq b\} .$$

We define the degree of approximation to f to be

$$E_n(f) = \inf \{||f - p||: p \in H_n\} ,$$

$n = 0, 1, 2, \cdots$. It is well-known that there is a unique $p_n \in H_n$ for which $||f - p_n|| = E_n(f)$. This p_n is called the *polynomial of best approximation to f on $[a, b]$ from H_n*. Unless specifically stated otherwise we will restrict ourselves to the interval $[-1, +1]$.

Define C^* *to be the class of continuous 2π-periodic functions* and H_n^* the trigonometric polynomials of degree n or less. Then

$E_n^*(f)$ is defined for $f \in C^*$ as *the degree of approximation to* f *by trigonometric polynomials* from H_n^*. That is,

$$E_n^*(f) = \inf\{\|f - T\|^*: T \in H_n^*\}$$

where

$$\|f\|^* = \sup\{|f(x)|: -\pi \leqq x \leqq \pi\}.$$

If $I = [-1, 1]$ or $I = [-\pi, \pi]$ and $f \in C[-1, +1]$ or $f \in C^*$ we define *the r-th modulus of smoothness* $\omega_r(f, h) = \sup\{|\Delta_t^r f(x)|: |t| \leqq h$ and $rh \leqq |I|\}$, where $\Delta_t^1 f(x) = f(x + t) - f(x)$ and $\Delta_t^r f(x) = \Delta_t^1(\Delta_t^{r-1} f(x))$, and $|I|$ is the length of I.

If $r = 1$ then $\omega_r(f, h)$ is called the *modulus of continuity* of f and is written $\omega(f, h)$.

Estimates for $E_n(f)$ are intimately related to $\omega_r(f, h)$ by the theorems of D. Jackson. These theorems are well-known and will not be given here. See [5].

As in [3] let $f[x_0, \cdots, x_r]$ denote the rth order divided difference of f. It is well-known that if $f \in C^r[x_0, x_r]$ and $x_0 < x_1 < \cdots < x_r$ then there is ξ in (x_0, x_r) for which

$$f^{(r)}(\xi) = r!\, f[x_0, \cdots, x_r].$$

It is also well-known that if all $(r + 1)$st order divided differences of f are nonnegative in $[-1, +1]$ then $f \in C^{r-1}(-1, +1)$. See [2].

In the following sections, p_n will always denote the polynomial from H_n of best approximation to f on the appropriate interval.

3. The main theorems.

The following theorems treat the situations where all $(r + 1)$st order divided differences of f are bounded away from zero on $[-1, +1]$ and $f \in C^{r-1}[-1, +1]$ or $f \in C^r[-1, +1]$. The first two theorems and their corollaries show that for all functions with nonnegative $(r + 1)$st order divided differences for which $E_n(f)$ does not get small too fast there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geqq 0$ on $[-1, +1]$. The last two theorems show that this will also occur for some functions with $(r + 1)$st order divided differences bounded away from zero even if $E_n(f)$ does get small faster than allowed in the first two theorems.

THEOREM 3.1. *Let* $f \in C[-1, 1]$ *have bounded* rth *order divided differences* (if $f \in C^r[-1, 1]$, then this happens) *and nonnegative* $(r + 1)$st *order divided differences on* $[-1, +1]$. *Assume that there is no* $C > 0$ *for which*

$$E_n(f) \leqq C/(n + 1)^{r+1} \textit{ for } n = 0, 1, \cdots.$$

Then there are infinitely many n *for which we do not have* $p_n^{\langle r+1\rangle}(x) \geqq 0$ *on* $[-1, +1]$.

COROLLARY 3.1(a). *Let* $f \in C^r[-1, +1]$ *and assume that* f *has nonnegative* $(r+1)$st *order divided differences on* $[-1, +1]$. *Define* $g(t) = f(\text{cost})$. *Assume that*

$$\limsup_{k\to\infty} k^{r+1}\omega_{r+1}\left(g, \frac{1}{k}\right)/\log k = +\infty . \tag{1}$$

Then there are infinitely many n *for which we do not have* $p_n^{\langle r+1\rangle}(x) \geqq 0$ *on* $[-1, +1]$.

COROLLARY 3.1(b). *If* f *has nonnegative* $(r+1)$st *order divided differences on* $(-1-\epsilon, 1+\epsilon)$ *for some* $\epsilon > 0$ *and if there is no* $C > 0$ *for which*

$$E_n(f) \leqq C/(n+1)^{r+1} \quad \textit{for} \quad n = 0, 1, \cdots$$

then there are infinitely many n *for which we do not have*

$$p_n^{\langle r+1\rangle}(x) \geqq 0 \quad \textit{on} \quad [-1, +1] .$$

THEOREM 3.2. *Let* $f \in C^{r-1}[-1, +1]$ *and assume that* f *has nonnegative* $(r+1)$st *order divided differences. Assume that there is no* $C > 0$ *for which*

$$E_n(f) \leqq C/(n+1)^r \quad \textit{for} \quad n = 0, 1, \cdots .$$

Then there are infinitely many n *for which we do not have* $p_n^{\langle r+1\rangle}(x) \geqq 0$ *on* $[-1, +1]$.

COROLLARY 3.2. *Let* $f \in C^{r-1}[-1, +1]$ *and assume that* f *has nonnegative* $(r+1)$st *order divided differences. Define*

$$g(t) = f(\cos t) .$$

Assume that

$$\limsup_{k\to\infty} k^r\omega_r\left(g, \frac{1}{k}\right)/\log k = +\infty . \tag{2}$$

Then there are infinitely many n *for which we do not have* $p_n^{\langle r+1\rangle}(x) \geqq 0$ *on* $[-1, +1]$.

THEOREM 3.3. *For each integer* $r \geqq 0$ *and modulus of continuity* ω *there exists* $f \in C^r[-1, +1]$ *with*

(3) $$f[x_0, \cdots, x_{r+1}] \geqq \delta > 0 \text{ for all } x_0 < \cdots < x_{r+1}$$

in $[-1, +1]$ *and with*

(4) $$\omega(h) \leqq \omega(f^{(r)}, h) \leqq K\omega(h)$$

and yet there are infinitely many n *for which we do not have* $p_n^{\langle r+1\rangle}(x) \geqq 0$.

THEOREM 3.4. *For each integer* $r \geqq 1$ *and modulus of continuity* ω *there exists* $f \in C^{r-1}[-1, +1]$ *with*

(5) $$f[x_0, \cdots, x_{r+1}] \geqq \delta > 0 \text{ for all } x_0 < \cdots < x_{r+1}$$

in $[-1, +1]$ *and with*

$$\omega(h) \leqq \omega(f^{(r-1)}, h) \leqq K\omega(h)$$

and yet there are infinitely many n *for which we do not have* $p_n^{\langle r+1\rangle}(x) \geqq 0$.

4. Proofs of the main theorems. We first state some known lemmas. The first lemma is due to Steckin [13] and is found in [5] page 59.

LEMMA 4.1. *There exist constants* M_p, $p = 1, 2, \cdots$, *such that for each* $f \in C^*$

(6) $$\omega_p(f, h) \leqq M_p h^p \sum_{0 \leqq n \leqq h^{-1}} (n+1)^{p-1} E_n^*(f) .$$

LEMMA 4.2. *Let* $f \in C[-1, +1]$ *and define* $g \in C^*$ *by* $g(t) = f(\cos t)$. *If*

(7) $$\limsup_{k \to \infty} k^{r+1} \omega_{r+1}\left(g, \frac{1}{k}\right) / \log k = +\infty ,$$

then there does not exist $M > 0$ *for which*

$$E_n(f) \leqq M/(n+1)^{r+1} , \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

Proof. Assume such a constant M exists. Then $E_n^*(g) = E_n(f) \leqq M/(n+1)^{r+1}$ for $n = 0, 1, \cdots$. Now use Lemma 4.1 with $h = 1/N$. This gives

$$\omega_{r+1}(g, 1/N) \leqq \frac{A_r}{N^{r+1}} \sum_{n=0}^{N} \frac{1}{n+1} \leqq \frac{K_r \log N}{N^{r+1}} .$$

Hence

$$N^{r+1}\omega_{r+1}(g, 1/N)/\log N \leqq K_r .$$

This is a contradiction.

The next lemma is stated in [12] and is a simple consequence of a theorem of Kadec [4].

LEMMA 4.3. *Let $f \in C[-1, +1]$ and for each $n = 0, 1, 2, \cdots$ let $x_{0,n} < \cdots < x_{n+1,n}$ be a Chebyshev alternation for f.*

Let $\delta_n = \max_{0 \leqq k \leqq n+1} |x_{k,n} - \cos(k\pi/(n+1))|$. Then there is a sequence $\{n_j\}_{j=0}^{\infty}$ of positive integers for which

$$\lim_{j\to\infty} \delta_{n_j} = 0 .$$

The next lemma is found in [5] page 45.

LEMMA 4.4. *Let ω be any modulus of continuity. Then there is a concave modulus of continuity $\bar{\omega}$ with the same domain of definition as ω for which*

$$\frac{1}{2}\bar{\omega}(h) \leqq \omega(h) \leqq \bar{\omega}(h) . \tag{8}$$

The next lemma is well-known. We first define for $r = 1, 2, \cdots$

$$x_+^r = \begin{cases} 0 & \text{for} \quad x \leqq 0 \\ x^r & \text{for} \quad x > 0 . \end{cases} \tag{9}$$

LEMMA 4.5. *There is a constant $C_r > 0$ for which*

$$E_n(x_+^r) \geqq C_r/(n+1)^r . \tag{10}$$

Proof. This is an easy consequence of a theorem of S. N. Bernstein [1].

LEMMA 4.6. *If there are m non-overlapping intervals $I_1, \cdots, I_m$ contained in $[a, b]$ each with length $l_i i = 1, \cdots, m$ respectively, then for each positive integer l there must be at least $[m(l-1)/l]$ intervals I_i for which $l_i \leqq (l(b-a)/m)$.*

Proof. The proof of this is elementary and is omitted.

LEMMA 4.7. *Let $m \geqq 2$ be an integer and let $z_0 < z_1 < \cdots < z_m$ be given. Define $h[z_0, \cdots, z_m] = \sum_{j=0}^{m} \prod_{k=0, k\neq j}^{m} |z_j - z_k|^{-1}$. Then*

$$(z_m - z_0)h[z_0, \cdots, z_m] \geqq (m+1)(z_m - z_0)^{-m+1} \tag{11}$$

$$(12) \qquad (z_m - z_0)(z_m - z_1)h[z_0, \cdots, z_m] \geqq (z_m - z_0)^{-m+2}$$

$$(13) \qquad (z_m - z_0)(z_{m-1} - z_0)h[z_0, \cdots, z_m] \geqq (z_m - z_0)^{-m+2} .$$

Proof. The proof of (11) is easy. The proofs of (12) and (13) are obtained by considering the terms $j = 1$ and $j = 0$ in the sum respectively.

LEMMA 4.8. *If* $f[x_0, \cdots, x_{r+1}] \geqq 0$ *for all* $x_0 < \cdots < x_{r+1}$ in $[-1 - \epsilon, 1 + \epsilon]$ *for some* $\epsilon > 0$ *then* $f[t_0, \cdots, t_r]$ *is bounded on* $[-1, +1]$.

Proof. Use the above mentioned result in [2] that

$$f \in C^{r-1}(-1 - \epsilon, 1 + \epsilon)$$

and therefore that $f^{(r-1)}$ is convex on $(-1 - \epsilon, 1 + \epsilon)$.

We now proceed with the proof of Theorem 3.1 and its corollaries. Let f have bounded rth order divided differences and nonnegative $(r + 1)$-st order divided differences on $[-1, +1]$. Assume that for n sufficiently large we have $p_n^{(r+1)}(x) \geqq 0$ on $[-1, +1]$. We will show that this gives a constant $M > 0$ for which

$$E_n(f) \leqq M/(n + 1)^{r+1} \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

This will give Theorem 3.1. Corollary 3.1(a) will then follow from Theorem 3.1 and Lemma 4.2. Corollary 3.1(b) follows from Theorem 3.1 and Lemma 4.8.

Proof of Theorem 3.1. Let $n \geqq r$ and let $x_0 < x_1 < \cdots < x_{n+1}$ be a Chebyshev alternation for f. Assume that there is a positive integer N so that for all $n \geqq N$ we have $p_n^{(r+1)}(x) \geqq 0$ on $[-1, +1]$, and let $n \geqq N$.

Now

$$f(x_i) = p_n(x_i) + \varepsilon(-1)^i E_n(f)$$

for $i = 0, 1, \cdots, n + 1$ where $\varepsilon = \pm 1$ is fixed relative to i. Let g be any function which satisfies

$$g(x_i) = (-1)^i \quad \text{for} \quad i = 0, 1, \cdots, n + 1 .$$

Then

$$(14) \qquad f(x_i) = p_n(x_i) + \varepsilon E_n(f)g(x_i)$$

for $i = 0, 1, 2, \cdots, n + 1$.

From [3] p. 247 we have the identity

$$F[x_0, \cdots, x_m] = \sum_{j=0}^{m} F(x_j) \prod_{\substack{k=0\\k\neq j}}^{m} (x_j - x_k)^{-1} . \tag{15}$$

If $i + r + 1 \leqq n + 1$ we have

$$g[x_i, \cdots, x_{i+r+1}] = \sum_{j=0}^{r+1} (-1)^{i+j} \prod_{\substack{k=0\\k\neq j}}^{r+1} (x_{i+j} - x_{i+k})^{-1} . \tag{16}$$

We note that all terms in the sum on the right of (16) have the same sign. If ε is as in (14) and if

$$(-1)^i \varepsilon \prod_{k=1}^{r+1} (x_i - x_{i+k})^{-1} > 0 \tag{17}$$

we have from (16)

$$\varepsilon g[x_i, \cdots, x_{i+r+1}] = h[x_i, \cdots, x_{i+r+1}] \tag{18}$$

where h is as in Lemma 4.7.

From (11) and (17) we have

$$\varepsilon(x_{i+r+1} - x_i) g[x_i, \cdots, x_{i+r+1}] \geqq (r + 2)(x_{i+r+1} - x_i)^{-r} . \tag{19}$$

Now using (14), (17), and (19) and the assumption that $p[x_i, \cdots, x_{i+r+1}] \geqq 0$ we have

$$(x_{i+r+1} - x_i) f[x_i, \cdots, x_{i+r+1}] \geqq E_n(f)(x_{i+r+1} - x_i)^{-r}(r + 2) . \tag{20}$$

There are at least $t_n = [(n - r + 1)/2]$ points x_i in $[-1, +1]$ for which (17) holds. We now consider non-overlapping sets $\{x_i, \cdots, x_{i+r+1}\}$ where (17) holds for x_i. There are at least

$$m = \left[\frac{t_n}{r + 2}\right]$$

such sets, and by Lemma 4.6 there are at least $[m/2]$ such sets with $x_{i+r+1} - x_i \leqq 4/m$. It is clear that there is a constant $K > 0$ for which

$$\frac{4}{m} \leqq \frac{K}{n} \quad \text{for} \quad m \geqq 1 . \tag{21}$$

Thus $x_{i+r+1} - x_i \leqq K/n$ for n sufficiently large.

Now we sum (20) over all such sets and use this to get

$$K_1\left[\frac{m}{2}\right]\left(\frac{n}{K}\right)^r E_n(f) \leqq \sum_i (x_{i+r+1} - x_i) f[x_i, \cdots, x_{i+r+1}] . \tag{22}$$

Clearly there is $K_2 > 0$ for which

$$E_n(f) \leqq \frac{K_2}{n^{r+1}} \sum_i (x_{i+r+1} - x_i) f[x_i, \cdots, x_{i+r+1}] \tag{23}$$
$$= \frac{K_2}{n^{r+1}} \sum_i (f[x_{i+1}, \cdots, x_{i+r+1}] - f[x_i, \cdots, x_{i+r}])$$
$$\leqq \frac{2K_2 M^*}{n^{r+1}}$$

where $M^* = \max\{|f[t_0, \cdots, t_r]| : -1 \leqq t_0 < \cdots < t_r \leqq 1\}$. This proves Theorem 3.1.

For the proof of Theorem 3.2 we use (12) and (13) and the fact that $f^{(r-1)}$ is of bounded variation. The proof proceeds as above except that $f[x_i, \cdots, x_{i+r+1}]$ is written in terms of $(r-1)$st order divided differences and therefore in terms of $f^{(r-1)}$. We omit the details here.

Corollary 3.2 is a simple consequence of Lemma 4.2 and Theorem 3.2.

For the proof of Theorems 3.3 and 3.4 we may as well assume that ω is concave in view of (8). The proofs will be done simultaneously. We will work on $[-2, 2]$ here instead of on $[-1, 1]$.

Proofs of Theorem 3.3 *and Theorem* 3.4. Let $\varepsilon > 0$ be given and let ω be any concave modulus of continuity. Define

$$g(x) = \begin{cases} \varepsilon(x^2 + 5x + 1) & \text{on } [-2, -1] \\ (x-1)^2 + |x| + (5 + 3\varepsilon)x & \text{on } [-1, +1] \\ 3(2+\varepsilon)x^2 + \omega(1) - \omega(2-x) & \text{on } [1, 2]\,. \end{cases}$$

g is easily seen to be continuous, increasing, and convex on $[-2, 2]$. Moreover, $g'(0)$ does not exist.

Let g_r be an rth order integral of g. Then $g_r \in C^r[-2, 2]$ and

$$g_r[t_0, \cdots t_{r+1}] \geqq \frac{\varepsilon}{(r+1)!}$$

for

$$-2 \leqq t_0 < \cdots < t_{r+1} \leqq 2$$

and

$$g_r[t_0, \cdots, t_{r+2}] \geqq \frac{2\varepsilon}{(r+2)!}$$

for

$$-2 \leqq t_0 < \cdots < t_{r+1} < t_{r+2} \leqq 2 .$$

We will show that there are infinitely many n for which we do not have $p_n^{\langle r+1 \rangle}(x) \geqq 0$ on $[-2, +2]$ and infinitely many n for which we do not have $p_n^{\langle r+2 \rangle}(x) \geqq 0$ on $[-2, +2]$, where p_n is the polynomial from H_n of best approximation to g_r. This will be sufficient for the proofs of both theorems in view of the fact that for $0 \leqq h \leqq 1$

$$\omega(h) \leqq \omega(g, h) \leqq K\omega(h) , \tag{24}$$

which is easy to show. The proof of (24) is essentially the same as the proof of (16) in [12]. It is easy to see that on $[-1, +1]$ we have $g_r(x) = Cx_+^{r+1} + Dq_r(x)$ where $q_r \in H_{r+2}$, and where C depends only on r. In view of this and Lemma 4.5 we have

$$E_n(g_r) \geqq \frac{K_r}{(n+1)^{r+1}} \quad \text{for} \quad n = 0, 1, \cdots , \tag{25}$$

where K_r depends only on r.

If $-2 \leqq t_0 < \cdots < t_{r+1} \leqq -1$ then

$$g_r[t_0, \cdots, t_{r+1}] \leqq \frac{3\varepsilon}{(r+1)!} \tag{26}$$

and if $-2 \leqq t_0 < \cdots < t_{r+2} \leqq -1$ then

$$g_r[t_0, \cdots, t_{r+2}] = \frac{2\varepsilon}{(r+2)!} . \tag{27}$$

Now assume that $p_n^{\langle r+1 \rangle}(x) \geqq 0$ on $[-2, +2]$ for n sufficiently large. Then as in the proof of Theorem 3.1 we choose a Chebyshev alternation for such n

$$-2 \leqq x_0 < x_1 < \cdots < x_{n+1} \leqq 2$$

and for g_r and obtain

$$g_r[x_i, \cdots, x_{i+r+1}] \geqq \sigma E_n(g_r) y[x_i, \cdots, x_{i+r+1}] \tag{28}$$

where $\sigma = \pm 1$ is independent of i, and y is any function for which $y(x_i) = (-1)^i i = 0, 1, \cdots, n+1$.

Now by Lemma 4.3 there is a sequence $\{n_j\}_{j=0}^{\infty}$ for which $\lim_{j \to \infty} \delta_{n_j} = 0$. Thus for j sufficiently large 1/4 of the $n_j + 2$ Chebyshev alternation points for g_r lie in $[-2, -1]$. Thus there is a constant K depending only on r such that for j sufficiently large there are $r+2$ alternation points $x_i, \cdots, x_{i+r+1}$ in $[-2, -1]$ with

$$x_{i+r+1} - x_i \leqq \frac{K}{n_j + 1} \tag{29}$$

and for which

$$\sigma y[x_i, \cdots, x_{i+r+1}] \geqq 0 . \tag{30}$$

An application of (11) now gives

$$\sigma y[x_i, \cdots, x_{i+r+1}] \geqq \frac{(r+2)}{K^{r+1}}(n_j + 1)^{r+1} . \tag{31}$$

Thus from (26), (28), and (31) we get for j sufficiently large

$$E_{n_j}(g_r) \leqq \frac{K^{r+1}}{(r+2)!} \cdot 3\varepsilon\left(\frac{1}{(n_j+1)^{r+1}}\right) . \tag{32}$$

This together with (25) gives

$$K_r \leqq \frac{3K^{r+1}}{(r+2)!}\varepsilon .$$

But for ε sufficiently small this can easily be violated. Thus we have a contradiction.

To show that we cannot have $p_n^{\langle r+2 \rangle}(x) \geqq 0$ for n sufficiently large we proceed in similar fashion. We use (27) and obtain a sequence $\{n_j\}_{j=0}^{\infty}$ for which

$$E_{n_j}(g_r) \leqq \frac{2C_r^{r+2}}{(r+3)!} \cdot \frac{\varepsilon}{(n_j+1)^{r+2}} . \tag{33}$$

This together with (25) gives an obvious contradiction. We omit the proof of (33) since it is the same as the proof of (32).

We remark that the existence of a $g \in C[-2, 2]$ such that (24) holds implies the existence of $A > 1$, $B > 0$ such that

$$\omega(h) \leqq \omega(Ag, h) \leqq B\omega(h) ,$$

for $0 \leqq h \leqq 4$. Thus both theorems are proven.

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