PACIFIC JOURNAL OF MATHEMATICS
Vol 66, No. 1, 1976

QUOTIENT-UNIVERSAL SEQUENTIAL SPACES

R. SIROIS-DUMAIS AND S. WILLARD

We produce 2^c mutually nonhomeomorphic countable sequential spaces. These are used

(1) to answer in the negative the following question of Michael and Stone [4]: is every regular T_1-space which is a quotient of some separable metric space and a continuous image of the space P of irrationals a quotient of P?

(2) to characterize c (with or without the continuum hypothesis) as the smallest cardinal κ with the property that a metric space of cardinality κ exists of which every sequential space of cardinality $\leqq \kappa$ is a quotient.

1. Introduction. We let Q denote the space of rationals, **P** the space of irrationals, **R** the real line, and **c** the cardinality of **R**. For any set X, the cardinality of X is denoted $|X|$.

We begin with the basic construction, which will be applied in the sequel in two different directions. Denote by Y the set $[Q \times (Q - \{0\})] \cup \{\infty\}$ and, for $E \subseteq \mathbf{R}$, denote by τ_E the quotient topology induced on Y by the obvious map from the subspace $[Q \times (Q - \{0\})] \cup (E \times \{0\})$ of $\mathbf{R} \times \mathbf{R}$. The set Y endowed with the topology τ_E will be denoted Y_E. Note that Y_E is a countable, regular, T_1-space which is, by construction, the quotient of a separable metric space. (Thus, see [3], Y_E is both an $\aleph_0$-space and a k-space.)

2. Quotients of P. In [4], Michael and Stone establish that every metrizable continuous image of **P** is a quotient of **P**. The question is raised there whether this result can be extended to nonmetrizable images of **P**, that is, whether a regular T_1-space which is at the same time a quotient of some separable metric space and a continuous image of **P** must be a quotient of **P**. The construction of §1 provides the negative answer. To see this, first note that the countable discrete space (hence, every countable space) is a continuous image of **P** (collapse each interval $(n, n+1)$ to a point). It follows that each space Y_E is a regular T_1-space which is a continuous image of **P** and a quotient of some separable metric space. But:

THEOREM. *Not every space Y_E is a quotient of* **P**.

Proof. If E and F are distinct subsets of **R**, the topologies τ_E and τ_F on Y are different, one containing a set containing ∞ which does not belong to the other.

Now let S be the set of all surjections $f\colon \boldsymbol{P} \to Y$ such that each $f^{-1}(y)$, $y \in Y$, is closed in $\mathbf{P}$, and let Φ be the set of all $\phi\colon Y \to 2^{\mathbf{P}}$, where $2^{\mathbf{P}}$ denotes the collection of closed subsets of $\mathbf{P}$. Then $f \to f^{-1}$ is a one-one map from S into Φ; since $|\Phi| = \mathbf{c}^{\aleph_0} = \mathbf{c}$, we have $|S| \leqq \mathbf{c}$. Let J be the set of all T_1 topologies τ on Y such that (Y, τ) is a quotient image of $\mathbf{P}$. Then each $\tau \in J$ is generated by some $f \in S$, so $|J| \leqq \mathbf{c}$. Since $|\{\tau_E \mid E \subseteq \mathbf{R}\}| = 2^{\mathbf{c}}$, and since each τ_E is T_1, it follows that (Y, τ_E) is not a quotient of $\mathbf{P}$ for some $E \subseteq \mathbf{R}$.

NOTES. (1) From the above, it is easily seen that there are $2^{\mathbf{c}}$ nonhomeomorphic spaces Y_E, at most $\mathbf{c}$ of which can be quotients of $\mathbf{P}$. This result can be sharpened, with some difficulty. In fact, *Y_E is a quotient* of $\mathbf{P}$ *iff E is an analytic subset of* $\mathbf{R}$.

(2) If, in the construction of Y, the set $Q \times (Q - \{0\})$ is replaced by a discrete space, say $\{(k/n, 1/n) \mid k, n \in \mathbf{N}\}$, the spaces Y_E which result still work, and have now the additional property that each has only one nonisolated point.

3. Quotient-universal sequential spaces. Let κ be an infinite cardinal and let $S(\kappa)$ denote the collection of all sequential spaces of cardinality $\leqq \kappa$. A sequential space S is *quotient-universal** for $S(\kappa)$ if $S \in S(\kappa)$ and every $T \in S(\kappa)$ is a quotient of S. We are particularly interested in the existence of metrizable quotient-universal spaces for $S(\kappa)$.

Whenever $\kappa^{\aleph_0} = \kappa$, the disjoint union of κ copies of the converging sequence will serve as a metrizable quotient-universal space for $S(\kappa)$. In particular, there is a metrizable quotient-universal space for $S(\mathbf{c})$. In this section, we use the construction of §1 to demonstrate that, whether or not the continuum hypothesis is true, $\mathbf{c}$ is the smallest cardinal for which this is true. In fact, we exhibit *a countable sequential space which is not a quotient of any metric space of cardinality* $< \mathbf{c}$.

LEMMA. *There exists a subset E of* $\mathbf{R}$ *with* $|E| = \mathbf{c}$ *which contains no uncountable closed subset of* $\mathbf{R}$.

Proof. Let $\{C_\alpha \mid \alpha < \mathbf{c}\}$ be a transfinite enumeration of the $\mathbf{c}$ uncountable closed subsets of $\mathbf{R}$. Pick p_0 and q_0 in C_0 with $p_0 \neq q_0$. If p_α and q_α have been chosen in C_α for $\alpha < \beta$ so that all p_α and q_α are distinct, choose p_β and q_β in C_β so that $p_\beta \neq q_\beta$ and p_β, q_β are distinct from all p_α, q_α for $\alpha < \beta$. This is possible since any uncountable closed subset of $\mathbf{R}$ has cardinal $\mathbf{c}$ so that $C_\beta - \{p_\alpha, q_\alpha \mid \alpha < \beta\} \neq \phi$.

* The term "universal" has been preempted by those who study spaces with a given property P which contain as subspaces every space (of appropriate cardinality or weight) having property P. See, for example, [2], [5] and [6].

Let $E = \{p_\alpha \mid \alpha < \mathbf{c}\}$. Then $|E| = \mathbf{c}$ and E contains no uncountable closed subset of $\mathbf{R}$ since $q_\alpha \in C_\alpha - E$ for each α.

Let $E \subseteq \mathbf{R}$ be the set of the lemma. Let M_E denote the subspace $[Q \times (Q - \{0\})] \cup (E \times \{0\})$ of $\mathbf{R} \times \mathbf{R}$. Recall that Y_E is the quotient of M_E obtained by collapsing $E \times \{0\}$ to a single point e. Let $q\colon M_E \to Y_E$ be the quotient map.

Y_E is a countable sequential space, but:

THEOREM. *Y_E is not the quotient of any metric space of cardinality $< \mathbf{c}$.*

Proof. Suppose there is a quotient map f of S onto Y_E, where S is a metric space and $|S| = \kappa < \mathbf{c}$. For each $p \in E$, let $\sigma_p = (x_{p1}, x_{p2}, \cdots)$ be a sequence in $Q \times (Q - \{0\})$ such that

$$|x_{pn} - (p, 0)| \leqq \min\left\{\frac{1}{n}, |x_{pn-1} - (p, 0)|\right\}.$$

Recall that q denotes the quotient map of M_E onto Y_E. For each n, let

$$z_{pn} = q(x_{pn})$$

and denote by η_p the sequence $(z_{p1}, z_{p2}, \cdots)$ in Y_E. Now $\eta_p \to e$. Hence, since f is a hereditary quotient map, there exists some $b_p \in f^{-1}(e)$ and a sequence $\sigma_p = (s_{p1}, s_{p2}, \cdots)$ in $S - f^{-1}(e)$ such that $\sigma_p \to b_p$ and $f(\sigma_p) = \eta_p$. Let

$$f^{-1}(e) = \{x_\alpha \mid \alpha < \kappa\}$$

and, for $\alpha < \kappa$, let

$$A_\alpha = \{p \in E \mid b_p = x_\alpha\}.$$

We claim some A_α must contain a sequence (p_i) converging to some element of $\mathbf{R} - E$. For otherwise $C1_{\mathbf{R}}(A_\alpha) \subset E$ for each $\alpha < \kappa$, whence E is the union of fewer than $\mathbf{c}$ closed sets. But since $|E| = \mathbf{c}$, one of these would be an uncountable closed set in E, contradicting the construction of E.

Without loss of generality, say A_1 contains a sequence (p_i) which is closed and discrete in E. Then the sequence $\eta_{p_i} = (z_{p_i1}, z_{p_i2}, \cdots)$ converges to e, for each i, and the sequence $\delta_{p_i} = (s_{p_i1}, s_{p_i2}, \cdots)$ converges to x_1, for each i. A diagonal sequence $(s_{p_1n_1}, s_{p_2n_2}, \cdots)$ with $n_k \geqq k$ for each k will then converge to x_1. Then $(z_{p_1n_1}, z_{p_2n_2}, \cdots)$ converges to e. Hence $(x_{p_1n_1}, x_{p_2n_2}, \cdots)$ must have a cluster point in M_E.

But $|x_{p_kn_k} - (p_k, 0)| \leqq |x_{p_kk} - (p_k, 0)| \leqq 1/k$, so any cluster point of $(x_{p_1n_1}, x_{p_2n_2}, \cdots)$ in M_E would be a cluster point of $((p_1, 0), (p_2, 0), \cdots)$, which is impossible by choice of the p_i.

We conclude with some observations on extension of the result above.

(1) As noted in §2, there are $2^{\mathbf{c}}$ mutually nonhomeomorphic spaces Y_E. Since there are at most $\mathbf{c}$ quotients of any single countable sequential space, there can exist no quotient-universal space (metrizable or not) for $S(\aleph_0)$. It is at least consistent with the usual (Zermelo-Fraenkel) axioms for set theory (with Choice) that this result extends to all cardinals $\kappa < \mathbf{c}$, for Martin's axiom entails $2^{\kappa} < 2^{\mathbf{c}}$ for $\kappa < \mathbf{c}$.

(2) Let $M(\kappa)$ denote the collection of metrizable spaces of cardinal $\leqq \kappa$. The space Q of rationals is a (metrizable) quotient-universal space for $M(\aleph_0)$, while the disjoint union of $\mathbf{c}$ copies of the converging sequence is a quotient-universal space for $M(\mathbf{c})$. For cardinals κ between $\aleph_0$ and $\mathbf{c}$ little is known. Baumgartner (**[1]**) has shown that it is consistent with Zermelo-Fraenkel set theory with choice that all $\aleph_1$-dense subsets of $\mathbf{R}$ are order-isomorphic. (A subset A of $\mathbf{R}$ is $\aleph_1$-*dense* if whenever $a < b$ in $\mathbf{R}$, $(a, b) \cap A$ has cardinal $\aleph_1$.) If this is the case, then every *separable* metric space M of cardinal $\leqq \aleph_1$ is a quotient of the unique $\aleph_1$-dense subset D of $\mathbf{R}$. For M is a quotient of $M \times D$, while (**[7]**, Theorem 76) $M \times D$ is homeomorphic to a subset of $\mathbf{R}$ and hence, by Baumgartner's result, to D.

REFERENCES

1. James E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math., **79** (1973), 103–106.

2. Jonsson, *Universal Relational Systems*, Math. Scand., **4** (1956), 193–208.

3. E. Michael, *$\aleph_0$-spaces*, J. Math. Mech., **15** (1966), 983–1002.

4. E. Michael and A. H. Stone, *Quotients of the space of irrationals*, Pacific J. Math., **28** (1969), 629–633.

5. R. Pol, *There is no universal totally disconnected space*, Fund. Math., **79** (1973), 265–267.

6. W. Sierpinski, *Sur les espaces métriques universels*, Fund. Math., **33** (1945), 123–136.

7. ———, *General Topology*, 2nd ed., U. of Toronto Press, 1956 (translated by C. C. Krieger).

Received March 12, 1976 and in revised form May 12, 1976.

UNIVERSITY OF ALBERTA