

A LIMIT-POINT CRITERION FOR EXPRESSIONS WITH OSCILLATORY COEFFICIENTS

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A criterion is given for the equation $-(py)' + qy = 0$ to have a solution on the interval $[a, \infty)$ which is not in $L_2(a, \infty)$. The criterion permits q (or $\operatorname{Re} q$ if q is complex-valued) to be decomposable into a sum $q = q_1 + q_2$ where the expression $-(py)' + q_1 y$ essentially satisfies the well-known limit-point criterion of Levinson and q_2 may be thought of as an oscillatory function whose amplitude may be large, but whose integral over $[a, x]$ increases relatively slowly as a function of x .

1. Introduction. Let p be a positive function and let q be a complex-valued function on the interval $[a, \infty)$, $a > -\infty$, such that $1/p$ and q are Lebesgue integrable on each finite interval $[a, b]$. Let M be the differential expression

$$(1.1) \quad M(y) = -(py)' + qy.$$

Then we shall obtain conditions on p and q sufficient for the equation $M(y) = 0$ to have a solution which is not in $L_2(a, \infty)$.

The conditions, stated precisely in Theorem 1 at the beginning of §2, place no restriction on the imaginary part of q . For real-valued q they extend the well-known limit point criterion of Levinson [8]. (See also Coddington and Levinson [3; page 229].) Roughly, for a given p , Levinson's condition limits the rate at which $q(x)$ can approach $-\infty$ as $x \rightarrow \infty$. Theorem 1 extends this by allowing q to be decomposable into a sum $q = q_1 + q_2$ where q_1 satisfies a condition very similar to Levinson's, and where the integral of q_2 over $[a, x]$ grows relatively slowly as a function of x .

A simple example of such an expression is

$$(1.2) \quad -(xy)' - (x + xe^x \sin(e^x))y.$$

A slightly more complicated one is

$$(1.3) \quad -y'' - (x + x^3(\sin x)^4 + x^5 \sin(x^6))y.$$

We shall return to these examples in §3.

Another very well-known limit-point criterion, due to Hartman and Wintner [5], states for $p = 1$ and q real-valued that if the negative part,

q^- , of q satisfies $\int_a^x q^- dt \leq Kx^3$ for all sufficiently large x , then (1.1) is limit-point. Their proof, which relies on an analysis of the frequency of the zeros of solutions of the equation $M(y) = 0$, has been refined by Eastham [4] to yield a criterion dependent only on estimates of the integral of q^- over a suitable sequence of pairwise disjoint intervals. We shall show in §4 that this interval criterion is a consequence of Theorem 1 by deriving an interval criterion for (1.1) that reduces to Eastham's result for $p = 1$.

For $p = 1$ and real-valued q one form of our result is essentially equivalent to a recent limit-point criterion of Knowles [7]. This equivalence, which is not entirely obvious, is demonstrated in §5.

Theorem 1 resembles Theorem 1 of Atkinson and Evans [1] in permitting complex-valued q and also the possibility of considering square-integrability with respect to a weight function. However, the conditions on $\operatorname{Re} q$ in the two theorems are independent.

Finally we note that the hypotheses of our theorem are largely invariant under the addition to (1.1) of a term Qy where the integral of Q is small in a suitable sense. We illustrate this in §6 with an extension of a theorem of Patula and Wong [9].

2. The main result.

THEOREM 1. *Suppose that there exists a nonnegative locally absolutely continuous function w on $[a, \infty)$ such that*

- (i) $pw'^2 \leq K_1^2$ a.e.,
- (ii) $\int_a^\infty wp^{-1/2} dt = \infty$,

$\operatorname{Re} q = q_1 + q_2$ where

- (iii) $-q_1 w^2 \leq K_2$,
- (iv) $\left| \int_a^x q_2 w^{1-b} dt \right| \{w(x)\}^b \{p(x)\}^{-1/2} \leq K_3$ for some b , $0 \leq b \leq 1$ and

all $x \geq a$.

Then the equation $-(py)' + qy = 0$ has a solution which is not of integrable square on $[a, \infty)$.

REMARKS. 1. If $q = q_1$ is real-valued, $q_2 = 0$, and if w is positive, then the change of notation $M = w^{-2}$ gives Levinson's limit-point criterion as stated in [3; page 229].

2. In general, if q is real-valued, then the conclusion of Theorem 1 may be rephrased to the statement that the expression (1.1) is limit-point. When q is complex-valued this may still be done in some circumstances (for instance if $\operatorname{Im} q$ is semi-bounded) provided the definition of the limit-point condition given in Kauffman [6] is used.

3. If k is a positive locally integrable function which is not in $L_1(a, \infty)$, then Theorem 1 can easily be adapted to a criterion for the equation $-(py')' + qy = 0$ to possess a solution y such that $\int_a^\infty |y|^2 k dt = \infty$. The right sides of (i), (iii), and (iv) then become $K_1^2 k$, $K_2 k$, and $K_3 k^{1/2}$ respectively and (ii) becomes $\int_a^\infty w(k/p)^{1/2} dt = \infty$. The manner in which the weight function enters (i), (ii), and (iii) is the same as in Theorem 1 of [1]. We shall comment at the end of the proof of Theorem 1 on the small alterations necessary to deal with this version.

Proof. Let y and z be solutions of $M(y) = 0$ such that

$$(2.1) \quad p(y'z - yz') = 1.$$

Set

$$r = (|y|^2 + |z|^2)^{1/2}.$$

We must show that r is not of integrable square. A straightforward calculation establishes that r satisfies the equation

$$(2.2) \quad (pr')' - (\operatorname{Re} q)r = S$$

where

$$S = (p/r^3)[(|y'|^2 + |z'|^2)(|y|^2 + |z|^2) - (\operatorname{Re}(y'\bar{y} + z'\bar{z}))^2].$$

We assert that $S \geq 1/2pr^3$. To see this, let u_1, u_2, u_3, u_4 be real-valued functions such that $y = u_1 + iu_2$, and $z = u_3 + iu_4$. Then

$$S = (p/r^3) \sum_{j < k} (u'_j u_k - u_j u'_k)^2.$$

The real part of (2.1) may be written as

$$p[(u'_1 u_3 - u_1 u'_3) - (u'_2 u_4 - u_2 u'_4)] = 1$$

Thus,

$$\begin{aligned} S &\geq (p/r^3)[(u'_1 u_3 - u_1 u'_3)^2 + (u'_2 u_4 - u_2 u'_4)^2] \\ &\geq (p/r^3)(1/2p^2) = 1/2pr^3. \end{aligned}$$

Now let w be as in the hypotheses of Theorem 1. By inserting the estimate just obtained for S into (2.2), multiplying by rw^2 , and integrating, we obtain that for each positive x ,

$$(2.3) \quad \int_a^x (pr')'rw^2 dt - \int_a^x q_1r^2w^2 dt - \int_a^x q_2r^2w^2 dt \\ \cong \int_a^x w^2/2pr^2 dt.$$

Suppose that $\|r\|^2 = \int_a^\infty r^2 dt$ is finite. Then by Schwarz's inequality,

$$\int_a^x w^2/2pr^2 dt \cong (1/2\|r\|^2) \left(\int_a^x wp^{-1/2} dt \right)^2.$$

Hence, by (ii), $\int_a^x w^2/2pr^2 dt \rightarrow \infty$ as $x \rightarrow \infty$. We shall see that this leads to a contradiction.

Set $H(x) = \int_a^x p(r')^2w^2 dt$. An integration by parts yields that

$$\int_a^x (pr')'rw^2 dt = prr'w^2 \Big|_a^x - H(x) - 2 \int_a^x prr'ww' dt.$$

By (i) and Schwarz's inequality again,

$$\int_a^x prr'ww' dt \cong K_1 \|r\| \{H(x)\}^{1/2}.$$

From another integration by parts,

$$(2.4) \quad \int_a^x q_2r^2w^2 dt = \{r(x)\}^2 \{w(x)\}^{1+b} \int_a^x q_2w^{1-b} dt - I(x)$$

where

$$I(x) = \int_a^x \left(\int_a^t q_2w^{1-b} ds \right) [2rr'w^{1+b} + (1+b)r^2w^bw'] dt.$$

Now

$$|I(x)| \cong 2K_3 \int_a^x p^{1/2}rr'w dt + (1+b)K_3 \int_a^x p^{1/2}r^2w' dt \\ \cong 2K_3 \|r\| \{H(x)\}^{1/2} + (1+b)K_1K_3 \|r\|^2.$$

The other term on the right side of (2.4) satisfies

$$\{r(x)\}^2 \{w(x)\}^{1+b} \left| \int_a^x q_2w^{1-b} dt \right| \cong K_3 \{r(x)\}^2 w(x) \{p(x)\}^{1/2}.$$

Finally, $\int_a^x q_1r^2w^2 dt \cong K_2 \|r\|^2$.

Thus (2.3) may be rewritten in the form

$$(2.5) \quad A(x) - [H(x) - K_4\{H(x)\}^{1/2} - K_5] \geq \int_a^x w^2/2pr^2 dt$$

where K_4 and K_5 are positive constants and

$$A = pw^2rr' + K_3p^{1/2}wr^2.$$

The expression in brackets on the left side of (2.5) either is bounded (if H is) or approaches $+\infty$ as $x \rightarrow \infty$ (if H does). In either case we must have that $A(x) \rightarrow +\infty$ as $x \rightarrow \infty$. We complete the proof by showing that this cannot occur.

Set $f = p^{1/2}w$ and $g = r^2$. Then $2A = f^2g' + 2K_3fg$. Note that g is in $L_1(a, \infty)$. We now use this decomposition of A to show that A cannot even be bounded away from 0 on an interval $[d, \infty)$. For suppose that $A(x) \geq c > 0$ for $x \geq d$. Set $N = \{x \geq d : g'(x) < 0\}$. Then on N , $2K_3fg > |f^2g'|$ and also $K_3fg > c$ so that $(K_3/c)g > 1/f$. Combining these yields that

$$\int_N -g'/g dt < (2K_3^2/c) \int_N g dt < \infty.$$

Hence $\log(g(x)/g(d)) > -(1/2c) \int_N g dt$ so that g is bounded away from 0. But this is impossible since g is in $L_1(a, \infty)$. Thus A cannot be bounded away from 0 and, in particular, cannot satisfy (2.5) for all positive x . Thus it must be the case that r is not of integrable square and the proof is complete.

We now comment briefly on the changes necessary to incorporate a weight function k as mentioned in Remark 3 above. Inequality (2.5) can be derived essentially as before under the assumption that $\int_a^\infty r^2k dt$ is finite, where now $2A = f^2g' + 2K_3fgk^{1/2}$ (f and g are as before). Then $A(x) \geq c$ for $x \geq d$ implies that on $N = \{x \geq d : g'(x) < 0\}$, $(2K_3^2/c)gk > -g'/g$ and one again has the contradiction that g is bounded away from 0. On the other hand it follows as before from Schwarz's inequality and the new form of (ii) that the right side of (2.5) approaches ∞ as $x \rightarrow \infty$. Thus again (2.5) cannot hold for all positive x and the conclusion follows as before.

3. Examples. In this section we return to the examples (1.2) and (1.3) mentioned in the introduction. For (1.2), set $q_1(x) = x$, $q_2(x) =$

$xe^x \sin(e^x)$, and $w(x) = x^{-1/2}$. Then $w(x)p(x)^{-1/2} = x^{-1}$ and $\int_1^x q_2 dt = -x \cos(e^x) + \cos e$. Thus the hypotheses of Theorem 1 are clearly satisfied on $[1, \infty)$ with $b = 1$. Note that with the same simple choice for w one could substitute $q_2(x) = x^d \sin(x^d)$, since it is easy to see that then $\left| \int_1^x q_2 dt \right| \leq Kx$.

To deal with (1.3) we must choose w more carefully. Here we take $q_1(x) = x + x^3(\sin x)^4$, and $q_2(x) = x^5 \sin(x^6)$. This example, with $q = q_1$, has been discussed by Eastham [4]. Inspired by his discussion we set $I_n = [n\pi - n^{-1/2}, n\pi + n^{-1/2}]$, $n = 1, 2, \dots$. Define w on I_n by $w(x) = x - n\pi + n^{-1/2}$ for $n\pi - n^{-1/2} \leq x \leq n\pi$ and $w(x) = w(2n\pi - x)$ on the rest of I_n . On the complement of the union of the I_n 's define $w(x) = 0$. On I_n , $w(x) \leq n^{-1/2}$, $q_1(x) \leq Kn$, and $\left| \int_0^x q_2 dt \right| \leq 1/6$. Thus (i), (iii), and (iv) are satisfied. Finally, $\int_{I_n} w dt = 1/n$ so (ii) is also satisfied and it follows from Theorem 1 that (1.3) is limit-point.

4. An interval criterion. We shall now establish a limit-point criterion for (1.1) (with q real-valued) in which the coefficients are restricted only on a sequence of pairwise disjoint intervals. The argument is a refinement of that used in the previous section to treat (1.3). For $p = 1$ the result is due to Eastham [4] by a quite different method. Thus Theorem 1 may be regarded as a step toward the integration of the "interval criteria" and the criteria of Levinson type into a common theory.

THEOREM 2. *Suppose that there is a sequence $\{I_n\}_{n=1}^\infty$, $I_n = [a_n, b_n]$, of pairwise disjoint intervals in $[a, \infty)$ and a sequence $\{v_n\}_{n=1}^\infty$ of positive numbers such that for each n ,*

- (i) $v_n P_n^2 \geq K > 0$ where $P_n = \int_{a_n}^{b_n} p^{-1/2} dt$,
- (ii) $\sum_{n=1}^\infty v_n^{-1} = \infty$,
- (iii) $\int_{a_n}^{b_n} q^- dt \leq C v_n^2 P_n^3 \min_{I_n} p^{1/2}$ where q^- is the negative part of q .

Then $-(py)' + qy$ is limit-point.

Proof. We may assume that $K \geq 1$. Divide each I_n into $[v_n P_n^2] + 1$ subintervals ($[\dots]$ is the greatest integer function) so that on each subinterval J ,

$$(2v_n P_n)^{-1} \leq \int_J p^{-1/2} dt \leq (v_n P_n)^{-1}.$$

On at least half of these,

$$(4.1) \quad \int_J q^- dt \leq 2Cv_n P_n \min_{I_n} p^{1/2}.$$

For each subinterval $J = [c, d] \subset I_n$ on which (4.1) is valid, choose $e \in (c, d)$ so that

$$\int_c^e p^{-1/2} dt = \left(\int_J p^{-1/2} dt \right) / 2.$$

Define w on each such subinterval by $w(t) = \int_c^t p^{-1/2} dt$ for $c \leq t \leq e$ and $w(t) = w(e) - \int_e^t p^{-1/2} dt$ for $e \leq t \leq d$. Then on J ,

$$\max w = (1/2) \int_J p^{-1/2} dt \leq (2v_n P_n)^{-1},$$

and

$$\int_J w p^{-1/2} dt = (1/4) \left(\int_J p^{-1/2} dt \right)^2 \geq (4v_n P_n)^{-2}.$$

Set $w = 0$ on the remaining subintervals of each I_n and on the complement of $\cup I_n$.

Then for each n , since (4.1) holds on at least $(1/2)([v_n P_n^2] + 1)$ subintervals of I_n ,

$$\int_{I_n} w p^{-1/2} dt \geq (4v_n P_n)^{-2} (1/2) v_n P_n^2 = (1/32) v_n^{-1}.$$

Hence by (ii), $\int_a^\infty w p^{-1/2} dt = \infty$. Also w is clearly absolutely continuous with $p(w')^2 \leq 1$ a.e.. Thus (i) and (ii) of Theorem 1 are satisfied. It remains to construct a suitable decomposition of q .

Define a step function q_0 which is constant on each of the subintervals $J = [c, d]$ constructed above (whether (4.1) holds or not) by

$$q_0 = \int_c^d q^- dt / (d - c)$$

on $[c, d]$.

Similarly, on the interval $[b_n, a_{n+1}]$ between I_n and I_{n+1} , set $q_0 = \int_{b_n}^{a_{n+1}} q^- dt / (a_{n+1} - b_n)$. Thus $[a, \infty)$ is the union of subintervals on each of which $\int (q^- - q_0) dt = 0$. Note that $q_0 w^2$ is bounded above on

$[a, \infty)$. This is clear outside the union of the subintervals where (4.1) holds, for then $w = 0$. On a subinterval $J = [c, d] \subset I_n$ for which (4.1) is valid,

$$q_0 w^2 \leq 2Cv_n P_n \min_{I_n} p^{1/2} / (2v_n P_n)^2 (d - c) \leq C$$

since $\min p^{1/2} / (d - c) \leq \left(\int_J p^{-1/2} \right)^{-1} \leq 2v_n P_n$.

Now decompose q by setting $q_1 = q^+ - q_0$, and $q_2 = -q^- + q_0$. From the previous paragraph, $-q_1 w^2 \leq C$ on $[a, \infty)$. Also $w p^{-1/2}(x) \int_a^x q_2 dt$ is nonzero only in the subintervals on which (4.1) holds, and on such a subinterval $[c, d]$, $\int_a^x q_2 dt = \int_c^x q_2 dt$ since $\int_a^c q_2 dt = 0$. Hence on such a subinterval

$$w p^{-1/2}(x) \left| \int_a^x q_2 dt \right| \leq 2Cv_n P_n \min_{I_n} p^{1/2} / 2v_n P_n p^{1/2}(x) \leq C.$$

Thus (iii) and (iv) of Theorem 1 are also satisfied for this decomposition and Theorem 2 now follows from that result.

5. A theorem of Knowles. In this section we shall derive from Theorem 1 the following result of Knowles [7] which was established by a refinement of a method due to Brinck [2]. Note that w is required here to be strictly positive (because of (iii)) rather than nonnegative as in Theorem 1.

THEOREM 3. *Suppose that there is a positive locally absolutely continuous function w such that*

(i) w' is bounded a.e.,

(ii) $\int_a^\infty w dt = \infty$,

(iii) *there exists a constant C such that $-\int_I qw dt \leq C$ for each*

interval I for which $\int_I w^{-1} dt \leq 1$.

Then $-y'' + qy$ is limit-point.

Proof. Hypotheses (i) and (ii) are identical to those of Theorem 1 for $p = 1$. It remains to consider (iii). For this we need the equivalence described in the following lemma. The author is grateful to Professor W. N. Everitt for bringing this equivalence to his attention.

LEMMA. Let μ be a positive continuous regular Borel measure on $[a, \infty)$, and let Q be a real-valued function defined on this interval. Then the following properties are equivalent.

(i) There is a constant C such that $-\int_I Q d\mu \leq C$ for each interval I for which $\mu(I) \leq 1$.

(ii) $Q = Q_1 + Q_2$ where $-Q_1 \leq C_1$ and $\left| \int_a^x Q_2 d\mu \right| \leq C_2$ for all $x \geq a$.

Proof of Lemma. It is clear that (ii) implies (i) with $C = C_1 + 2C_2$ since for any interval $[c, d]$,

$$-\int_c^d Q_2 d\mu = \int_a^c Q_2 d\mu - \int_a^d Q_2 d\mu \leq 2C_2.$$

For the other direction, set $x_0 = a$ and for each n choose x_n so that $\mu([x_{n-1}, x_n]) = 1$. If $\mu([x_{n-1}, \infty)) < 1$, set $x_n = \infty$. The proof will not be affected by whether the sequence $\{x_n\}$ is finite or infinite. Suppose that on each $[x_{n-1}, x_n]$ we can write $Q = Q_1 + Q_2$ where $-Q_1 \leq C$, $\left| \int_a^x Q_2 d\mu \right| \leq C$ for all $x_{n-1} \leq x \leq x_n$, and $\int_{x_{n-1}}^{x_n} Q_2 d\mu = 0$. Then, combining the decompositions, we will have that $-Q_1 \leq C$ on $[a, \infty)$ and that if $x_{n-1} < x \leq x_n$, then

$$\left| \int_a^x Q_2 d\mu \right| = \left| \int_{x_{n-1}}^x Q_2 d\mu \right| \leq C.$$

Thus the lemma will be proved.

To simplify notation slightly we consider $[a, x_1]$. Let $\int_a^{x_1} Q d\mu = \alpha C$. Note that (i) then implies that $\int_a^x Q d\mu \leq (\alpha + 1)C$ for all $x \leq x_1$. Suppose first that $\alpha \leq 0$. Define $Q_1(x) = \alpha C$, $a \leq x \leq x_1$, and $Q_2 = Q - Q_1$. Then $-Q_1 \leq C$ and for $a \leq x \leq x_1$, $\int_a^x Q_2 d\mu$ satisfies

$$\begin{aligned} -C &\leq \int_a^x Q d\mu \leq \int_a^x Q_2 d\mu = \int_a^x Q d\mu - \alpha C \mu([a, x]) \\ &\leq (\alpha + 1)C - \alpha C = C. \end{aligned}$$

Now suppose that $\alpha > 0$. Let m be the integer such that $m < \alpha \leq m + 1$. For $j = 0, 1, \dots, m$, set

$$a_j = \max \left\{ x < x_1 : \int_a^x Q d\mu = jC \right\}.$$

Set $a_{m+1} = x_1$. Then $a \leq a_0 < a_1 < \dots < a_{m+1}$. Choose $b_j > a_j$, $j = 0, 1, \dots, m-1$ so that

$$\int_{a_j}^{b_j} Q^+ d\mu = C$$

and b_m so that $\int_{a_m}^{b_m} Q^+ d\mu = (\alpha - m)C$. Here $Q^+ = \max\{Q, 0\}$ as usual. Note that $a_j < b_j \leq a_{j+1}$ and that $Q^- = Q^+ - Q$ satisfies $\int_{a_j}^{b_j} Q^- d\mu < C$.

Define $Q_1 = Q^+$ on $\bigcup_{j=0}^m [a_j, b_j]$, $Q_1 = 0$ otherwise, and $Q_2 = Q - Q_1$. Then $Q_1 \geq 0$ so we need only verify that Q_2 has the required property. If $x < a_0$, then $Q_2 = Q$ where

$$-C \leq \int_a^x Q d\mu \leq C.$$

The last inequality follows from the fact that $\int_a^{a_0} Q d\mu = 0$. If $a_j \leq x \leq b_j$, $j = 0, 1, \dots, m$, then $\int_a^x Q_2 d\mu = \int_{a_j}^x Q_2 d\mu$ and

$$-C < \int_{a_j}^{b_j} Q_2 d\mu \leq \int_{a_j}^x Q_2 d\mu \leq 0$$

since $Q_2 = Q - Q^+$ on $[a_j, b_j]$.

If $b_j \leq x \leq a_{j+1}$, $j \leq m-1$, then

$$\int_a^x Q_2 d\mu = \int_a^x Q d\mu - (j+1)C \leq (j+2)C - (j+1)C = C.$$

Similarly, if $b_m \leq x \leq a_{m+1}$, then $\int_a^x Q_2 d\mu = \int_a^x Q d\mu - \alpha C \leq C$.

Finally, for $b_j \leq x \leq a_{j+1}$,

$$\int_a^x Q_2 d\mu = \int_{a_j}^{b_j} Q_2 d\mu + \int_{b_j}^x Q d\mu \geq -C$$

since otherwise $\int_a^x Q d\mu = jC$ for some $x > a_j$. Thus the decomposition

$Q = Q_1 + Q_2$ has the property (ii) with $C_1 = C_2 = C$ and the proof of the lemma is complete.

We now conclude the proof of Theorem 3. Applying the lemma with $Q = qw^2$ and $d\mu = w^{-1} dt$, we have that $q = q_1 + q_2$ where $-q_1 w^2 \leq C_1$ and $\left| \int_a^x q_2 w dt \right| \leq C_2$. This is (iii) and (iv) of Theorem 1 with $b = 0$. Thus Theorem 3 is a consequence of Theorem 1.

6. A perturbation result. There have been some investigations recently into whether the limit-point property for a differential expression is preserved under the addition of a term Qy where the integral of Q is small in some sense (generally Q is assumed in $L_r(a, \infty)$ for some $r \geq 1$). In this direction we offer the following extension of a result of Patula and Wong [9].

THEOREM 4. *Suppose that (1.1) satisfies the hypotheses of Theorem 1 with $b = 1$ and that p is bounded away from 0. Suppose that Q is a locally integrable function such that for some $r \geq 1$ and all $x \geq a$,*

$$(6.1) \quad \int_x^{x+1} |Q|^r dt \leq K.$$

Then $-(py)' + (q + Q)y$ satisfies the hypotheses of Theorem 1 with $b = 1$.

Proof. We may assume that Q is real-valued. (In fact the result is true if Q is replaced by $\text{Re } Q$ in (6.1).) It then follows from (6.1) and Schwarz's inequality that $\int_x^{x+d} Q dt \geq -K^{1/r}$ for each $x \geq a$ and each positive $d \leq 1$. Thus by the lemma of §5 with $d\mu = dt$, $Q = Q_1 + Q_2$ with $-Q_1 \leq C_1$ and $\left| \int_a^x Q_2 dt \right| \leq C_2$ for all $x \geq a$. Then $-(q_1 + Q_1) \leq -q_1 + C_1$ and

$$\left| \int_a^x q_2 + Q_2 dt \right| \leq \left| \int_a^x q_2 dt \right| + C_2.$$

Let w be the nonnegative locally absolutely continuous function of Theorem 1 for (1.1). If w is bounded then it is clear from the previous paragraph that the hypotheses of Theorem 1 are also satisfied for the new expression with the same choice for w .

Suppose, then, that w is not bounded. Set

$$W(x) = w(x) / \left(1 + \int_a^x wp^{-1/2} dt \right).$$

We assert that W is bounded and has the other necessary properties. Suppose for some x_1 that $w(x_1) > 1$. Set $x_0 = \max \{x < x_1 : w(x) \leq 1\}$. Then on $[x_0, x_1]$, $|w'| \leq K_1 w p^{-1/2}$ so that

$$w(x_1) \leq 1 + K_1 \int_a^{x_1} w p^{-1/2} dt.$$

This inequality also holds if $w(x_1) \leq 1$. Thus $\limsup W \leq K_1$.

We now verify the hypotheses of Theorem 1. For $x \geq a$,

$$W'(x) = w'(x) / \left(1 + \int_a^x w p^{-1/2} dt\right) - \{W(x)\}^2 \{p(x)\}^{-1/2}$$

so that $p^{1/2} W'$ is bounded. Also

$$\int_a^x W p^{-1/2} dt = \log \left(1 + \int_a^x w p^{-1/2} dt\right)$$

so that this integral tends to infinity with x .

Finally, $W \leq w$ so that

$$-(q_1 + Q_1)W^2 \leq K_2 - Q_1 W^2 \leq K_2 + C_1(\max W)^2$$

and

$$\{p(x)\}^{-1/2} W(x) \left| \int_a^x q_2 + Q_2 dt \right| \leq K_3 + C_2 \max \{p^{-1/2} W\}.$$

Thus the hypotheses of Theorem 1 are satisfied for $-(py') + (q + Q)y$ and the proof is complete.

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