

INSERTION OF A CONTINUOUS FUNCTION

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Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a continuous function between two comparable real valued functions with a certain pair of a general class of properties. The class of properties is defined by being preserved when added to a continuous function and by being possessed by any constant function.

A property P defined relative to a real valued function on a topological space is a C property provided any constant function has property P and provided the sum of a function with property P and any continuous function also has property P . If P_1 and P_2 are C properties, the following terminology is used: (i) A space X has the *weak C insertion property for (P_1, P_2)* if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a continuous function h on X such that $g \leq h \leq f$. (ii) A space X has the *C insertion property for (P_1, P_2)* if and only if for any functions g and f on X such that $g < f$, g has property P_1 and f has property P_2 , then there exists a continuous function h on X such that $g < h < f$. (iii) A space X has the *strong C insertion property for (P_1, P_2)* if and only if for any functions g and f on X such that $g \leq f$, then there exists a continuous function h on X such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X , then $g(x) < h(x) < f(x)$. If a space X has the weak C insertion property for (P_1, P_2) , necessary and sufficient conditions in terms of lower cut sets are given for the space to have the C insertion property for (P_1, P_2) . For a space with the weak C insertion property, a sufficient condition for the space to have the strong C insertion property is given. In certain situations, this condition is necessary. Several insertion theorems, some of which are known, are obtained as corollaries of these two results.

1. Examples of C properties and C insertion. The following are examples of C properties: lower semicontinuous (lsc), upper semicontinuous (usc), normal lsc, normal usc, continuity, measurable, almost continuous (in the sense of Husain [5]), almost continuous (in the sense of Singal and Singal [12]), Baire class one. If f_* denotes the lower limit functions of f , then f is normal lsc in case $f = (f^*)_*$ and f is normal usc in case $f = (f_*)^*$; see Dilworth [3] for amplification. Observe that any set of functions that satisfies a given C property contains the set of all continuous functions on the space.

The following examples of C insertion are known:

(a) A space X has the weak C insertion property for (usc, lsc) if and only if X is normal.

(b) A space X has the C insertion property for (usc, lsc) if and only if X is normal and countably paracompact.

(c) A space X has the strong C insertion property for (usc, lsc) if and only if X is perfectly normal.

(d) A space X has the C insertion property for (usc, continuous) if and only if X is a cb -space.

(e) A space X has the C insertion property for (normal usc, continuous) if and only if X is a weak- cb space.

(f) A space X has the weak C insertion property for (normal usc, normal lsc) if and only if X is mildly normal.

(g) A space X has the weak C insertion property for (normal usc, lsc) [respectively, (usc, normal lsc)] if and only if X is almost normal.

(h) A space X has the weak C insertion property for (lsc, usc) if and only if X is extremally disconnected.

Result (a) is due independently to Tong [13] and to Katětov [6]. Example (b) was proved by Katětov [6] and by Dowker [4]. Result (c) is a consequence of Theorem 3.1''', Example 1.2*, and Proposition 2.3 of Michael [11]. Theorem 1 (c) of Mack [9] gives example (d), and (e) is an immediate corollary of Theorem 3.1 (c) of Mack and Johnson [10]. Results (f), (g), and (h) are from Lane [8]. A space is mildly normal in case disjoint regular closed subsets are separated by disjoint open sets, and a space is called almost normal in case disjoint closed sets, at least one of which is regular closed, are separated by disjoint open sets.

If a space has the strong C insertion property for (P_1, P_2) , then it has the weak C insertion and the C insertion property for (P_1, P_2) . In order to see when the C insertion property implies the weak C insertion property, a technique used by Dieudonné [2] is employed to prove the following result.

THEOREM 1.1. *Let P_1 and P_2 be C properties and assume that X satisfies the C insertion property for (P_1, P_2) . If*

(a) *the transformation $f \rightarrow f/(1+|f|)$ preserves P_1 and P_2 , and*

(b) *if $\inf(f, h)$ has property P_2 and $\sup(g, h)$ has property P_1 whenever f has property P_2 , g has property P_1 and h is any continuous function,*

then X satisfies the weak C insertion property for (P_1, P_2) .

Proof. Let g and f be functions on X such that $g \leq f$, g has property P_1 and f has property P_2 . If $G = g/(1+|g|)$ and $F = f/(1+|f|)$, then G has property P_1 and F has property P_2 and $-1 < G \leq$

$F < 1$. If $G_0 = G - 1$ and if $F_0 = F - 1$, then by hypothesis there exists a continuous function h_0 on X such that $G_0 < h_0 < F_0$. Let

$$f_1 = \inf(F + 1/2, h_0 + 1/2), \quad g_1 = \sup(G - 1/2, h_0 - 1/2).$$

Then $g_1 < f_1$ and by hypothesis, g_1 has property P_1 and f_1 has property P_2 . Inductively, let

$$f_n = \inf(F + 1/2^n, h_{n-1} + 1/2^n), \quad g_n = \sup(G - 1/2^n, h_{n-1} - 1/2^n),$$

where h_{n-1} is continuous and $g_{n-1} < h_{n-1} < f_{n-1}$. Again by hypothesis there exists a continuous function h_n on X such that $g_n < h_n < f_n$. Since $h_n < f_n \leq h_{n-1} + 1/2^n$ and since $h_n > g_n \geq h_{n-1} - 1/2^n$, then $-1/2^n \leq h_n - h_{n-1} \leq 1/2^n$. Since $|h_n - h_{n-1}| \leq 1/2^n$, the sequence $\{h_n\}$ converges uniformly on X by the Cauchy condition. Thus there exists a continuous function H on X such that $\{h_n\}$ converges uniformly on X to H . Since

$$G - 1/2^n \leq g_n < h_n < f_n < F + 1/2^n$$

and since $\{h_n\}$ converges to H , it follows that $G \leq H \leq F$. If $h = H/(1 - |H|)$, then h is continuous and $g \leq h \leq f$. Thus X satisfies the weak C insertion property for (P_1, P_2) .

2. C insertion. If f is a real valued function defined on a space X and if

$$\{x \mid f(x) < r\} \subset A(f, r) \subset \{x \mid f(x) \leq r\},$$

for a real number r , then $A(f, r)$ is a *lower cut set* in the domain of f at the level r . This definition is due to Brooks [1], where the terminology lower indefinite cut set is used. A sufficient condition in terms of lower cut sets for a space to satisfy the weak C insertion property for arbitrary comparable functions was given by Lane [8]. The main result of this section uses lower cut sets and gives a necessary and sufficient condition for a space that satisfies the weak C insertion property to satisfy the C insertion property.

THEOREM 2.1. *Let X be a space that satisfies the weak C insertion property for C properties P_1 and P_2 . The space X has the C insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each n , $X - D_n$ and $A(f - g, 3^{-n+1})$ are completely separated.*

Proof. Assume that X has the weak C insertion property for C properties P_1 and P_2 . Let g and f be functions such that $g < f$, g has property P_1 and f has property P_2 . By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence $\{D_n\}$ such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and such that for each n , $X - D_n$ and $A(f - g, 3^{-n+1})$ are completely separated. Let k_n be a continuous function from X into $[0, 1]$ such that $k_n = 0$ on $A(f - g, 3^{-n+1})$ and $k_n = 1$ on $X - D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Weierstrass M -test the function k is continuous. Since $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and since $k_n = 1$ on $X - D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f - g, 3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If x is any point in X , then $x \notin A(f - g, 1)$ or for some n ,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $k(x) \leq 1/4(3^{-n}) \leq f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are C properties, g_1 has property P_1 and f_1 has property P_2 . Since X has the weak C insertion property for (P_1, P_2) , there exists a continuous function h on X such that $g_1 \leq h \leq f_1$. Since $g < h < f$, it follows that X satisfies the C insertion property for (P_1, P_2) . (The technique of this proof is from Katětov [6].)

Conversely, let g and f be functions on X such that g has property P_1 , f has property P_2 and $g < f$. By hypothesis, there exists a continuous function h on X such that $g < h < f$. Since the constant function 0 has property P_1 , since $f - h$ has property P_2 , and since X has the C insertion property for (P_1, P_2) , there exists a continuous function k on X such that $0 < k < f - h$. Let $A(f - g, 3^{-n+1})$ be any lower cut set for $f - g$ and let $D_n = \{x \mid k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Since

$$A(f - g, 3^{-n+1}) \subset \{x \mid (f - g)(x) \leq 3^{-n+1}\} \subset \{x \mid k(x) \leq 3^{-n+1}\}$$

and since $\{x \mid k(x) \leq 3^{-n+1}\}$ and $\{x \mid k(x) \geq 3^{-n+2}\} = X - D_n$ are completely separated by k , it follows that for each n , $A(f - g, 3^{-n+1})$ and $X - D_n$ are completely separated.

The first corollary of this theorem is one half of the well known characterization of normal and countably paracompact spaces which is due independently to Dowker [4] and to Katětov [6].

COROLLARY 2.2. *If X is normal and countably paracompact, then X has the C insertion property for (usc, lsc).*

Proof. Since X is normal, X has the weak C insertion property for (usc, lsc). (See Example (a) of section 1.) Let g and f be functions on X with $g < f$, g usc and f lsc. For each n , let

$$A(f - g, 3^{-n+1}) = \{x \mid (f - g)(x) \leq 3^{-n+1}\}.$$

Since $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of closed sets and since X is normal and countably paracompact, there exists a decreasing sequence $\{D_n\}$ of open subsets of X such that $D_n \supset A(f - g, 3^{-n+1})$ and such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Since X is normal, $A(f - g, 3^{-n+1})$ and $X - D_n$ are completely separated. Thus by Theorem 2.1, there exists a continuous function h on X such that $g < h < f$.

For the second corollary, the following result is needed. The proof, which is directly analogous to the proof of Theorem 2 in Dowker's paper [4], is omitted.

LEMMA 2.3. *If X is a cb -space and if $\{F_n\}$ is a decreasing sequence of closed sets with empty intersection, then there exists a sequence of regular open sets $\{H_n\}$ such that for each n , $F_n \subset H_n$ and such that $\bigcap_{n=1}^{\infty} \bar{H}_n = \emptyset$.*

COROLLARY 2.4. *A space X has the C insertion property for (usc, normal lsc) [respectively (normal usc, lsc)] if X is almost normal and a cb -space.*

Proof. The proof is given for (usc, normal lsc). Since X is almost normal, X has the weak C insertion property for (usc, normal lsc) (Lane [8]). If g and f are functions on X such that $g < f$, g is usc, and f is normal lsc, and if

$$A(f - g, 3^{-n+1}) = \{x \mid (f - g)(x) \leq 3^{-n+1}\},$$

then $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of closed lower cut sets with empty intersection. Since X is a cb -space, it follows from Lemma 2.3 that there exists a decreasing sequence $\{H_n\}$ of regular open subsets of X such that $A(f - g, 3^{-n+1})$ is contained in H_n and such that $\bigcap_{n=1}^{\infty} \bar{H}_n = \emptyset$. Since X is almost normal, $A(f - g, 3^{-n+1})$ and the regular closed set $X - H_n$ are completely separated. Thus $A(f - g, 3^{-n+1})$ and $X - \bar{H}_n$ are completely separated. By Theorem 2.1, there exists a continuous function h on X such that $g < h < f$.

The proof of the next corollary is omitted since the technique is the same as in the preceding two corollaries. The main result needed is that

an extremally disconnected space has the weak C insertion property for (lsc, usc) (Lane [8]).

COROLLARY 2.5. *A space X has the C insertion property for (lsc, usc) if X is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of closed subsets of X with vacuous intersection such that $F_n \supset G_n$ for each n .*

In each of the preceding three corollaries, the sufficient condition for C insertion is also a necessary condition.

3. Strong C insertion.

THEOREM 3.1. *Let P_1 and P_2 be C properties and consider the following condition:*

(α) *If g and f are functions on X such that $g \leq f$, g satisfies property P_1 and f satisfies property P_2 , then there exists a sequence $\{A(f - g, 2^{-n})\}$ of lower cut sets for $f - g$ and there exists a sequence $\{F_n\}$ of subsets of X such that*

- (i) $\{x \mid (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$, and
- (ii) *for each n the sets $A(f - g, 2^{-n})$ and F_n are completely separated.*

If X satisfies the weak C insertion property for (P_1, P_2) and if X satisfies condition (α), then X satisfies the strong C insertion property for (P_1, P_2) . Conversely, if X satisfies the strong C insertion property for (P_1, P_2) and if $f - g$ satisfies property P_1 , then X satisfies condition (α).

Proof. Suppose that there is a sequence $\{A(f - g, 2^{-n})\}$ of lower cut sets for $f - g$ and suppose there is a sequence $\{F_n\}$ of subsets of X such that

$$\{x \mid (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n there exists a continuous function k_n from X into $[0, 2^{-n}]$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f - g, 2^{-n})$. The function k from X into $[0, 1/4]$ which is defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} k_n(x)$$

is continuous, (1) $k^{-1}(0) = \{x \mid (f - g)(x) = 0\}$ and (2) if $(f - g)(x) > 0$ then $k(x) < (f - g)(x)$: In order to verify (1), observe that if $(f - g)(x) =$

0, then $x \in A(f - g, 2^{-n})$ for each n and hence $k_n(x) = 0$ for each n . Thus $k(x) = 0$. Conversely, if $(f - g)(x) > 0$, then there exists an n such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \mid (f - g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f - g, 2^{-n})$$

and that $\{A(f - g, 2^{-n})\}$ is a decreasing sequence. Thus if $(f - g)(x) > 0$ then either $x \notin A(f - g, 1/2)$ or there exists a smallest n such that $x \notin A(f - g, 2^{-n})$ and $x \in A(f - g, 2^{-j})$ for $j = 1, \dots, n - 1$. In the former case,

$$k(x) = 1/2 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f - g)(x),$$

and in the latter,

$$k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \leq 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f - g)(x).$$

Thus $0 \leq k \leq f - g$ and if $(f - g)(x) > 0$ then $(f - g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and let $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$ then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are C properties, g_1 has property P_1 and f_1 has property P_2 . Since X has the weak C insertion property for (P_1, P_2) by hypothesis, there exists a continuous function h on X such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Therefore X has the strong C insertion property for (P_1, P_2) .

Conversely, assume that X satisfies the strong C insertion property for (P_1, P_2) and assume that g and f are functions on X such that $g \leq f$ and such that $f - g$ satisfies property P_1 . There exists a continuous function h on X such that $0 \leq h \leq f - g$ and such that if $(f - g)(x) > 0$ then

$$(f - g)(x) > h(x) > 0.$$

For each n , let

$$A(f - g, 2^{-n}) = \{x \mid (f - g)(x) \leq 2^{-n}\},$$

$$F_n = \{x \mid h(x) \geq 2^{-n+1}\}, \text{ and}$$

$$k_n = \sup\{\inf\{h, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Then $\{x \mid (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and k_n is a continuous function from X into $[0, 2^{-n}]$ which completely separates F_n and $A(f - g, 2^{-n})$.

The first corollary of this theorem is the result of Michael [11] which was listed above.

COROLLARY 3.2. *A space X satisfies the strong C insertion property for (usc, lsc) if and only if X is perfectly normal.*

Proof. The proof of the sufficiency of the insertion property for perfect normality is omitted. To see how the necessity of the condition follows from Theorem 3.1, suppose that X is perfectly normal and let f and g be functions on X such that $g \leq f$, g is usc and f is lsc. Since X is normal, X satisfies the weak C insertion property for (P_1, P_2) . Since $f - g$ is lsc, the lower cut sets for $f - g$ defined by

$$A(f - g, 2^{-n}) = \{x \mid (f - g)(x) \leq 2^{-n}\}$$

are closed sets. Since X is perfectly normal, there exists a sequence $\{F_n\}$ of closed subsets of X such that

$$\{x \mid (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n , F_n and $A(f - g, 2^{-n})$ are completely separated. Hence by Theorem 3.1 X satisfies the strong C insertion property for (usc, lsc).

COROLLARY 3.3. *A space X satisfies the strong C insertion property for (usc, normal lsc) [respectively, (normal usc, lsc)] if and only if X is almost normal and if each closed subset of X is an intersection of a sequence of regular open sets.*

Proof. Assume that X is almost normal and assume that each closed subset of X is an intersection of a sequence of regular open sets. Since X is almost normal, X satisfies the weak C insertion property for (usc, normal lsc) as noted in Example (g) above. Let g and f be functions on X such that $g \leq f$, and such that g is usc and f is normal lsc. Since $f - g$ is lsc, the lower cut set for $f - g$ defined by

$$A(f - g, 2^{-n}) = \{x \mid (f - g)(x) \leq 2^{-n}\}$$

is a closed set for each n . By hypothesis, each open subset of X is a union of a sequence of regular closed sets; thus it is possible to choose a

sequence of regular closed sets $\{F_n\}$ such that for each n , F_n and $A(f - g, 2^{-n})$ are disjoint and such that

$$\{x \mid (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

Since X is almost normal, the regular closed set F_n is completely separated from the closed set $A(f - g, 2^{-n})$ by Theorem 3.5 of Lane [7]. Thus it follows from Theorem 3.1 that X satisfies the strong C insertion property for (usc, normal lsc).

If X satisfies the strong C insertion property for (usc, normal lsc) then X satisfies the weak C insertion property for (usc, normal lsc) and by Example (g) above, X is almost normal. In order to see that each closed subset of X is an intersection of a sequence of regular open sets, let F be a closed subset of X . Let $f = 1$ on X , and let $g = 1$ on F and $g = 0$ on $X - F$. Since $g \leq f$, and since g is usc and f is normal lsc, there is a continuous function h on X such that $g \leq h \leq f$ and such that if $x \in X - F$ then $0 < h(x) < 1$. If G_n is the interior of the closed set $\{x \mid h(x) \geq 1 - 1/n\}$, then $\{G_n\}$ is a sequence of regular open sets whose intersection is F .

The proof for the (normal usc, lsc) is analogous and is omitted.

The proofs of the following two results are essentially the same as the preceding corollary and are not given. Note that in both cases, the space trivially satisfies the weak C insertion property.

COROLLARY 3.3. *An almost normal space X satisfies the strong C insertion property for (usc, continuous) [respectively, (continuous, lsc)] if and only if each closed subset of X is an intersection of a sequence of regular open sets.*

COROLLARY 3.4. *If X is almost normal and if each closed subset of X is an intersection of a sequence of regular open sets, then X satisfies the strong C insertion property for (normal usc, continuous) [respectively, (continuous, normal lsc)].*

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