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CONDITIONS FOR SIMULTANEOUS APPROXIMATION AND INTERPOLATION WITH NORM PRESERVATION IN $C[a, b]$

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This paper gives necessary and sufficient conditions for a triple (X, M, Γ) to have property SAIN (simultaneous approximation and interpolation which preserves the norm), X being an arbitrary Banach space. The best previous result concerned X, a reflexive, rotund Banach space. The paper proceeds to use this result to yield geometric proofs of the work of D. J. Johnson concerning property SAIN and $C[a, b]$.

The concept of simultaneous approximation and interpolation with norm preservation (SAIN) was introduced by F. Deutsch and P. D. Morris [2].

DEFINITION 0.1. The triple (X, M, Γ) satisfies the hypotheses of the SAIN problem if X is a normed linear space, M a dense subspace of X, and Γ a finite dimensional subspace of X^*. The triple (X, M, Γ) has a SAIN solution at x in X if given $\epsilon > 0$, there exists y in M such that $\|x - y\| < \epsilon$, $\|x\| = \|y\|$, and $\gamma(x) = \gamma(y)$ for every γ in Γ. The triple (X, M, Γ) is said to have property SAIN if (X, M, Γ) has a SAIN solution for every x in X.

The papers **[4]** and **[10]** took a geometrical approach to the SAIN problem and along with **[9]** extended the concept by allowing M to be a dense convex subset of X. In **[7, 8]**, specific questions posed in **[2, 4]** were resolved in the standard Banach spaces ℓ_1, L_1 and $C(T)$. In **[5]**, a weak SAIN problem was formulated.

In **[2]**, a necessary and sufficient condition was given for (X, M, Γ) to have property SAIN, where X was a Hilbert space. In **[4]**, such a condition was presented for X a reflexive and rotund Banach space. One of the purposes of this paper is to give necessary and sufficient condition for (X, M, Γ) to have property SAIN, where X is an arbitrary Banach space.

The approach to the problem will be a geometrical attack similar to those in **[4, 7, 8]**. The study of certain extremal subsets of the unit sphere will yield the result.

Significant work by D. Johnson **[6]** in $C[a, b]$ was the key in formulating the correct statement of the desired condition. The work in **[6]** was analytic and constructive in nature. The hope was to find geometrical proofs for these results. These proofs were found and they led to the formulation of the necessary and sufficient condition for property SAIN. The second purpose of this paper is to give these shorter alternate proofs of the results in **[6]** and to provide alternate conditions to see if the triple $(C[a,b],\Pi,\Gamma)$ has property SAIN.

In this paper, we will use the following notation. If X is a Banach space, $U(X)$ and $S(X)$ are respectively, the closed unit ball and its boundary in X. R denotes the real number field. A set E contained in F is F-extremal if whenever $tx+(1-t)y \in E$ with x, y in F, $0<t<1$, then x, y are in E. The convex hull of a set A is denoted $\mathrm{co}(A)$. If $\Gamma \subset X^*$, then $\Gamma_\perp = \{x \in X \mid \gamma(x)=0 \ \forall \gamma \in \Gamma\}$. The extreme points of a set A will be denoted $\mathrm{ext}\, A$.

If $f \in C[a,b]$, $\mathrm{crit}(f) = \{t \in [a,b] \mid |f(t)| = \|f\|\}$. The extreme points of $U(C[a,b]^*)$ are the point evaluation functionals e_t, defined via $e_t(f) = f(t)$.

1. Property SAIN in arbitrary Banach spaces.

The following definitions were stated in **[1]** and **[8]**:

Definition 1.1. If $x \in S(X)$, then the set $E(x) = \{y \in S(X) \mid x = \lambda y + (1-\lambda)z,\ 0<\lambda<1,\ z \in S(X)\}$ is the minimal $U(X)$ extremal subset containing x, the intersection of all $U(X)$ extremal subsets containing x. $F(x)$ is the minimal closed $U(X)$ extremal subset containing x. The set $P(x) = \{\varphi \in S(X^*) \mid \varphi(x) = \|x\|\}$ is called the conjugate set of x. $Q(x) = \{y \in S(X) \mid \varphi(y) = 1 \ \forall \varphi \in P(x)\}$ is the intersection of all exposed sets containing x.

It was shown in 8, Theorem 1.1 that if (X, M, Γ) is a given triple and if $F(x) \cap M$ is dense in $F(x)$, then there is a SAIN solution at x. It will be shown that by looking at larger extremal sets we can obtain necessary and sufficient conditions for a SAIN solution at x.

Definition 1.2. Let A be a convex subset of $S(X)$. Let $E(A)$ be the minimal $U(X)$ extremal subset containing A and $F(A)$ be the minimal closed $U(X)$ extremal set containing A. Further let $P(A) = \bigcap_{a \in A} P(a)$. Define $Q(A) = \{y \in S(X) \mid \varphi(y) = 1 \ \forall \varphi \in P(A)\}$.

It is necessary to check that $P(A)$ is a nonempty set, since otherwise the definition of $Q(A)$ is meaningless.

Lemma 1.3. *The set $P(A)$ is nonempty.*

Proof. If a_0 is in A, then $P(a_0)$ is a weak star compact convex subset of $S(X^*)$. The sets $P(a_i) \cap P(a_0)$, $i \in I$, a finite index set, have the finite intersection property. This can be seen since if b is in $\mathrm{co}(a_i, a_0 \mid i \in I)$

then $b = \Sigma_{i \in I} t_i a_i + (1 - \Sigma_{i \in I} t_i) a_0$. Thus if φ is in $P(b)$ then φ is in $\bigcap_{i \in I}(P(a_i) \cap P(a_0))$. But $P(a_0)$ is a compact topological space and $\bigcap_{a \in A} P(a)$ is nonempty.

One should also note that if E is an extremal subset of $S(X)$ and if y is in E, then $E(y)$ is contained in E. This fact coupled with a simple two dimensional argument showing that $\bigcup_{a \in A} E(a)$ is a $U(X)$ extremal subset, yields that $E(A)$ equals $\bigcup_{a \in A} E(a)$.

For the discussion of the SAIN problem we will take $A_x = x + \Gamma_\perp \cap S(X)$. Work in **[4]** showed that if $P(x) \cap \Gamma$ was empty, then a solution of the SAIN problem is guaranteed at x. In this case A_x is not a convex subset of $S(X)$ and by convention we take $E(A_x) = F(A_x) = Q(A_x) = \phi$.

THEOREM 1.4. *Let (X, M, Γ) satisfy the hypotheses of the SAIN problem. If $F(A_x) \cap M$ is dense in $F(A_x)$ then the SAIN problem has a solution at x.*

Proof. If $F(A_x)$ is empty, the result is immediate from **[4]**. If $F(A_x)$ is not empty, assume without loss of generality that $\Gamma = \operatorname{span}\{\varphi_i \mid i = 1, \cdots, n\}$ and $\Gamma_0 = \{\varphi \in \Gamma \mid \varphi(x) = \varphi(y)\ \forall y \in F(A_x)\} = \operatorname{span}\{\varphi_i \mid i = 1, \cdots, \sigma\}$. If $\Gamma_0 = \phi$, $\sigma = 0$. If $\Gamma = \Gamma_0$, the result is trivial. For notational simplicity set $K = F(A_x)$. Define $\Phi: K \to R^{n-\sigma}$ via $\Phi(y) = (\varphi_{\sigma+1}(y), \cdots, \varphi_n(y))$. We assert the existence of $\{m_\alpha \mid \alpha \in B\}$ in $K \cap M$ with $\|x - m_\alpha\| < \epsilon$, B an arbitrary index set such that in $R^{n-\sigma}$, $\Phi(x) \in \operatorname{co}(\Phi(m_\alpha) \mid \alpha \in B)$. Since if this were not the case, then in $R^{n-\sigma}$, there exists a linear functional τ, a linear combination of φ_i, $i > \sigma$, such that without loss of generality $\tau(m) \leqq \tau(x)$ for all m in $K \cap M$, with $\|x - m\| < \epsilon$. But this implies $\tau(m) \leqq \tau(x)$ for all m in $K \cap M$, since if there exists an m_0 in $K \cap M$ with $\|x - m_0\| > \epsilon$ and $\tau(m_0) > \tau(x)$, then the set $\{y \in K \mid \tau(y) > \tau(x), \|y - x\| < \epsilon\}$ would be relatively open and nonempty (choose a suitable combination of x and m_0). Thus it would contain an m in $K \cap M$, contradicting $\tau(m) \leqq \tau(x)$ for $\|x - m\| < \epsilon$. Since $K \cap M$ is dense in K, this yields $\tau(y) \leqq \tau(x)$ for all y in K. But then the set $\{y \in K \mid \tau(y) = \tau(x)\}$ is a closed extremal subset of K, a contradiction unless $\tau \in \Gamma_0$, but this contradicts $\tau \notin \Gamma_0$. Thus the collection $\{m_\alpha \mid \alpha \in B\}$ exists such that $\Phi(x) \in \operatorname{co}(\Phi(m_\alpha) \mid \alpha \in B)$. The convexity of M and $\Phi(M)$ yields the result.

Before proceeding to the major theorem of this paper, we need a technical result concerning extremal subsets.

LEMMA 1.5. *Let $A_x = x + \Gamma_\perp \cap S(X)$ and $A_y = y + \Gamma_\perp \cap S(X)$. If $y \in A_x$, then $E(A_y) \subset E(A_x)$.*

Proof. Let $z \in E(A_y)$. Then there exists t such that $0 < t < 1$ with $tz + (1-t)s \in A_y$ for some $s \in S(X)$. Since $y \in A_x$, $y - x \in \Gamma_\perp$ and by adding $tz + (1-t)s - y \in \Gamma_\perp$, one obtains $tz + (1-t)s - x \in \Gamma_\perp$. Thus $tz + (1-t)s \in x + \Gamma_\perp \cap S(X)$ and $z \in E(A_x)$.

THEOREM 1.6. *Let* (X, M, Γ) *satisfy the hypotheses of the SAIN problem. The following are equivalent*:

(a) $E(A_x) \cap M$ *is dense in* $E(A_x)$ *for all* $x \in S(X)\backslash M$

(b) $F(A_x) \cap M$ *is dense in* $F(A_x)$ *for all* $x \in S(X)\backslash M$

(c) (X, M, Γ) *has property SAIN.*

Proof. To show that (a) implies (b), let y be in $F(A_x)$. Given $\epsilon > 0$, there exists z in $E(A_x)$ and m in $E(A_x) \cap M$ such that $\|y - z\| < \epsilon/2$ and $\|z - m\| < \epsilon/2$. Thus $F(A_x) \cap M$ is dense in $F(A_x)$.

The implication (b) implies (c) follows from Theorem 1.4. To show that (c) implies (a), set x in $S(X)$ and let y be in $E(A_x)$. Since (X, M, Γ) has property SAIN, $y + \Gamma_\perp \cap S(X) \cap M$ is dense in $y + \Gamma_\perp \cap S(X)$. If we can show that $y + \Gamma_\perp \cap S(X) \subset E(A_x)$ the result follows. Let $p \in y + \Gamma_\perp \cap S(X)$. Then $p - y \in \Gamma_\perp$. Since y is in $E(A_x)$, there exists v in $x + \Gamma_\perp \cap S(X)$ such that y is in $E(v)$. Therefore $ty + (1-t)z = v$ for some $0 < t < 1$ and some $z \in S(X)$. Thus $ty + (1-t)z - x \in \Gamma_\perp$. Adding $tp - ty$ we have $tp + (1-t)z - x$ is in $\Gamma_\perp$. Thus $tp + (1-t)z \in x + \Gamma_\perp$. Since p is in $E(A_y)$, p is in $E(A_x)$ by Lemma 1.5. The point z is also in $E(A_x)$ and thus a convex combination of p and z must have norm one. But then p is in $E(tp + (1-t)z)$, which is contained in $E(A_x)$. Thus $y + \Gamma$

2. A geometrical approach to SAIN in $C[a, b]$.

D. Johnson in [6] formulated the following definitions.

DEFINITION 2.0. Let X be a normed linear space, M a dense subset of X. A linear functional $x^* \in X^*$ is said to be a SAIN functional if (X, M, x^*) has property SAIN. A finite sequence $x_1^*, x_2^*, \cdots, x_n^*$ is said to be a SAIN sequence in case every $x^* \in \operatorname{span}\{x_i^* \mid i = 1, \cdots, n\}$ is a SAIN functional.

The dense subset to be used in this investigation will be Π the set of polynomials in $C[a, b]$.

Our first goal will be to show that $F(A_x) = Q(A_x)$ in $(C[a, b], \Pi, \Gamma)$, where $A_x = x + \Gamma_\perp \cap S(X)$, $x \in S(C[a, b])$.

LEMMA 2.1. $P(A_x) = P(F(A_x)) = P(Q(A_x))$.

Proof. Since $A_x \subset F(A_x) \subset Q(A_x)$, one has $P(Q(A_x)) \subset P(F(A_x)) \subset P(A_x)$. If one assumes the lemma is false, then there exists a $\tau \in P(A_x) \backslash P(Q(A_x))$ and a y in $Q(A_x)$ such that $\tau(y) < 1$. Set $B = \{z \in Q(A_x) \mid \tau(z) = 1\}$. B is strictly contained in $Q(A_x)$, contradicting the definition of $Q(A_x)$.

In $C[a, b]$ we can use some properties of the closed extremal subsets of the unit ball discussed in **[1]**.

PROPOSITION A. [**1**, Corollary 2.2]. *Let X be a separable Banach space with the weak Q property. If E is a closed face in $S(X)$, then there exists an x in E such that $E = Q(x)$.*

PROPOSITION B. [**1**, Theorem 3.3]. *$C[a, b]$ has the weak Q property.*

PROPOSITION C. [**1**, Lemma 3.1]. *In $C[a, b]$, if x is in $S(C[a, b])$ then $Q(x) = \{y \in S(X) \mid y = x \text{ on } \operatorname{crit} x\}$.*

LEMMA 2.2. *Let x be in $S(C[a, b])$. Then $F(A_x) = Q(A_x)$.*

Proof. Since $F(A_x)$ is a closed face of $S(C[a, b])$, $F(A_x) = Q(f)$ for some f in $F(A_x)$ by Proposition C. Similarly, $Q(A_x) = Q(q)$ for some q in $Q(A_x)$. $P(F(A_x))$ is the weak star closure of the set $\{\delta(t) \mid t \in \operatorname{crit} f,\ \delta(t) = e_t \text{ if } f(t) = +1,\ \delta(t) = -e_t \text{ if } f(t) = -1\}$. $P(Q(A_x))$ has a similar representation. Since $P(Q(A_x)) = P(F(A_x))$ by Lemma 2.1, we must have $\operatorname{ext} P(Q(A_x)) = \operatorname{ext} P(F(A_x))$. This forces f to be in $Q(A_x)$ and q to be in $F(A_x)$. Thus $F(A_x) = Q(A_x)$.

To obtain the main result of this section, we must recall some of the work in **[6]**.

PROPOSITION D. [**6**, Lemma 2]. *If $[a, b]$ is a compact interval and x^* is a bounded linear functional on $C[a, b]$, then $(C[a, b], \Pi, x^*)$ has property SAIN (x^* is a SAIN functional) if and only if either*

(i) *x^* has finitely atomic support, or*

(ii) *$x^* \in \pm \mathcal{P}$ and $\operatorname{supp} x^* = [a, b]$ where $\mathcal{P}$ denotes the cone of positive linear functionals defined on a function space, or*

(iii) $\operatorname{supp} x^{*+} \cap \operatorname{supp} x^{*-} \neq \varnothing$.

Furthermore, the only SAIN functionals which attain their norm are those satisfying property (i).

LEMMA 2.3. *Let $x \in S(C[a, b]) \backslash \Pi$. If Γ is a SAIN sequence in $C[a, b]$ with respect to Π, then $\operatorname{ext}(P(A_x))$ has finite cardinality.*

Proof. As always, we assume that $P(x)\cap\Gamma\neq\varnothing$, for otherwise A_x is the empty set by convention. Assume that $\mathrm{ext}(P(A_x))$ does not have finite cardinality. Let

$$B=\{y\in S(C[a,b])\mid \varphi(y)=1,\ \forall\varphi\in P(x)\cap\Gamma\}.$$

A_x is contained in B and $\mathrm{ext}(P(B))$ has finite cardinality since the only extremal SAIN functionals are finitely purely atomic by Proposition D. Thus, there exists $\varphi\in\Gamma\backslash P(A_x)\cap\Gamma$ such that $\sup_{b\in B}\varphi(b)=\varphi(x)<1$. If such a φ did not exist then $F(A_x)$ would equal B and $\mathrm{ext}(P(A_x))$ would have finite cardinality. Define $B_0=\{y\in B\mid \varphi(y)=\varphi(x)\}$. Let $I=[a,b]$. By Proposition D, we need to examine three possible cases.

Case (i). φ is finitely purely atomic. Such a functional would be extremal contradicting the action of φ on the set B.

Case (ii). $\varphi\in\mathcal{P}$. Assume that φ is in $+\mathcal{P}$, the argument for $-\mathcal{P}$, being similar. Let $C^+=\{t\in I\mid f(t)=x(t)=+1,\ \forall f\in B_0\}$ and $C^-=\{t\in I\mid f(t)=x(t)=-1,\ \forall f\in B_0\}$. The above sets are measurable since B has a representation $B=Q(b)$ for some b in B. If $C^-=\varnothing$, then $p=1$ is in B_0. Thus φ attains its norm and $B_0=\{1\}$, a contradiction. If $C^-\neq\varnothing$ let $\varphi(x)=\alpha<1$, and by Uhrysohn's lemma define $g(t)=1$, if $t\in C^+\cup K$, $g(t)=-1$, if $t\in C^-$, $g\in C[I]$, where K is a compact subset of $(I\backslash C^+\cup C^-)\equiv Z$ such that $\varphi(Z)-\varphi(K)<\varepsilon$ whose existence is guaranteed since φ is a regular measure. Then

$$\begin{aligned}\int_I g\,d\varphi &= \varphi(C^+)-\varphi(C^-)+\varphi(K)+\int_{Z\backslash K} g\,d\varphi\\ &\geqq \varphi(C^+)-\varphi(C^-)+\varphi(Z)-\epsilon-\varphi(Z\backslash K)\\ &\geqq \varphi(C^+)-\varphi(C^-)+\varphi(Z)-2\epsilon.\end{aligned}$$

Note that $\beta=\varphi(C^+)-\varphi(C^-)+\varphi(Z)$ is strictly greater than $\alpha=\int f\,d\varphi$ for all f in B_0. By choosing ϵ sufficiently small, we can construct a function g as above in B such that $\int g\,d\varphi>\alpha$. Hence φ does not exist such that $\sup_{b\in B}\varphi(b)=\varphi(x)$ and we obtain that $F(A_x)=B$.

Case (iii). $\mathrm{supp}\,\varphi^+\cap\mathrm{supp}\,\varphi^-\neq\varnothing$. Let C^+ and C^- be defined as in Case (ii). Let $C=C^+\cup C^-$. $\int_C x\,d\varphi=\varphi(C^+)-\varphi(C^-)$. Define $D=I\backslash C$. The restriction of φ to D yields a relative measure space. Thus there exist measurable sets D^+ and D^- contained in D such that $D^+ \cup D^- = D$ and $D^+ \cap D^- = \emptyset$, and $\varphi(D) = \varphi^+(D^+) - \varphi^-(D^-)$. Let

$$\alpha = \int_I x d\varphi = \int_C x d\varphi + \int_D x d\varphi = \varphi(C^+) - \varphi(C^-) + \int_D x d\varphi$$
$$\leqq \varphi(C^+) - \varphi(C^-) + \varphi(D^+) - \varphi(D^-) = \beta.$$

Define $g \in C[I]$, by Uhrysohn's lemma such that $g(t) = +1$, $t \in C^+ \cup K^+$, $g(t) = -1$, $t \in C^- \cup K^-$ where K^+ and K^- are compact sets contained in D^+ and D^- respectively, such that $\varphi^+(D^+ \backslash K^+) < \epsilon$ and $\varphi^-(D^- \backslash K^-) < \epsilon$. Then

$$\int_I g d\varphi = \varphi(C^+) - \varphi(C^-) + \varphi(K^+) - \varphi(K^-) + \int_{D^+ \backslash K^+} g d\varphi + \int_{D^- \backslash K^-} g d\varphi$$
$$\geqq \varphi(C^+) - \varphi(C^-) + \varphi(D^+) - \epsilon - (\varphi(D^-) - \epsilon) - \epsilon - \epsilon \geqq \beta - 4\epsilon.$$

Thus as in Case (ii) with ϵ being arbitrarily small, one can contradict the existence of such a φ and $F(A_x) = B$.

THEOREM 2.4. [**6**, Theorem 2]. *$(C[a, b], \Pi, \Gamma)$ has property SAIN if and only if Γ is a SAIN sequence.*

Proof. The necessity of Γ being a SAIN sequence is an obvious consequence of the definition. We must show that if Γ is a SAIN sequence then $(C[a, b], \Pi, \Gamma)$ has property SAIN. By Theorem 1.5 and Lemma 2.2, it is sufficient to show that $Q(A_x) \cap \Pi$ is dense in $Q(A_x)$ for all x in $S(C[a, b]$. Fix x in $S(C[a, b] \backslash \Pi)$ and assume that $Q(A_x)$ is not empty, (i.e. $P(x) \cap \Gamma \neq \emptyset$) since otherwise the condition of property SAIN is trivially satisfied. Since $P(A_x)$ is a weak star convex compact subset of the unit ball of a dual space, $Q(A_x)$ is equivalent to the set $\{y \in S(C[a, b]) \mid \varphi(y) = 1 \ \forall \varphi \in \operatorname{ext} P(A_x)\}$. These extreme points are the point evaluation functionals. By well known results [**1**, Corollary 4.1], $(C[a, b], \Pi, \Gamma)$ has property SAIN at a given point x if and only if there are at most finitely many prescribed interpolatory points for the function x. The $Q(A_x)$ will be dense in $Q(A_x)$ if and only if $P(A_x)$ has at most a finite number of extreme points. But Lemma 2.3 yields this fact.

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