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EXISTENCE OF FIXED POINTS OF NONEXPANSIVE MAPPINGS IN A SPACE WITHOUT NORMAL STRUCTURE

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A mapping $T: C \to X$ defined on a subset C of a Banach space X, with norm $\|\cdot\|$, is said to be *nonexpansive* if $\|Tx - Ty\| \leqq \|x - y\|$ for all $x, y \in C$. If C is assumed to be convex and weakly compact and if $T: C \to C$ then one of the main open questions is whether T has a fixed point in C, i.e., whether there exists $x \in C$ so that $Tx = x$. If X is reflexive and uniformly convex or, more generally, if X is reflexive and has normal structure then the answer is affirmative. Our purpose is to give an example of a classical reflexive space which does not have normal structure and for which the answer is nevertheless affirmative.

Nonexpansive mappings have played an important role in recent developments of nonlinear functional analysis (e.g., Browder **[1]**, de-Figueiredo **[3]**). The analysis of the uniformly convex and normal structure cases is due to Browder **[2]**, Göhde **[4]** and Kirk **[6]**.

Let X_J be the space l_2 renormed according to

$$\|x\| = \max\{\|x\|_\infty, \|x\|_2/\sqrt{2}\},$$

where $\|\cdot\|_\infty$ denotes the l_∞ norm and $\|\cdot\|_2$ the l_2 norm. This space originates with R. C. James. A space is said to have *normal structure* if for each bounded convex subset C consisting of more than one point there is a point $w \in C$ so that $\sup\{\|w - y\|: y \in C\} < \sup\{\|x - y\|: x, y \in C\}$. The set $\{x: x = x(i),\ x(i) \geqq 0,\ \|x\|_2 \leqq 1\}$ in X_J contains no such point w.

THEOREM. *Let C be a bounded closed convex subset of X_J. Let $T: C \to C$ be nonexpansive. Then T has a fixed point in C.*

One of the main tools in the proof is the following general lemma which has been used in Karlovitz **[5]**.

LEMMA. *Let X be a Banach space. Let C_0 be a weakly compact convex subset of X. Let $T: C_0 \to C_0$ be nonexpansive. Suppose that C_0 is minimal in the sense that it contains no proper closed convex subset which is* *invariant under T. Let $\{x_n\}$ be a sequence of approximate fixed points, i.e., $x_n \in C_0$ and $\| Tx_n - x_n \| \to 0$. Then for each $x \in C_0$*

$$\lim_n \| x - x_n \| = \text{diameter } C_0. \tag{1}$$

If X is reflexive then a standard argument establishes the existence of a subset $C_0 \subset C$ which is minimal in the sense of the lemma. If C_0 consists of more than one point, another standard argument yields a sequence of approximate fixed points in C_0. It follows from (1) that X does not have normal structure. Hence if X does have normal structure then C_0 is necessarily a singleton and thus a fixed point of T. This is an alternate proof of the known result. In order to use the lemma to show that in X_J C_0 is necessarily a singleton, we need to make more explicit use of the nonexpansiveness of T and of the geometry of X_J. While some modest generalizations will become apparent, the implications for the general reflexive case are not yet clear.

Proof of the lemma. Choose $y \in C_0$ and let $s = \limsup \| y - x_n \|$. Let $D = \{x : x \in C_0, \limsup \| x - x_n \| \leqq s\}$, which is nonempty closed and convex. It is also invariant under T. For

$$\| Tx - x_n \| \leqq \| Tx - Tx_n \| + \| Tx_n - x_n \| \leqq \| x - x_n \| + \| Tx_n - x_n \|$$

and $\| Tx_n - x_n \| \to 0$. By the minimality of C_0, $D = C_0$. Extract a subsequence $\{x_{n'}\}$ so that $\lim \| y - x_{n'} \| = s'$ exists. Suppose that there exists $z \in C_0$ and a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ so that $\lim \| z - x_{n''} \| = t$. Let $E = \{x : x \in C_0, \limsup \| x - x_{n''} \| \leqq \min \{t, s'\}\}$. Repeating the argument, we find $E = C_0$. Hence $y, z \in E$, and so $t = s'$. Thus for each $x \in C_0$, $\lim \| x - x_{n'} \|$ exists and equals s'.

We complete the proof by showing that $s' = r = \text{diam } C_0$. From this it follows that $\| y - x_{n'} \| \to r$ whenever $\{\| y - x_{n'} \|\}$ converges. Whence, by boundedness, $\| y - x_n \| \to r$ for the entire sequence. By repeating the argument above with $\{x_{n'}\}$ replaced by the entire sequence $\{x_n\}$, we shall have proved (1).

To this end, consider $F = \{u : u \in C_0, \| u - x \| \leqq s'$ for each $x \in C_0\}$. F is nonempty because we can extract a weakly convergent subsequence, again denoted by $\{x_{n'}\}$, with limit z. Since $\| x - x_{n'} \| \to s'$ for each $x \in C_0$ it follows that $\| x - z \| \leqq s'$ for each $x \in C_0$; hence $z \in F$. Now if $s' < r$ then $F \subsetneqq C_0$. However, this contradicts the minimality of C_0 because F is invariant under T. To see the latter, we first note that as a consequence of minimality closed convex hull $(TC_0) = C_0$. Hence if u is an arbitrary element of C_0 then for given $\epsilon > 0$ we can choose $v = \sum_{i=1}^{k} \lambda_i T x_i$ with $x_i \in C_0$, $\lambda_i > 0$, $\sum \lambda_i = 1$ and $\| u - v \| \leqq \epsilon$. Choose arbitrary w in F. Then

$$\begin{aligned}\|Tw - u\| &\leqq \|Tw - v\| + \|v - u\| \\ &\leqq \Sigma\lambda_i \|Tw - Tx_i\| + \|v - u\| \\ &\leqq \Sigma\lambda_i \|w - x_i\| + \|v - u\| \leqq s' + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ and $u \in C_0$ are arbitrary this shows $Tw \in F$. Thus F is invariant under T and hence $s' = r$. This finishes the proof.

Proof of the theorem. Given $x \in X_J$ we represent its components by $x(j)$, $j = 1, 2, \cdots$. Since X_J is a renorming of l_2 there exists a component $x(j)$ so that $\|x\|_\infty = |x(j)|$.

By a standard Zorn's lemma argument there exists a closed convex subset C_0 of C which is invariant under T and minimal in the sense of the lemma. We propose to show that C_0 consists of a single point. By invariance this is then a fixed point of T.

We proceed by contradiction. Suppose that C_0 consists of more than one point. We may assume without loss of generality that $0 \in C_0$, and we let diameter $C_0 = r > 0$. For each s, $0 < s < 1$, we define $T_s = (1 - s)T$. Clearly $T_s\colon C_0 \to C_0$ and it is a strict contraction. Hence by the Banach contraction principle there exists a unique $x_s \in C_0$ so that $T_s x_s = x_s$. Thus

$$Tx_s = (1 - s)^{-1} x_s, \quad \text{for} \quad 0 < s < 1.$$

By the minimality of C_0, $x_s \neq 0$. The desired contradiction results from a study of the points x_s. Several propositions are needed.

PROPOSITION 1. *For each* $x \in C_0$, $\lim_{s \to 0} \|x - x_s\|_\infty = r$.

Proof. By contradiction. Suppose that for some $x \neq 0 \in C_0$ and sequence $\{x_{s_n}\}$ with $s_n \to 0$, denoted by $\{x_n\}$, $\|x - x_n\|_\infty \leqq r - \delta$, $\delta > 0$. Since $\|0 - x_n\|_\infty \leqq \operatorname{diam} C_0 = r$ it follows that $\|x/2 - x_n\|_\infty \leqq r - \delta/2$ for all n. By the uniform convexity of $\|\cdot\|_2$, it follows from $\|x - x_n\|_2/\sqrt{2}$, $\|0 - x_n\|_2/\sqrt{2} \leqq \operatorname{diam} C_0 = r$ that $\|x/2 - x_n\|_2/\sqrt{2} \leqq r - \tau$ for some $\tau > 0$. Hence, $\|x/2 - x_n\| \leqq r - \min\{\tau, \delta/2\}$ for all n, which contradicts (1) because $\|Tx_n - x_n\| = s_n(1 - s_n)^{-1}\|x_n\| \to 0$.

PROPOSITION 2. *For each* s, $0 < s < 1$, $\lim_{t \to s} \|x_t - x_s\| = 0$.

Proof. We denote x_s by x and x_t by y. Suppose that $\|x - y\| = \|x - y\|_\infty = |x(k) - y(k)|$. By nonexpansiveness

$$(2) \qquad |(1 - s)^{-1} x(k) - (1 - t)^{-1} y(k)| \leqq |x(k) - y(k)|.$$

For $s \neq t$ it readily follows that $\operatorname{sgn} x(k) = \operatorname{sgn} y(k) = \sigma = \pm 1$. Suppose that $0 < t < s$. If $\sigma x(k) > \sigma y(k)$ then

$$(1-s)^{-1}\sigma x(k) - (1-t)^{-1}\sigma y(k) > (1-t)^{-1}(\sigma x(k) - \sigma y(k)),$$

contradicting (2). Hence $\sigma y(k) \geqq \sigma x(k)$. If $(1-t)^{-1}\sigma y(k) - (1-s)^{-1}\sigma x(k) \geqq 0$ then, by (2), $(1-s)t(1-t)^{-1}s^{-1}\sigma y(k) \leqq \sigma x(k) \leqq \sigma y(k)$. If $(1-s)^{-1}\sigma x(k) - (1-t)^{-1}\sigma y(k) \geqq 0$ then, directly, $(1-s)(1-t)^{-1}\sigma y(k) \leqq \sigma x(k) \leqq \sigma y(k)$. If $s < t < 1$, analogous inequalities are derived. It follows that

$$(3) \qquad |x(k) - y(k)| \leqq \begin{cases} (s-t)s^{-1}(1-t)^{-1}|y(k)|, & 0 < t < s, \\ (t-s)t^{-1}(1-s)^{-1}|x(k)|, & s < t < 1. \end{cases}$$

Hence if $\|x_t - x_s\| = \|x_s - x_t\|_\infty$ and $s/2 < t < (1+s)/2$,

$$(4) \qquad \|x_s - x_t\| \leqq A\,|s-t|, \quad \text{for some} \quad A = A(s) > 0.$$

Now suppose that $\|x - y\| = \|x - y\|_2/\sqrt{2}$. By nonexpansiveness

$$\|(1-s)^{-1}x - (1-t)^{-1}y\|_2 \leqq \|x - y\|_2.$$

We divide the positive integers according to:

$$I_1 = \{i : |(1-s)^{-1}x(i) - (1-t)^{-1}y(i)| \leqq |x(i) - y(i)|\}$$

and

$$I_2 = \{i : |(1-s)^{-1}x(i) - (1-t)^{-1}y(i)| > |x(i) - y(i)|\}.$$

Then

$$(5) \quad \sum_{I_2} [((1-s)^{-1}x(i) - (1-t)^{-1}y(i))^2 - (x(i) - y(i))^2] \leqq \sum_{I_1} (x(i) - y(i))^2.$$

By definition, (2) holds for $k \in I_1$. We can deduce, as above, that (3) holds. Whence, for $s/2 < t < (1+s)/2$

$$(6) \qquad \sum_{I_1} (x(i) - y(i))^2 \leqq B(s-t)^2 \quad \text{for some} \quad B = B(s) > 0.$$

We note the identity·

$$(1-t)^{-1}y(i) - (1-s)^{-1}x(i) = (1-s)^{-1}(y(i) - x(i) - \gamma(s,t)y(i)),$$

where $\gamma(s,t)=(s-t)\,(1-t)^{-1}$. Substitution into (5) yields

$$\sum_{I_2}\,[(1-s)^{-2}((y(i)-x(i))-\gamma(s,t)y(i))^2-(x(i)-y(i))^2]$$
$$\leqq \sum_{I_1}\,(x(i)-y(i))^2.$$

By the Schwarz inequality and some simple manipulation

$$\sum_{I_2}\,(x(i)-y(i))^2 \leqq s^{-1}(1-s)^2\sum_{I_1}\,(x(i)-y(i))^2+2s^{-1}\gamma(s,t)\|y\|_2\|x-y\|_2.$$

Combining this with (6) we find that if $\|x_s-x_t\|=\|x_s-x_t\|_2/\sqrt{2}$ and $s/2<t<(1+s)/2$ then

$$\text{(7)}\qquad \|x_s-x_t\|\leqq K(s-t)^{1/2}\quad\text{for some}\quad K=K(s)>0.$$

The proposition now follows from (4) and (7).

For each positive integer i and $\epsilon>0$ we introduce the notation:

$$A^{\epsilon}(i)=\{s\colon 0<s<1,\ |x_s(i)|\geqq r-\epsilon\}\quad\text{and}\quad \alpha^{\epsilon}(i)=\inf A^{\epsilon}(i).$$

PROPOSITION 3. *For each positive integer i and ϵ, $0<\epsilon\leqq r/4$, there exists s_1, $0<s_1<1$, with the property that for each s, $0<s\leqq s_1$, there exists a positive integer $k(s)$ such that $k(s)\neq i$ and $|x_s(k(s))|\geqq r-\epsilon$.*

Proof. If $A^{\epsilon}(i)=\varnothing$ this follows from Proposition 1 with $x=0$. Otherwise choose $s_0\in A^{\epsilon}(i)$. Let $\epsilon_1=\min\{s_0(1-s_0)^{-1}(r-\epsilon),\ \epsilon/2,\ rs_0/2,\ s_0(1-s_0)^{-1}\epsilon/2\}$. By Proposition 1 choose s_1 so that $\|x_{s_0}-x_s\|_\infty\geqq r-\epsilon_1$ for all $0<s\leqq s_1$. By virtue of $Tx_{s_0}=(1-s_0)^{-1}x_{s_0}\in C_0$ we have $\|x_{s_0}\|_\infty\leqq r(1-s_0)$. Choose s, $0<s\leqq s_1$. Suppose that $\operatorname{sgn}x_{s_0}(i)=\operatorname{sgn}x_s(i)$ or $x_s(i)=0$. Then from $3r/4\leqq|x_{s_0}(i)|\leqq r(1-s_0)$ we deduce $|x_{s_0}(i)-x_s(i)|\leqq r-rs_0$, $r/4<r-\epsilon_1$. If $\operatorname{sgn}x_{s_0}(i)=-\operatorname{sgn}x_s(i)$ then

$$\begin{aligned} r\geqq\|Tx_{s_0}-Tx_s\|&\geqq|(1-s_0)^{-1}x_{s_0}(i)-(1-s)^{-1}x_s(i)|>|x_{s_0}(i)-x_s(i)|\\ &\quad+s_0(1-s_0)^{-1}|x_{s_0}(i)|\geqq|x_{s_0}(i)-x_s(i)|\\ &\quad+s_0(1-s_0)^{-1}(r-\epsilon)\geqq|x_{s_0}(i)-x_s(i)|+\epsilon_1,\end{aligned}$$

and hence $|x_{s_0}(i)-x_s(i)|<r-\epsilon_1$. Thus there exists a positive integer $j\neq i$ so that $\|x_{s_0}-x_s\|_\infty=|x_{s_0}(j)-x_s(j)|\geqq r-\epsilon_1$. We assert that $k(s)=j$ satisfies the proposition. If $\operatorname{sgn}x_{s_0}(j)=\operatorname{sgn}x_s(j)$ then $r-\epsilon_1\leqq|x_{s_0}(j)-x_s(j)|<\max\{|x_{s_0}(j)|,|x_s(j)|\}$. Since $|x_{s_0}(j)|\leqq r(1-s_0)<r-\epsilon_1$ it follows that $|x_s(j)|>r-\epsilon_1>r-\epsilon$, as desired. If $\operatorname{sgn}x_{s_0}(j)\neq\operatorname{sgn}x_s(j)$, then

$$\begin{aligned} r &\geqq \| Tx_{s_0} - Tx_s \| \geqq |(1-s_0)^{-1}x_{s_0}(j) - (1-s)^{-1}x_s(j)| = (1-s_0)^{-1}|x_{s_0}(j)| \\ &\quad + (1-s)^{-1}|x_s(j)| \geqq |x_{s_0}(j)| + |x_s(j)| + s_0(1-s_0)^{-1}|x_{s_0}(j)| \\ &= |x_{s_0}(j) - x_s(j)| + s_0(1-s_0)^{-1}|x_{s_0}(j)| \geqq r - \epsilon_1 + s_0(1-s_0)^{-1}|x_{s_0}(j)|. \end{aligned}$$

Hence $|x_{s_0}(j)| \leqq s_0^{-1}(1-s_0)\epsilon_1$. So $|x_{s_0}(j) - x_s(j)| \geqq r - \epsilon_1$ implies $|x_s(j)| \geqq r - \epsilon_1 - s_0^{-1}(1-s_0)\epsilon_1 \geqq r - \epsilon_1 - \epsilon/2 \geqq r - \epsilon$, as desired. Since $s \leqq s_1$ was arbitrarily chosen this finishes the proof.

PROPOSITION 4. *Suppose $\alpha^\epsilon(i) = \alpha^\delta(j) = 0$ for some ϵ, δ with $0 < \epsilon$, $\delta \leqq r/64$. Then $i = j$.*

Proof. For i and ϵ we choose s_1 according to Proposition 3. Thus if $s \in A^\epsilon(i)$ and $s \leqq s_1$ then $|x_s(m)| \geqq r - \epsilon$ for $m = i, k(s)$.. Since $i \neq k(s)$ and $\|x_s\|_2^2 \leqq 2r^2$ we readily find that $|x_s(m)| \leqq r/4$ for $m \neq i, k(s)$. By Proposition 1 we choose $s_2, s_3 \in A^\epsilon(i)$, $s_2, s_3 < s_1$, so that $\|x_{s_p} - x_{s_q}\|_\infty \geqq r - \epsilon$ for $p \neq q$ and $p, q = 1, 2, 3$. Now suppose that $k(s_1) = k(s_2)$. Then $|x_{s_1}(m) - x_{s_2}(m)| \leqq |x_{s_1}(m)| + |x_{s_2}(m)| \leqq r/2 < r - \epsilon$ for $m \neq i, k(s_1)$. Moreover, $\operatorname{sgn} x_{s_1}(i) = \operatorname{sgn} x_{s_2}(i)$; otherwise $\|x_{s_1} - x_{s_2}\|_\infty \geqq |x_{s_1}(i)| + |x_{s_2}(i)| \geqq 2r - 2\epsilon > r$. Thus $|x_{s_1}(i) - x_{s_2}(i)| \leqq r - (r - \epsilon) = \epsilon$. By the same argument $|x_{s_1}(k(s_1)) - x_{s_2}(k(s_1))| \leqq \epsilon$. Thus $|x_{s_1}(i) - x_{s_2}(i)| < r - \epsilon$ for all positive integers i, which is a contradiction. Hence $k(s_1) \neq k(s_2)$. Similarly $k(s_3) \neq k(s_1), k(s_2)$. Now if $i \neq j$ we repeat the argument and find $t_1, t_2, t_3 \in A^\delta(j)$ so that $|x_{t_p}(j)|$, $|x_{t_p}(k(t_p))| \geqq r - \delta$, $p = 1, 2, 3$ and so that $j, k(t_1), k(t_2)$ and $k(t_3)$ are disjoint. Thus we can find s_q and t_p so that $\{i, k(s_q)\} \cap \{j, k(t_p)\} = \varnothing$. Then from $|x_{s_q}(i)|$, $|x_{s_q}(k(s_q))| \geqq r - \epsilon$, $|x_{t_p}(j)|$, $|x_{t_p}(k(t_p))| \geqq r - \delta$ and $\|x_{t_p}\|_2$, $\|x_{s_q}\|_2 \leqq \sqrt{2}r$ it follows that $\|x_{s_q} - x_{t_p}\|_2/\sqrt{2} > r$ which contradicts $x_{s_q}, x_{t_p} \in C_0$. Hence $i = j$.

Completion of the proof of the theorem. Let $\epsilon = r/128$. If there exists a positive integer i so that $\alpha^\epsilon(i) = 0$, let $i_0 = i$. Otherwise $\alpha^\epsilon(i) > 0$ for each i and we let $i_0 = 1$. Apply Proposition 3 to find $s_1 = s_1(i_0, \epsilon)$. In the sequel the positive integers $k(\cdot)$ will be those given by the proposition for this s_1. Denote $k(s_1)$ by k_1. Let $s_2 = \alpha^\epsilon(k_1)$. If $\alpha^\epsilon(i_0) = 0$ it follows from $i_0 \neq k_1$ and Proposition 4 that $s_2 > 0$; otherwise $s_2 > 0$ by hypothesis. By Proposition 2 $\|x_{s_2-\mu} - x_{s_2}\| \to 0$ as $\mu \to 0$. Hence we can choose $\mu > 0$ so that $r - 2\epsilon \leqq |x_{s_2-\mu}(k_1)|$. Since $s_2 - \mu < s_2 < s_1$, $k(s_2 - \mu)$ is well defined. Since $s_2 - \mu < \alpha^\epsilon(k_1)$, $|x_{s_2-\mu}(k_1)| < r - \epsilon$; and hence $k(s_2 - \mu) \neq k_1$. Denote $x_{s_2-\mu}$ by y and $k(s_2 - \mu)$ by k_2. Thus $|y(k_1)|$, $|y(k_2)| \geqq r - 2\epsilon$. Since $k_1, k_2 \neq i_0$, reasoning as above, $\alpha^\epsilon(k_1)$, $\alpha^\epsilon(k_2) > 0$. Hence we can choose s_3, $0 < s_3 < \alpha^\epsilon(k_1)$, $\alpha^\epsilon(k_2)$. Then $|x_{s_3}(k_1)|$, $|x_{s_3}(k_2)| < r - \epsilon$, and hence $k_3 = k(s_3) \neq k_1, k_2$. Repeating the argument we find $z = x_{s_3-\eta}$, $\eta > 0$, and $k_4 = k(s_3 - \eta) \neq k_3$ so that $|z(k_3)| \geqq r - 2\epsilon$ and $|z(k_4)| \geqq$

$r - \epsilon$. Moreover, $s_3 - \eta < \alpha^\epsilon(k_1)$, $\alpha^\epsilon(k_2)$, hence $k_4 \neq k_1, k_2$. Thus k_1, k_2, k_3 and k_4 are disjoint. Hence from $\| y \|$, $\| z \| \leqq r$ and $| y(k_1) |$, $| y(k_2) |$, $| z(k_3) |$, $| z(k_4) | \geqq r - 2\epsilon$ we readily calculate $\| y - z \|_2 / \sqrt{2} > r$ which contradicts $y, z \in C_0$. This contradiction proves that C_0 cannot consist of more than one point and finishes the proof of the theorem.

Added in proof. Additional applications of the lemma as well as some of its points of contact with work of M. Edelstein will be discussed elsewhere. P. M. Fitzpatrick has informed us that he independently developed the lemma.

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