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AN IMBEDDING THEOREM FOR INDETERMINATE HERMITIAN MOMENT SEQUENCES

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Hermitian moment sequences are generalizations of classical power moment sequences to bounded operators on a Hilbert space. The main result is that every indeterminate Hermitian moment sequence on a complex Hilbert space can be imbedded in a determinate Hermitian moment sequence on an enlarged Hilbert space in the sense that the first sequence is a compression of the second. This implies the existence of determinate Hermitian moment sequences which, when compressed, are indeterminate and leads to the following questions: Which orthogonal projections on the Hilbert space give rise to determinate compressions of a fixed, determinate sequence? What structure do these projections induce on the underlying Hilbert space?

Background. Let $\mathscr{H}$ be a complex Hilbert space, with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and let $B(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$. A sequence $\{T_j\}_{j=0}^{\infty}$, $T_j \in B(\mathscr{H})$, is said to be an *Hermitian moment sequence* (cf. J. S. MacNerney **[4]**) if there exists a positive operator-valued measure (cf. Berberian **[2]**), $\mu(\cdot)$, defined on the Borel sets of $(-\infty,\infty)$, such that

$$T_j = \int_{-\infty}^{\infty} t^j d\mu(t), \qquad j = 0, 1, 2, \cdots. \tag{1}$$

Necessary and sufficient conditions for a sequence of operators to be of the form (1) and, in addition, be such that $\mu(\cdot)$ has support over a finite interval were first given by B. Sz.-Nagy **[6]**. General necessary and sufficient conditions for a sequence to be of the form (1), with no restrictions on the support of $\mu(\cdot)$, were first given by J. S. MacNerney **[4]**.

An Hermitian moment sequence (1) will be said to be *determinate* if $\mu(\cdot)$ is unique, and *indeterminate* otherwise (cf. Akhiezer **[1]**, Dubois–Violette **[3]**, and Shohat and Tamarkin **[7]**).

Main result. A determinate moment sequence $\{T_j\}$ can be "imbedded" in an indeterminate moment sequence on $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$. For example, let $\{a_j\}_{j=0}^{\infty}$ be a scalar-valued indeterminate moment sequence, I be the identity operator on $\mathscr{H}$, and P be the orthogonal projection from $\mathscr{K}$ to $0 \oplus \mathscr{H} = \mathscr{H}$. The sequence,

$$T'_j = \begin{pmatrix} a_j \cdot I & 0 \\ 0 & T_j \end{pmatrix},$$

is easily seen to be an indeterminate Hermitian moment sequence, while the compression of $\{T'_j\}$ (cf. Putnam **[5]**, pg. 76),

$$PT'_jP = T_j, \qquad j=0,1,2,\cdots,$$

is determinate. What is not at all obvious is that every indeterminate moment sequence on $\mathscr{H}$ can be imbedded in (that is, arises as a compression of) a determinate moment sequence on $\mathscr{H} \oplus \mathscr{H}$. The main purpose in writing this paper is to establish

THEOREM 1. (*Imbedding Theorem*) *Let* T_j, $\mu(\cdot)$ *be as in* (1) *and let*

$$R_j = \int_{-\infty}^{\infty} t^j e^{-|t|} d\mu(t), \quad S_j = \int_{-\infty}^{\infty} t^j e^{-|t|/2} d\mu(t). \tag{2}$$

Then the sequence,

$$\tilde{T}_j = \begin{pmatrix} R_j & S_j \\ S_j & T_j \end{pmatrix}, \qquad j=0,1,2,\cdots, \tag{3}$$

defined on $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$, *is a determinate Hermitian moment sequence.*

The proof is postponed.

The Imbedding Theorem raises an interesting question: Let $\{T_j\}$ be a determinate moment sequence and let P be the orthogonal projection onto a closed subspace, $\mathscr{H}_p$, of $\mathscr{H}$. Under what conditions will the compression of the moment sequence, $\{PT_jP\}$, be determinate?

For any given moment sequence, $\{T_j\}_{j=0}^{\infty}$, whether it is determinate or not, the subspace $\mathscr{H}_p$, will be said to be *determinate* or *indeterminate* depending on whether $\{PT_jP\}$ is determinate or indeterminate. A moment sequence for which every subspace is determinate will be called a *completely determinate* moment sequence. In general, a nontrivial characterization of the determinate subspaces of a moment sequence is not known. For completely determinate sequences, the problem is much simpler:

LEMMA 1. *A moment sequence of the form* (1) *is completely determinate if and only if it is determinate in all of the one dimensional subspaces of* $\mathscr{H}$. *A sufficient condition for this to occur is that*

$$\sum_{j=1}^{\infty} \| T_{2j} \|^{-1/2j} = +\infty. \tag{4}$$

Proof. The necessity of the condition of being determinate in each one dimensional subspace follows from the definition. To establish sufficiency, let $\{T_j\}$ obey this condition, then for any $\phi \in \mathcal{H}$, $\|\phi\| = 1$, $\tau_j(\phi) = (T_j\phi, \phi)$ is a determinate scalar moment sequence. If $\mu(\cdot)$ and $\hat{\mu}(\cdot)$ both generate $\{T_j\}$, then

$$(5) \qquad \tau_j(\phi) = \int_{-\infty}^{\infty} t^j d(\mu(t)\phi, \phi) = \int_{-\infty}^{\infty} t^j d(\hat{\mu}(t)\phi, \phi).$$

The determinacy of $\{\tau_j(\phi)\}$ then implies that for any Borel set Δ,

$$(6) \qquad (\mu(\Delta)\phi, \phi) = (\hat{\mu}(\Delta)\phi, \phi).$$

Coupling (6) with the fact that $\mathcal{H}$ is a complex Hilbert space then implies $\mu(\Delta) = \hat{\mu}(\Delta)$. Hence, the full moment sequence is determinate.

Now, let P be a projection onto a closed subspace, $\mathcal{H}_p$, of $\mathcal{H}$ and consider the corresponding one dimensional compressions of PT_jP:

$$(7) \qquad \sigma_j(\phi) = (PT_jP\phi, \phi), \qquad j = 0, 1, 2, \cdots,$$

where $\phi \in \mathcal{H}$, $\|\phi\| = 1$. From (7), it is clear that $\sigma_j(\phi) = 0$ if $\|P\phi\| = 0$ and

$$(8) \qquad \sigma_j(\phi) = \|P\phi\|^2 \tau_j\left(\frac{P\phi}{\|P\phi\|}\right), \qquad j = 0, 1, 2, \cdots,$$

if $\|P\phi\| \neq 0$. In the former case, $\{\sigma_j(\phi)\}$ is obviously determinate. In the latter case, $\{\sigma_j(\phi)\}$ is determinate because, by (8), it is a positive constant multiple of a determinate moment sequence. A direct application of a Theorem of Hamburger (cf. Shohat and Tamarkin **[7]**, p. 70) shows all such sequences are determinate. Thus, $\{PT_jP\}$ is determinate in each of its one dimensional subspaces and, by the preceeding discussion, is determinate. This, in turn, implies $\{T_j\}$ is completely determinate.

Finally, that (4) is a sufficient condition for $\{T_j\}$ to be completely determinate follows from the fact that for $\phi \in \mathcal{H}$, $\|\phi\| = 1$,

$$|\tau_j(\phi)| = |(T_j\phi, \phi)| \leqq \|T_j\|, \qquad j = 0, 1, 2, \cdots.$$

This, together with (4), implies $\{\tau_j(\phi)\}$ satisfies the ordinary Carleman condition (cf. **[7]**, pg. 19) and is, therefore, determinate for each ϕ. By the preceding discussion, $\{T_j\}$ is completely determinate. This ends the proof.

Condition (4) is certainly not a necessary condition for complete determinacy. Consider the following example (A. E. Nussbaum, private communication): Let $\{a_j\}$ be an indeterminate scalar (Hamburger) moment sequence and set

$$\delta = \lim_{n\to\infty} \min_{x_0=1} \left(\sum_{j=0}^{n} \sum_{k=0}^{n} a_{j+k} x_j x_j \right).$$

Then by the Theorem of Hamburger mentioned earlier (cf. [7], pg. 70), the sequence

$$\begin{aligned} b_0 &= a_0 - \delta \\ b_j &= a_j, \qquad j = 1, 2, 3, \cdots, \end{aligned} \tag{9}$$

is easily seen to be a determinate moment sequence. Moreover, since $\{a_j\}$ was indeterminate to begin with, it cannot satisfy the Carleman condition and, therefore, $\{b_j\}$ is a determinate moment sequence which does not satisfy the Carleman condition. Thus, the Carleman condition is not a necessary condition for determinacy, even in the scalar case.

The determinate moment sequence $\{b_j\}$ also has two other interesting properties, which can be demonstrated via the Theorem of Hamburger. The translates $\{b_{j+2k}\}_{j=0}^{\infty}$ are also moment sequences, but they are indeterminate if $k \geqq 1$. Finally, adding the two determinate moment sequences $\{b_j\}$ and $\{\delta, 0, 0, \cdots\}$ results in the indeterminate moment sequence $\{a_j\}$. Thus determinacy is not necessarily preserved when two determinate moment sequences are added.

Proof of Theorem 1. Since the positive $B(\mathscr{K})$-valued Borel measure,

$$d\nu(t) = \begin{pmatrix} e^{-|t|} & e^{-|t|/2} \\ e^{-|t|/2} & 1 \end{pmatrix} d\mu(t), \tag{10}$$

generates the sequence $\{\tilde{T}_j\}$, it is a moment sequence. If $d\hat{\nu}(t)$ is any positive $B(\mathscr{K})$-valued Borel measure which generates $\{\tilde{T}_j\}$, and if

$$d\omega(t) = d\hat{\nu}(t) - d\nu(t), \tag{11}$$

then

$$\int_{-\infty}^{\infty} t^j d\omega(t) = 0, \qquad j = 0, 1, 2, \cdots. \tag{12}$$

What needs to be shown is that $d\omega(t) = 0$, for then $d\hat{\nu} = d\nu$ and $\{\tilde{T}_j\}$ is determinate.

By considering vectors of the form $(\phi, 0)^T \in \mathcal{K}$, it is possible to show that the upper left component of $d\hat{\nu}(t)$, $d\hat{\nu}_{1,1}(t)$, is a positive $B(\mathcal{H})$-valued Borel measure. Moreover, (12) implies that

$$R_j = \int_{-\infty}^{\infty} t^j d\hat{\nu}_{1,1}(t) = \int_{-\infty}^{\infty} t^j e^{-|t|} d\mu(t), \tag{13}$$

for $j = 0, 1, 2, \cdots$. However, using the definition of R_j, the inequality

$$\| R_j \| \leqq j^j e^{-j} \| T_0 \| \tag{14}$$

can be easily obtained. An application of the Carleman condition (4), given in Lemma 1, then implies R_j is completely determinate. This fact, coupled with (13), implies that the two measures in (13) are equal. Hence,

$$d\omega_{1,1}(t) = 0. \tag{15}$$

Again by considering vectors of a special form, $(\alpha\phi, \beta\phi)^T$, $\phi \in \mathcal{H}$, $\alpha, \beta \in \mathcal{C}$, it is clear that the 2×2 matrix-valued measures,

$$\begin{aligned} &d\hat{\nu}_\phi(t) = (d\hat{\nu}(t)\phi, \phi), \quad d\nu_\phi(t) = (d\nu(t)\phi, \phi), \\ &d\omega_\phi(t) = (d\omega(t)\phi, \phi), \end{aligned} \tag{16}$$

satisfy

$$d\hat{\nu}_\phi(t) \geqq 0, \quad d\nu_\phi(t) \geqq 0, \quad \int_{-\infty}^{\infty} t^j d\omega_\phi(t) = 0. \tag{17}$$

Fix $\phi \in \mathcal{H}$. By (15) and (16),

$$d\omega_\phi(t) = \begin{pmatrix} 0 & d\sigma(t) \\ d\bar{\sigma}(t) & d\tau(t) \end{pmatrix}, \tag{18}$$

where $d\sigma(t) = (d\omega_{1,2}\phi, \phi)$, $d\tau(t) = (d\omega_{2,2}\phi, \phi)$. Because the measures $d\hat{\nu}$ and $d\nu$ are bounded operator valued measures, all the measures $d\sigma$, $d\tau$, $d\mu_\phi(t) = (d\mu(t)\phi, \phi)$ are bounded and, respectively, are complex, real, and positive Borel measures. By the Radon–Nikodym Theorem, there exist $f(t), g(t) \in L^1(d\mu_\phi)$, and bounded Borel measures $d\tau_s(t)$, $d\sigma_s(t)$, singular with respect to $d\mu_\phi(t)$, such that

$$\begin{aligned} d\tau(t) &= f(t) d\mu_\phi(t) + d\tau_s(t) \\ d\sigma(t) &= g(t) d\mu_\phi(t) + d\sigma_s(t). \end{aligned} \tag{19}$$

Combining (16), (18), and (19) gives

$$\text{(20)}\quad d\hat{\nu}_\phi(t) = \begin{pmatrix} e^{-|t|} & e^{-|t|/2} + g(t) \\ e^{-|t|/2} + \overline{g(t)} & 1 + f(t) \end{pmatrix} \cdot d\mu_\phi(t) + \begin{pmatrix} 0 & d\sigma_s(t) \\ \overline{d\sigma_s}(t) & d\tau_s(t) \end{pmatrix}.$$

Let Δ be any subset of the real line such that $\mu_\phi(\Delta) = 0$. Since $d\hat{\nu}_\phi(t) \geqq 0$,

$$\hat{\nu}_\phi(\Delta) = \begin{pmatrix} 0 & \sigma_s(\Delta) \\ \overline{\sigma_s}(\Delta) & \tau_s(\Delta) \end{pmatrix} \geqq 0.$$

The facts that trace and determinant of a nonnegative matrix are themselves nonnegative imply that

$$\text{(21)}\quad \begin{aligned} &\sigma_s(\Delta) = 0 \\ &\tau_s(\Delta) \geqq 0. \end{aligned}$$

Since the support of σ_s is singular with respect to $d\mu_s(t)$, this implies that

$$\text{(22)}\quad d\sigma_s(t) = 0, \qquad d\tau_s(t) \geqq 0.$$

To complete the proof of the theorem, it need only be shown that $f(t) = g(t) = 0$, $d\tau_s(t) = 0$. If this can be done, it follows from (20) that $d\hat{\nu}_\phi(t) = d\nu_\phi(t)$ for every ϕ in $\mathscr{H}$. Hence, for any Borel set Δ, $\hat{\nu}_\phi(\Delta) = (\hat{\nu}(\Delta)\phi, \phi) = \nu_\phi(\Delta) = (\nu(\Delta)\phi, \phi)$. Since the underlying space is a a complex Hilbert space, this is sufficient to imply that $\hat{\nu}(\Delta) = \nu(\Delta)$ and that $\{\tilde{T}_j\}$ is determinate.

To see that $g(t) = 0$, first note that the matrix multiplying $d\mu_\phi(t)$ in (20) is nonnegative almost everywhere with respect to μ_ϕ. Otherwise, the integral of $d\hat{\nu}_\phi$ over a set which is both disjoint from the support of τ_s and has positive μ_ϕ-measure would be negative, which is impossible. Applying the condition that the determinant of a nonnegative matrix must itself be nonnegative and then multiplying the resultant inequality by $e^{|t|}$ yields

$$\text{(23)}\quad |1 + g(t)e^{|t|/2}|^2 \leqq 1 + f(t).$$

Since both 1, $f(t)$ belong to $L^1(d\mu_\phi(t))$, the function $1 + g(t)e^{|t|/2}$ belongs to $L^2(d\mu_\phi(t))$. But this in turn implies $g(t)e^{|t|/2} \in L^2(d\mu_\phi(t))$. Finally, this implies

$$\int_{-\infty}^{\infty} e^{|t|}|g(t)|^2 d\mu_\phi(t) = \int_{-\infty}^{\infty} e^{2|t|}|g(t)|^2 e^{-|t|} d\mu_\phi(t),$$

and

$$e^{|t|}g(t) \in L^2(e^{-|t|}d\mu_\phi(t)). \tag{24}$$

Combining (17), (18), (19), and (22) yields

$$\int_{-\infty}^{\infty} t^j g(t) d\mu_\phi(t) = 0 = \int_{-\infty}^{\infty} t^j e^{|t|} g(t) e^{-|t|} d\mu_\phi(t). \tag{25}$$

Thus by (24) and (25), $e^{|t|}g(t)$ is in $L^2(e^{-|t|}d\mu_\phi(t))$ and orthogonal to all polynomials. Since $\{R_j\}$ is a completely determinate moment sequence, $\{(R_j\phi, \phi)\}$ is a determinate moment sequence. The polynomials are therefore complete in $L^2(e^{-|t|}d\mu_\phi(t))$ (cf. Akhiezer **[1]**, pg. 45) and $e^{|t|}g(t) = 0$. Hence,

$$g(t) = 0 \text{ a.e. } \mu_\phi. \tag{26}$$

It is now possible to show that $f(t)$ and $d\tau_s(t)$ are both 0. Combining (20), (22), and (26), the measure $d\hat{\nu}_\phi$ has the form

$$d\hat{\nu}_\phi(t) = \begin{pmatrix} e^{-|t|} & e^{-|t|/2} \\ e^{-|t|/2} & 1 + f(t) \end{pmatrix} d\mu_\phi(t) + \begin{pmatrix} 0 & 0 \\ 0 & d\tau_s(t) \end{pmatrix}. \tag{27}$$

Again, it follows from the nonnegative character of the matrix multiplying $d\mu_\phi(t)$ that

$$e^{-|t|} \leqq e^{-|t|} + e^{-|t|} f(t),$$

and, hence,

$$0 \leqq f(t), \qquad \text{a.e. } \mu_\phi(\,\cdot\,). \tag{28}$$

Now, however, because $d\hat{\nu}$ and $d\nu$ both generate the moment sequence $\{\tilde{T}_j\}$, it is obvious that

$$\begin{aligned} (T_j\phi, \phi) &= \int_{-\infty}^{\infty} t^j(1 + f(t)) d\mu_\phi(t) + \int_{-\infty}^{\infty} t^j d\tau_s(t) \\ &= \int_{-\infty}^{\infty} t^j d\mu_\phi(t). \end{aligned} \tag{29}$$

From (29), it follows that for $j = 0$,

$$\int_{-\infty}^{\infty} f(t) d\mu_\phi(t) + \int_{-\infty}^{\infty} d\tau_s(t) = 0. \tag{30}$$

By (25), (28), and (30),

$$f(t) = 0, \quad \text{a.e. } \mu_\phi, \quad d\tau_s = 0, \tag{31}$$

and, by the remarks made earlier, $\{\tilde{T}_j\}$ is a determinate moment sequence.

Reducing subspaces. If an Hermitian moment sequence, $\{T_j\}$, is determinate and can be expressed as

$$T_j = U_j + V_j, \qquad j = 0, 1, 2, \cdots,$$

where $\{U_j\}$ and $\{V_j\}$ are also Hermitian moment sequences, then both sequences are determinate (the converse is, of course, false—see the remarks following Lemma 1). If this were not the case and, say, $\{U_j\}$ were generated by both $d\rho(t)$ and $d\hat{\rho}(t)$, then if $d\sigma(t)$ generates $\{V_j\}$, both $d\rho(t) + d\sigma(t)$ and $d\hat{\rho}(t) + d\sigma(t)$ would generate $\{T_j\}$. Since $\{T_j\}$ is assumed determinate, these two measures are equal. This implies $d\hat{\rho}(t) = d\rho(t)$, and $\{U_j\}$ is determinate. These considerations lead to

THEOREM 2. *If $\{T_j\}$ is a determinate Hermitian moment sequence, P is an orthogonal projection onto a closed subspace, $\mathcal{H}_P$, of $\mathcal{H}$, and P reduces $\{T_j\}$,*

$$PT_j = T_jP = PT_jP, \qquad j = 0, 1, 2, \cdots,$$

then both $\{PT_jP\}$ and $\{(I - P)T_j(I - P)\}$ are determinate and $\mathcal{H}_P$, $\mathcal{H}_{I-P}$ are determinate subspaces of $\mathcal{H}$ with respect to $\{T_j\}$.

Proof. Note that,

$$T_j = PT_jP + (I - P)T_j(I - P), \qquad j = 0, 1, 2, \cdots.$$

Both terms on the right are Hermitian moment sequences. Since $\{T_j\}$ is assumed determinate, both sequences are, by the discussion given above, determinate. The rest of the theorem follows from the definition of determinate subspace.

It should be noted that the theorem just proved, along with the Imbedding Theorem, show that subspaces of indeterminacy are associated with "off-diagonal" elements being present when a 2×2 matrix representation is used.

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